# FROM CHINA TO PARIS: 2000 YEARS TRANSMISSION OF MATHEMATICAL IDEAS

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BENNO VAN DALEN

FRANZ STEINER VERLAG STUTTGART

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# BOETHIUS

TEXTE UND ABHANDLUNGEN ZUR
GESCHICHTE DER MATHEMATIK
UND DER NATURWISSENSCHAFTEN

BEGRÜNDET VON JOSEPH EHRENFRIED HOFMANN
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# Introduction

> The mathematical *Daoshu* is brief in expression but extensive in use, showing a wise perception of generality. That from one kind all others are arrived at is called knowledge of the *Dao*. Classification is indispensable for the sage to learn and master knowledge.
>
> — Chenzi to Rong Fang, *Zhou Bi Suanjing*

Communication and the exchange of goods, information, and ideas between cultures has been a part of the world's history since the beginning of time despite barriers of geography, language, and local custom. While it is often possible to trace the fortunes of artifacts as they moved between different parts of the world, the exchange of ideas is less tangible and therefore more difficult to document. How is it possible, for example, to account for the fact that in various mathematical texts — from ancient China to medieval Europe — more or less the same problems arise, often using the same examples, parameters, and methods? If virtually identical problems and procedures were not rediscovered from culture to culture independently, then how were they transmitted, and to what effect?

It was in hopes of beginning to provide answers to such questions that in July of 1997, a conference was held at the Mathematisches Forschungsinstitut in Oberwolfach, Germany. There a group of eleven scholars began the task of examining together primary sources that might shed some light on exactly how and in what forms mathematical problems, concepts, and techniques may have been transmitted between various civilizations, from antiquity down to the European Renaissance following more or less the legendary silk routes between China and Western Europe. As the Oberwolfach meeting did not include either a Sanskrit expert or an historian of Indian mathematics, it made the rationale for a second, larger meeting all the stronger. With this in mind we approached the International Commission on the History of Mathematics, the International Mathematical Union, the US National Science Foundation, and the Rockefeller Foundation, in hopes of attracting sufficient support to hold a subsequent meeting at the Rockefeller Foundation's Study and Conference Center in Bellagio, Italy. The purpose of the Bellagio meeting was to bring together a larger group of scholars than had been possible at Oberwolfach, and to focus attention on early mathematical works, especially those in China, India, Mesopotamia, the Arabic/Islamic world, and the late Middle Ages/Renaissance in Europe.

The focus of both the Oberwolfach and Bellagio meetings was upon a core of rather simple practical or textbook problems that turn up in different cultures and times, often but not always in different guises. For example, the problem familiar in every schoolroom from ancient Babylonia to modern Beijing: how high will a ladder of given length reach up against a wall if its base is a certain number of feet away from the wall? Other procedures that appear in many similar contexts are derived from problems of surveying, practical arithmetic problems involving the conversion of money between different currencies or amounts of grain of various grades or value, or geometric problems concerning right triangles, circles, or spheres. By examining such analogous methods, procedures, and problems, it is sometimes possible to determine or at least to suggest with some degree of certainty where and how the problems and methods in question originated, and how they may have migrated from one location or text to others in different places and later periods. Did such information flow predominantly in one direction at certain times, or were ideas being freely exchanged between cultures on a more-or-less continuous basis?

Only recently could such meetings as those held in Oberwolfach and Bellagio have been possible, for it is only in the last decade or two that numerous primary sources of Chinese, Indian, Mesopotamian, and Arabic or Persian origins, among others, and a variety of ancient and medieval Western works, have been sufficiently identified, studied, edited, and published to make such cross-cultural studies possible. While no single individual might have a fluent command of Chinese, Sanskrit, Arabic, Persian, Greek, Latin, Hebrew, Italian, French, German, and Spanish — to name just the most obvious languages that would be necessary to study the complexities of transmission of ideas between East and West in any comprehensive and satisfactory way — it was possible to assemble an international group of scholars with shared interests who did have collectively a command of all these languages, and who could in turn examine together paradigmatic cases which served to help us, as a group, to piece together the history of the transmission of mathematics in the early days when trade routes and patterns of intellectual migration were the primary loci for exchange of both material and intellectual goods.

A paradigmatic example of the kind of study we wanted to pursue through the conference at Bellagio was written nearly twenty years ago by Kurt Vogel. In his article, "Ein Vermessungsproblem reist von China nach Paris," professor Vogel studied the history of a specific surveying problem in which the height of a mountain or tower as well as its distance from an observer is to be determined in cases where the distance between the observer and object is not directly traversable or measurable. The problem arises in very similar forms, with virtually the same methods of solution, in works of mathematicians as diverse as Liu Hui (late third century A.D.), Āryabhaṭa, Brahmagupta, al-Bīrūnī, and Hugh of St. Victor. As late as the fourteenth century, in Paris, it turned up again in another treatise, the *Practica geometriae* of Dominicus de Clavasio. Consequently, we felt it was appropriate to make Vogel's now classic paper available to a wider readership with the English translation that opens this collection of papers presented at the Bellagio conference.

It was exactly such points as those raised above that we expected the Bellagio meeting to explore in greater and more satisfying detail. Specifically, we hoped to document with as many diverse examples as possible exactly what the nature of communication was between mathematicians working in China and India on the one hand, and between India and the West on the other. While the studies in this volume do not resolve these matters completely, they make valuable contributions to clarifying what the routes of transmission were, and how mathematical knowledge, be it practical, theoretical, or purely recreational, passed from one culture to another.


## Acknowledgments

We are grateful to the International Commission on the History of Mathematics, the International Union for History and Philosophy of Science, the International Mathematical Union, the US National Science Foundation, and the Rockefeller Foundation, for providing the financial resources to hold a one-week conference at the Rockefeller Foundation's Research and Conference Center in Bellagio, Italy, in May of 2000. We also wish to express, on behalf of the twenty-three participants in the Bellagio meeting, our thanks to Susan Garfield, Director of the New York Office of the Bellagio Conference Center, and to Gianna Celli, Director of the Center in Bellagio, for all their help in making sure that the conference ran smoothly. The staff at the Bellagio Conference Center did everything possible to maximize our efforts during the week of our meeting, and did so in an extraordinarily hospitable and welcoming atmosphere.

We are likewise pleased to acknowledge here our sincere thanks to Matthias Kreck, Director of the Mathematisches Forschungsinstitut Oberwolfach, Germany, for his support of the earlier meeting on the Transmission of Mathematics in July of 1997. The relative seclusion of the Institute in the Black Forest, and its excellent scientific resources provide an especially stimulating atmosphere for scholarly discussion and exchange. Indeed, it was the initial success of that first meeting in Oberwolfach that motivated us to organize the follow-up conference at Bellagio.

We also wish to thank Steiner Verlag and the Kurt Vogel Stiftung for support in making publication of the proceedings of the Bellagio conference possible in the series *Boethius*. To Kim Williams and Markus Barow we wish to acknowledge our thanks for their help in the editorial process, and in preparing the final version of the individual manuscripts for publication. The Arabic citations in the article by Jan P. Hogendijk were typeset with the package ArabTEX developed by Klaus Lagally of Stuttgart University.



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Menso Folkerts (Munich)
Benno van Dalen (Frankfurt am Main)

# A Surveying Problem Travels from China to Paris *

by KURT VOGEL

This paper treats the history of a surveying problem in which both the height of a mountain or tower and its distance from the observer are to be determined in cases where the distance between observer and object cannot be crossed. This problem arises in works by the Chinese (Liu Hui), Indians (Āryabhaṭa, Brahmagupta), Arabs (al-Bīrūnī) and the Christians of the Middle Ages (*Geometria incerti auctoris*, Hugh of St. Victor), all of which present similar examples and methods of solution.

The determination of distances in the plane like that of the width of a river, or the measurements of the height of a mountain or a tower, belong to the tasks of civil or military experts, which would include the Egyptian *harpedonaptae*, the Indian rope-stretchers, and the Roman surveyors. Heron [1903: 187–315] wrote the earliest work on this topic, including the necessary instrument. However, in his theory of measurement, the *Dioptra*, there is yet a problem missing that was much studied later. It concerns the measurement of the height and distance of a mountain or tower when the terrain between the observer's position and the object cannot be crossed.

Such an exercise appears for the first time in the *Sea Island Mathematical Classic* (*Haidao suanjing*) [Van Hée 1938; Swetz 1992], which Liu Hui provided as a continuation of the last book of the *Nine Chapters on the Art of Mathematics* (*Jiuzhang suanshu*) [Vogel 1968; Shen *et al.* 1999], in which surveying problems also appear. The text of the first of the nine exercises reads:

> Now an island in the sea is to be observed. One puts two poles of the same height of three *zhang* [= 30 feet = 5 paces] 1,000 paces apart, such that the pole in back is aligned with the pole in front. If one moves 123 paces back from the front pole, and looks from the ground up to the top of the mountain on the island, this coincides with the top end of the [front] pole. If one moves 127 paces back from the rear pole, and looks from the ground up to the top of the mountain on the island, this again coincides with the top end of the [rear] pole. The question is to find the height of the island and its distance from the [front] pole? [Van Hée 1938: 269]

Then Liu Hui gives the answer: the height of the island is 4 miles, 55 paces (1 mile = 300 paces) and its distance from the first pole is 102 miles, 150 paces.

Thereafter, Liu Hui describes the solution procedure, which in terms of the labeling in Figure 1 — Liu Hui had no diagram — may be represented as follows:

* The present paper is a translation by Joseph W. Dauben and Benno van Dalen of the German original [Vogel 1983]. The figures were scanned and redrawn by Susanne Krömker of the Interdisciplinary Institute for Scientific Computing (IWR) of Heidelberg University.

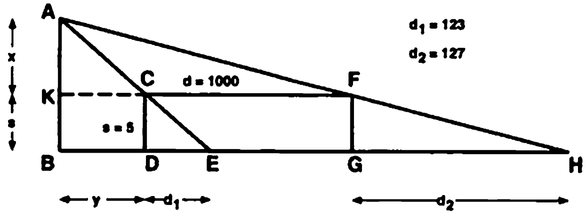


**Figure 1**

$$\text{Height of the island:} \quad x+s=\frac{s\cdot d}{d_2-d_1}+s \qquad (F_1)$$

$$\text{Distance:} \quad y=\frac{d_1\cdot d}{d_2-d_1} \qquad (F_2)$$

These two formulas can be derived in various ways from the similarity of the three pairs of triangles ($AKC$ and $ABE$; $AKF$ and $ABH$; $ACF$ and $AEH$). The easiest derivation is:

$$CF:EH=AC:AE=AK:AB=BD:BE \quad \text{or}$$

$$\frac{d}{d+d_2-d_1}=\frac{x}{x+s}=\frac{y}{y+d_1},$$

which immediately gives the formulas ($F_1$) and ($F_2$).

Liu Hui wrote nothing about the derivation of his procedure; his commentary on the *Sea Island* may perhaps have given some explanation. It was lost as were so many ancient mathematical writings. These have only survived because all works that could be found were collected in the "Great Encyclopedia" (*Yongle dadian*, 1403–1407) of the Ming Dynasty. Also in a later project, the "Complete Library of the Four Branches" (*Siku quanshu*, beginning in 1773), the ancient mathematical works appeared reedited, as did the *Sea Island* in an edition by Dai Zhen (1724–1777) or the commentary on this work by Dai's younger contemporary Li Huang [Kogelschatz 1981].

From Li Huang's diagram of our surveying problem (Figure 2), one sees that he begins with a line $FJ$ parallel to $ACE$. If we substitute Li Huang's technical terms for the separate lines with letters, then his derivation of the formulas ($F_1$) and ($F_2$) appears as follows:

> Extend $FCK$ running parallel to $GEDB$, then $KB$, $CD$ and $FG$ are all equal. Extend $FJ$ parallel to $EC$, then $JG$ and $ED$ are equal. The figure $FGH$ is similar to the figure $AKF$ ("equal pattern"). The figure $FGJ$ is similar to the figure $AKC$. The figure $FGJ$ as part of the figure $FGH$ has the same short-side [*Kurzkathete*] $FG$, while $HJ$ represents the difference between the long-sides [*Langkatheten*]. The figure $AKC$ as part of the figure $AKF$ has the same short-side $AK$, while $FC$ represents the difference between the long-sides. Now the small difference between the long-sides $HJ$ has the same relation to the small short-side $FG$ as the

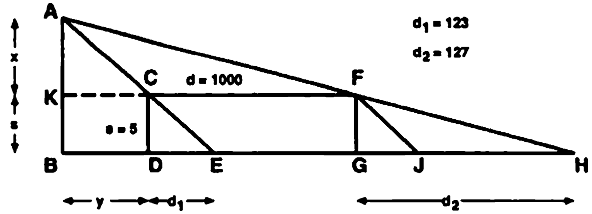


**Figure 2**

large difference between the long-sides $FC$ has to the large short-side $AK$, so that $AK$ may be derived. If one adds the height of the pole $KB$ to this, one obtains $AB$ and thus the height of the island. [Unpublished translation by Kogelschatz].

The calculation of the distance $BD$ is also described in the same way. It is $HJ : JG = FC : CK$. "In this way one obtains the distance of the island from the forward pole." Then follows the substitution of the numerical values in ($F_1$) and ($F_2$).

Two hundred years after Liu Hui this problem appears in India, with Āryabhaṭa [Clark 1930; Shukla & Sarma 1976]. In verse 16 of the mathematical chapter in the *Āryabhaṭīya*, it says:

> The shadow, multiplied by the distance between the tips of the shadows, divided by the difference (between the shadows) is one side (*koṭī*); *koṭī* multiplied by the gnomon, divided by the shadow, is *bhujā* (the arm) [Elfering 1975: 121].

The horizontal side (see Figure 2) is thus

$$BE = y + d_1 = \frac{d_1 \cdot d}{d_2 - d_1} + d_1 = \frac{d_1 \cdot (d + d_2 - d_1)}{d_2 - d_1},$$

and the vertical side is:

$$AB = \frac{s \cdot (d + d_2 - d_1)}{d_2 - d_1}.$$

Moreover, something else has also changed, namely the presentation of the problem. It is no longer an exercise in surveying. From a lamp on a lampstand the light falls on two poles (*śaṅku*), and what now has to be determined is the distance of the lampstand from the ends of the shadows.

The commentator Parameśvara introduces a numerical example, wherein the second gnomon stands inside the shadow of the first gnomon (Figure 3). His numerical values are: $d_1 = 10$; $d_2 = 16$; the distance between the ends of the two shadows is $d + d_2 - d_1 = 12$, thus $d = 6$. It is not mentioned that $s = 12$; the gnomon is always divided into 12 parts (fingers). Thus the side *bhujā* is 24 and the two *koṭī* are 20 and 32.

The shadow problem also appears with later Indian mathematicians, for example with Brahmagupta [Colebrooke 1817: 318]. For his general rule Pṛthūdakasvāmin gives the numerical values $d_1 = 15$, $d_2 = 18$, $d = 15 + 7 = 22$. For $s$, 12 is again assumed. Thus the height of the lamp is 100, and the lengths of the two lines on the

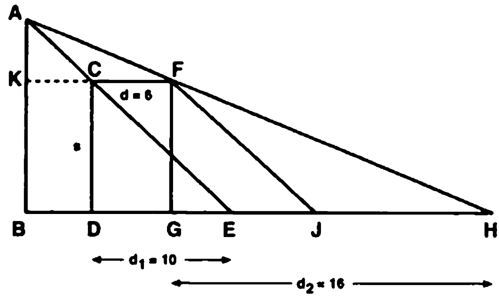


Figure 3

ground from the base of the gnomons to the ends of the shadows are 125 and 150 respectively.

An example in Bhāskara's *Līlāvatī* [Banerji 1927: 163 f., Section 245] uses $d_1 = 8$, $d_2 = 12$, $d + d_2 - d_1 = 52$, thus $d = 48$. Again with $s = 12$, the height of the lamp is 156 fingers = $6\frac{1}{2}$ ells, and the distances to the ends of the shadows are 104 and 156.

The Indian knowledge was transmitted further. It went on to the Arabs through translations from the Sanskrit, for example by al-Bīrūnī, one of the greatest scholars of Islam. In his astronomical work, *The Exhaustive Treatise on Shadows*, there is also a chapter "On the Determination of Distances on Land and the Heights of Mountains" [Kennedy 1976]. Therein he introduces, with explicit reference to Brahmagupta, whom he also frequently cites elsewhere, the problem already mentioned in which light from a minaret shines on two gnomons [Kennedy 1976, I: 279; II: Section 149, 161 f.]. In the diagram (Figure 4) the parallel to the ray of light $ACE$ is again drawn through the tip of the second pole, and the numerical values selected are also the same as Pṛthūdakasvāmin's. In other chapters al-Bīrūnī determines the directions by means of the astrolabe.

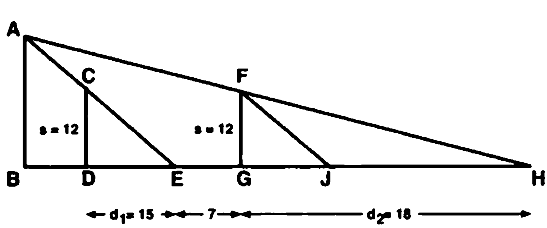


Figure 4

The route of our problem from the Arabs to the Christian Middle Ages, where we know that it has arrived, is still hidden in the dark. One might suspect that the problem was known to Leonardo of Pisa, who was an expert on Arabic mathematics and astronomy. He was familiar with the quadrant and described it [Boncompagni 1862:

204 f.]. However, his examples are limited to only the simple case of measurement from a single standpoint.

The situation is different with the manuscript in which our problem reappears for the first time. It is the third book of the *Geometria incerti auctoris*, which previously was ascribed to Gerbert [Bubnov 1899: 310–355]. No one knows where it came from; it is not contained in the corpus of the Roman surveyors [Bubnov 1899: 311: "III cc. 1–26 ~ ex incerto opusculo"]. The earliest manuscripts (London, Munich, Paris) are from the 11th century. But since the text of these has already been rearranged, they must be based on an earlier version, perhaps going back to a time shortly before Gerbert. Is it conceivable that the author may have stayed in Luxeuil or Corbie, in whose manuscript workshops (*scriptorium*) many manuscripts were produced, including gromatic and geometric ones, as early as the eighth, ninth, and tenth centuries? [Ullman 1964] The author had knowledge of the astrolabe and the Arabic word *(h)alhidada* for the moveable dioptric ruler through the aperture (*foramina*) of which one sights on the object to be surveyed.

For the problem of determining the height of a mountain over a terrain that cannot be crossed, he had various methods.

In one example [Bubnov 1899: 331–332 (Exercise 21), Fig. 60 in Tab. III], the height of the observer is also taken into consideration since — as it says in another place, *quia visum humi adjungere difficile mensori* [Bubnov 1899: 324] — it is difficult to observe on the ground with the eye. In the diagram *ab* represents the gnomon and *cd* the height of the observer up to eye level (see Figure 5).

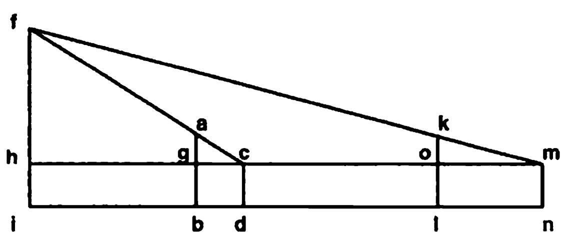


**Figure 5**

From the triangles *agc* and *kom* the ratio of the sides is determined. One sees that $hc$ is the same multiple of $hf$ as $gc$ of $ga$, thus $hc = \kappa \cdot hf$. Likewise $hm = \rho \cdot hf$. Since the distance $cm$ is known, we have $(\rho - \kappa) \cdot hf = cm$. The text uses only the specific numerical values $\kappa = 2$ (resp. 3) and $\rho = 4$ (resp. 7). Since the difference is taken to be equal to XXX *ells*, the base *ni* in the first case is twice as large $[\rho - \kappa = 2]$, in the second case four times as large $[\rho - \kappa = 4]$ as the height of the mountain that is sought.

Another method is applied in Exercises 2 and 3 [Bubnov 1899: 318–320, Figs. 43, 44 in Tab. II] (see Figure 6). One can see from the diagram that this time the astrolabe is used, represented symbolically by means of a little circle. Along both sight lines it says that according to the reading of the plumb line the setting is 4 (resp. 3). We indicate two quadrants in Figure 7.

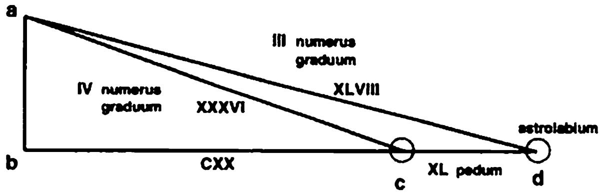


**Figure 6**

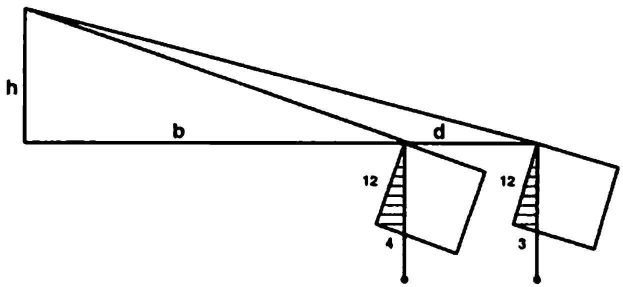


**Figure 7**

Since the side of the quadrant is usually divided into 12 parts, it follows from the similarity of the triangles that:

$$(1) \quad b : h = 12 : 4 \qquad \text{and} \qquad (2) \quad (b+d) : h = 12 : 3 \,.$$

Thus $b + d - b = d = 4h - 3h = h$. In this case the height of the mountain is therefore equal to the distance, which one knows and in the exercise appears as 40. But in the text an instrument is assumed with a side of 144 units. Thus:

$$d = \frac{144\,h}{36} - \frac{144\,h}{48} = \frac{144\,h\,(48-36)}{36 \cdot 48} = h \,.$$

In Exercise 3 the author gives further specific numerical examples.

From all of these diagrams a geometric problem arises, namely the determination of the height, of the projection (*ejectura*), and of the area of an oblique triangle, which is worked out in the second exercise of Book IV of the *Geometria incerti auctoris*. This might confirm the conjecture of Ullman [1964: 285], who sees the many medieval writings on surveying as a sign that these not only served the acquisition of practical knowledge, but that from these the material was also derived for teaching the *quadrivium* (so long as the entirety of Euclid was not yet available for use).

Another early gromatic-geometric treatise is the *Practica geometriae* of Hugh of St. Victor [Baron 1956]. In the exercises he has everything that was to be learned from the *Geometria incerti auctoris*: he also has the exercise in which the *numerus graduum* (cf. Figure 6) was 4 and 3, and although he says that the side of the quadrant is divided into 12 degrees, he applies — like the *Geometria incerti auctoris* — the superfluous numbers 36 and 48. Hugh, who came from Northern France, first lived in Marseille and then taught in Paris, where he died in 1141. In the middle of the

14th century the *Practica geometriae* of Dominicus of Clavasio also originated in Paris and was distributed widely in numerous manuscripts [Busard 1965].

Thus our travels from China to Paris come to an end.

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# Seleucid Innovations in the Babylonian "Algebraic" Tradition and their Kin Abroad

by JENS HØYRUP

Seleucid and Demotic mathematical sources, along with problems and techniques that continue older Babylonian and Egyptian traditions, both present us with a number of innovations: the treatment of "quasi-algebraic" problems about rectangular sides, diagonals and areas, and summations of series "until 10." This paper characterises these two problem types and investigates their presence in certain Neopythagorean and agrimensorial Greco-Roman writings, and in Mahāvīra's compendium of Jaina mathematics, and discusses their possible influence on the Chinese *Nine Chapters on Arithmetic*.

## The Surveyors' Tradition and its Impact in Mesopotamia

Much of my work during the last decade or so has dealt with what I have called the "surveyors' tradition" and its impact on Babylonian, Greek and Islamic "literate" mathematics. Most of the results have been published elsewhere,[1] for which reason I shall restrict this introduction to what is crucial for the theme I have been asked to deal with.

The tradition in question is a "lay," that is, non-scribal and for long certainly also in the main an oral tradition (barring a single known exception, non-scribal literacy only became a possibility with the advent of alphabetic writing); elsewhere, when discussing the characteristics of such traditions, I have termed them "sub-scientific."[2] Its early traces are found in the Mesopotamian written record of the late third and early second millennium BCE; it is plausible but not supported by any positive evidence that it was also present in the Syrian and the core Iranian regions in the second millennium BCE; influence in Egypt at that date was probably absent.[3] In the later first millennium BCE it influenced both Demotic Egyptian,

1 A concise presentation will be found in [Høyrup 1997d], with a fuller discussion in [Høyrup 2001]; both concentrate on the proto-algebraic aspects of this material. Aspects related to practical geometry proper are dealt with in [Høyrup 1997a].

2 See [Høyrup 1990a, 1997b, 1997c].

3 Firstly, neither the Rhind Mathematical Papyrus nor the Moscow Mathematical Papyrus betrays any acquaintance with this "surveyors' tradition," even though surveying mathematics is well represented in both. Secondly, one problem group in the Rhind Papyrus (the filling problems, no. 35–38) turns out to be related to (for linguistic reasons indeed to be derived from) West Asian practitioners' mathematics (see [Høyrup 1999a: 124]), indicating that such borrowed problems were not *a priori* ostracized, which would otherwise be a possible explanation of their absence from the two papyri; but this problem type has nothing to do with mensuration. Thirdly, the problems pointing to West Asia that turn up in Demotic sources are exclusively related to the innovations of the Seleucid age (third to second century BCE) — see below, p. 20.

Greek and Hellenistic mathematics. Whether the altar geometry of the *Śulbasūtras* was also influenced is less easily decided (quite apart from the question whether the constructions described in these first millennium writings go back to early Vedic, that is, second-millennium practice); however, the indubitable traces of the tradition in Mahāvīra's ninth-century CE *Gaṇita-sāra-saṅgraha* are likely to go back to early Jaina times and thus to the late first millennium BCE or the first centuries CE (see below, p. 24). Even though the intermediaries are not identified, the Syrian, Iranian and Central Asian regions are unlikely not to have been involved at that moment. Possible influence on the Chinese *Nine Chapters on Arithmetic* is a question to which we shall return.

Many traces of the tradition are found in Arabic sources from the later first and early second millennium CE; some, finally, in Italian *abbaco* writings from the late Middle Ages — in part derived from earlier Latin translations of classical Arabic works, in part however from what must have been still living tradition in the Islamic world.

One of the Arabic works — perhaps written around 800 CE, perhaps only a witness of the terminology and thus of the situation of that period, immediately preceding the work of al-Khwārizmī — is a *Liber mensurationum* written by one Abū Bakr, so far known only in a Latin translation due to Gherardo da Cremona[4] (whence the Latin title). Strangely, it still conserves much of the phraseology already found in Old Babylonian (but not late Babylonian) problems derived from the tradition — so much indeed that transmission within some kind of institutionalised teaching network seems plausible, perhaps supported by writing since the advent of non-scribal Aramaic alphabetic literacy. Even though we know nothing about the geographic origins of Abū Bakr, this consideration seems to locate him within the Iraqo-Syrian area.

Basic practical mathematics is often so unspecific that shared methods constitute no argument in favour of transmission or borrowing. Once area measures are based on the square of the length unit, there is only one reasonable way to find the area of a rectangle, and only one intuitively obvious way to approximate the area of an almost-rectangular area (*viz* the "surveyors' formula," average length times average width). Certain geometric procedures are less constrained by the subject-matter and may serve (we shall see an example in note 32), but the best evidence for links is normally supplied by recreational problems, those mathematical riddles that in pre-modern mathematical practical professions served to confirm professional identity — "if you are an accomplished calculator/surveyor/..., tell me ...," as runs a formula that recurs in various versions in many cultures.

The main evidence for the existence and long-term survival of the surveyors' tradition is indeed provided by a set of geometrical riddles that turn up in all the contexts mentioned above, in formulations that often exclude mere transmission within the literate traditions and from one literate tradition (as known to us, at least) to another.

4 Ed. [Busard 1968].

The original core of this set of riddles contained a group dealing with a single square ("???" indicates doubt as to whether the problem was already present in the second millennium BCE; $s$ designates the side, $_4s$ "all four sides", $d$ the diagonal and $Q$ the area of the square; here and everywhere in the following, Greek letters stand for given numbers):

$$\begin{array}{lll} s + Q = \alpha & (= 110) \\ {}_4s + Q = \alpha & (= 140) \\ Q - s = \alpha & \\ Q - {}_4s = \alpha & (???) \\ S - Q = \alpha & \\ {}_4s - Q = \alpha & (???) \\ {}_4s = Q & \\ d - s = 4 & (???)^{5} \end{array}$$

Other early riddles must have treated of two squares:

$$Q_1 + Q_2 = \alpha, \quad s_1 \pm s_2 = \beta$$
$$Q_1 - Q_2 = \alpha, \quad s_2 \pm s_1 = \beta$$

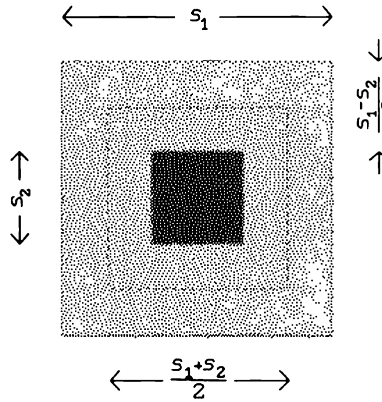


**Figure 1.** Illustration of the principle that the area of a quadratic border equals a rectangle contained by $l$ and $w$, where $l$ is the (dotted) "mid-length" of the border and $w$ its width.

When the difference between the areas is given, the two squares were thought of as concentric and the difference thus as the area of a quadratic border; at least in later (classical and medieval) times the areas of such quadratic and circular borders were determined as the product of the "mid-length" $l$ (in the quadratic case $2s_1 + 2s_2$) and the width $w$ (in the quadratic case $(s_1 - s_2)/2$) — cf. Figure 1. It would be very strange if the same intuitively evident rule (a "naive" version of *Elements* II.8) were not used in early times.

These problems dealing with a rectangle (length $l$, width $w$, diagonal $d$, area $A$) must have circulated in the earliest second millennium:[6]

5 The problem $d - s = 4$ turns up (with solution $s = 4 + \sqrt{32}$) in the *Liber mensurationum* in a way that shows it to be somehow traditional; evidently it refers to the practical assumption that $d = 14$ if $s = 10$. But the evidence is insufficient to prove that this was a traditional *riddle* — even a traditional *approximation* might inspire a problem once it was discovered how to solve it correctly.

6 In fact, the two problems $A = \alpha$, $l \pm w = \beta$ are likely to constitute, together with the problems $A = \alpha$, $l = \beta$ and $A = \alpha$, $w = \beta$, the very beginning of the riddle tradition. These four constitute a cluster in several Old Babylonian texts (ca. 1700 BCE) and still go together in Ibn Thabāt's *Reckoner's Wealth* from ca. 1200 CE (ed., trans. in [Rebstock 1993: 124]). The problems $A = \alpha$, $l = \beta$ and $A = \alpha$, $w = \beta$ are already found in Akkadian tablets from the 22nd century BCE, but the other two not yet (which is part of the evidence that the trick of the quadratic completion was only discovered somewhere between 2100 BCE and 1900 BCE).
One may add that area metrologies were sufficiently complex to make the apparently innocuous "division problems" $A = \alpha$, $l = \beta$ and $A = \alpha$, $w = \beta$ unpleasantly complex — something like a division of acres by feet.

$$A = \alpha, \quad l \pm w = \beta$$
$$A + (l \pm w) = \alpha, \quad l \pm w = \beta$$
$$A = \alpha, \quad d = \beta$$

On circles (with diameter $d$, perimeter $p$ and area $A$), at least the problem

$$d + p + A = \alpha$$

must have circulated in the early second millennium BCE, probably also the simpler version where $A$ is omitted.

Most of the problems must have been solved in the original environment in the same way as in the Old Babylonian school, *viz* by means of a "naive" (that is, non-critical though reasoned) analytical cut-and-paste technique. When a length and a width were involved, the solution made use of

$$\text{average } \mu = \frac{l+w}{2} \quad \text{and deviation } \delta = \frac{l-w}{2},$$

a technique that is familiar from *Elements* II.5–6, and which is illustrated in Figure 2 for the case $A = 15$, $l - w = 2$ (corresponding to *Elements* II.6):

The excess $2\delta = 2$ of length over width is bisected and the corresponding area moved around so as to bring forth a gnomon, still with area 15. To this is added a quadratic complement $\delta \times \delta = 1 \times 1 = 1$, which gives us a square $\mu \times \mu = 15 + 1 = 16$. Hence $\mu = 4$, whence $w = \mu - \delta = 4 - 1 = 3$, $l = \mu + \delta = 4 + 1 = 5$.

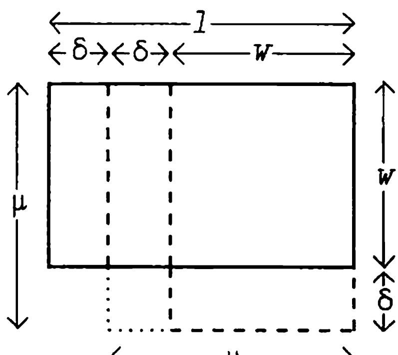


**Figure 2.** A geometrical solution of the problem $A = 15$, $l - w = 2$ by means of average $\mu$ and deviation $\delta$.

This restricted set of riddles was adopted by the Old Babylonian school not long before 1800 BCE, and developed into a genuine mathematical discipline of algebraic character, in which measurable segments served to represent entities of other kinds (at times areas or volumes, at times prices or numbers of men and days' work), and where coefficients varied freely. But this discipline died with the school institution itself at the collapse of the Old Babylonian social structure around 1600 BCE, and only one trace of its sophistication can be found in later sources (a problem type where the sides of a rectangle represent *igûm* and *igibûm*, "the reciprocal and its reciprocal," a pair of numbers belonging together in the table of reciprocals). When "algebraic" problems turn up sparingly in Late Babylonian but apparently pre-Seleucid texts (maybe ca. 500 BCE), they appear to have been a fresh borrowing from the surveyors tradition,[7] and do not go beyond the restricted range of original riddles.[8] Methods are still traditional, based on naive cut-and-paste geometry and on average and deviation.

## Seleucid Innovations

Innovation, instead, is unmistakable in the two Seleucid tablets AO 6484 and BM 34568.[9] The former of these is from the early second century BCE,[10] the latter — and most interesting — is undated but roughly contemporary.

One problem from the undated text treats of alloying, all the others deal with rectangular sides, diagonals and areas;[11] apart from determinations of $d$ or $A$ from $l$ and $w$ or of $w$ from $d$ and $l$, everything is new in some way. Two problems are traditional as such, giving $A$ and either $l+w$ or $l-w$; but the procedures differ from traditional ways, finding for instance in the former case $l-w$ as $\sqrt{(l+w)^2-4A}$, $w$ next as $\frac{(l+w)-(l-w)}{2}$, and $l$ finally as $(l+w)-w$; the method of average and deviation is nowhere used (not even in the reverse version where $l$ and $w$ are found from $l+w$ and $l-$ w as average and deviation).

The remaining problems belong to totally new types. In total, the content of the tablet is as follows:[12]

(1) $l=4$, $w=3$; $d$ is found as $\frac{1}{2}l+w$ — first formulated as a general rule, next done on the actual example.

(2) $l=4$, $d=5$; $w$ is found as $\sqrt{d^2-l^2}$.

(3) $d+l=9$, $w=3$; $l$ is found as $\frac{\frac{1}{2}\cdot([d+l]^2-w^2)}{d+l}$, $d$ as $(d+l)-l$.

(4) $d+w=8$, $l=4$; solution corresponding to (3).

(5) $l=60$, $w=32$; $d$ is found as $\sqrt{l^2+w^2}=68$.

(6) $l=60$, $w=32$; $A$ is found as $l\cdot w$.

7 In any case, the transmission was not due to the Babylonian scribal tradition proper, since the technical use of Sumerograms in the terminology is discontinuous. But we cannot exclude the possibility that part of the transmission was the result of peripheral scribal groups (Hittite or Syrian), as was the case for astrology, cf. [Farber 1993: 253 f.].

8 See [Friberg 1997]. For supplementary (linguistic) evidence that the texts in question antedate the Seleucid era, see [Høyrup 1999a: 161].

9 Ed. Neugebauer in [*MKT* I: 96–99] and Waschow in [*MKT* III: 14–17], respectively.

10 See [Høyrup 1990b: 347, n. 180].

11 One is dressed as a problem about a reed leaned against a wall. In the general context of rectangular problems it is obvious, however, that the underlying problem is $d-l=3$, $w=9$ (symbols as above).

12 I interpret the sexagesimal place value numbers in the lowest possible integer order of magnitude and transcribe correspondingly into Arabic numerals. Entities that are found but not named in the text are identified in ⟨ ⟩.

(7) $l = 60$, $w = 25$; $d$ is found as $\sqrt{l^2 + w^2} = 65$.

(8) $l = 60$, $w = 25$; $A$ is found as $l \cdot w$.

(9) $l + w = 14$, $A = 48$; $\langle l - w\rangle$ is found as $\sqrt{(l+w)^2 - 4A} = 2$, $w$ as $\frac{1}{2}\cdot([l + w] - \langle l - w\rangle)$, and $l$ finally as $w + \langle l - w\rangle$.

(10) $l + w = 23$, $d = 17$; $\langle 2A\rangle$ is found as $([l + w]^2 - d^2) = 240$, $\langle l - w\rangle$ next as $\sqrt{(l+w)^2 - 4A} = 7$— whence $l$ and $w$ follow as in (9).

(11) $d + l = 50$, $w = 20$; solved as (3),[13] $l = 21$, $d = 29$.

(12) $d - l = 3$, $w = 9$ (the reed problem translated into a rectangle problem, cf. note 11); $d$ is found as $\frac{\frac{1}{2}\cdot(w^2 + [d-l]^2)}{d-l} = 15$, $l$ as $\sqrt{d^2 - w^2} = 12$.

(13) $d + l = 9$, $d + w = 8$; first $\langle l + w + d\rangle$ is found as $\sqrt{(d+l)^2 + (d+w)^2 - 1} = 12$, where 1 obviously stands for $(l - w)^2 = ([d + l] - [d + w])^2$; next, $w$ is found as $\langle l + w + d\rangle - (d + l) = 3$, $d$ as $(d + w) - w$, and $l$ as $(d + l) - d$.

(14) $l + w + d = 70$, $A = 420$; $d$ is found as $\frac{\frac{1}{2}\cdot([l+w+d]^2 - 2A)}{l+w+d} = 29$.

(15) $l - w = 7$, $A = 120$; first $\langle l + w\rangle$ is found as $\sqrt{(l-w)^2 + 4A} = 23$, next $w$ as $\frac{1}{2}(\langle l + w\rangle - [l - w]) = 8$, and finally $l$ as $w + (l - w)$.

(16) A cup weighing 1 *mina* is composed of gold and copper in ratio 1 : 9.

(17) $l + w + d = 12$, $A = 12$; solved as (14), $d = 5$.

(18) $l + w + d = 60$, $A = 300$; not followed by a solution but by a rule formulated in general terms and corresponding to (14) and (17).

(19) $l + d = 45$, $w + d = 40$; again, a general rule is given which follows (13).

(Several more problems follow, too damaged however to allow interpretation.)

Although problems of the simple form of (2), (5) and (7) do not occur in the Old Babylonian record, the Pythagorean theorem (or, as I would prefer to call it in a context where "theorems" have no place, the "Pythagorean rule") is well represented in the Old Babylonian mathematical corpus.[14] So is of course the area computation of (6) and (8). Numbers (2), (5), (6), (7) and (8) thus present us with no innovation beyond the numerical parameters.

The use of a special rule for the case (1) where $l$ and $w$ are in ratio 4 : 3, on the other hand, is certainly an innovation, even though this ratio was the standard assumption of the Old Babylonian calculators and the reason that 1;15 (= 1¼)

13 With the only difference being that the division in (3) is formulated as the question "9 steps of what shall I go in order to have 36," whereas the present problem multiplies by the reciprocal of $d + l$. Since this method is distinctively Babylonian and thus irrelevant for the questions of borrowing and influence I shall not mention it further on.

14 [Høyrup 1999b] presents a complete survey of its published appearances.

appears in tables of technical coefficients as "the coefficient for the diagonal of the length and width".[15] In later times, a standard rule told that the situation where $d-l=l-w$ corresponds to the ratios $w:l:d=3:4:5$;[16] it is a reasonable assumption that (1) is a first extant witness of this rule.

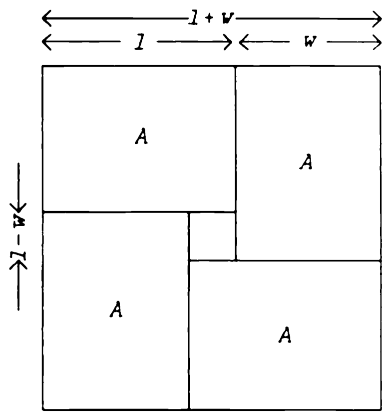


Figure 3. The diagram underlying BM 34568, no. (9) and (15) (but not drawn in the tablet).

As problems, (9) and (15) are familiar from Old Babylonian texts, and they are likely to represent the very beginning of the tradition for mixed second-degree riddles — cf. note 6. The solutions, however, are not the traditional ones based on

$$\text{average } \mu=\frac{l+w}{2} \quad \text{and}$$

$$\text{deviation } \delta=\frac{l-w}{2};$$

instead, they follow the diagram of Figure 3, which is also likely to have been the actual basis for the reasoning. It is a striking stylistic feature (and also a deviation from earlier habits) that even the possibility of determining $l$ and $w$ by a symmetric procedure (namely as

$$\frac{(l+w)+(l-w)}{2} \text{ and } \frac{(l+w)-(l-w)}{2},$$

respectively) is not used.

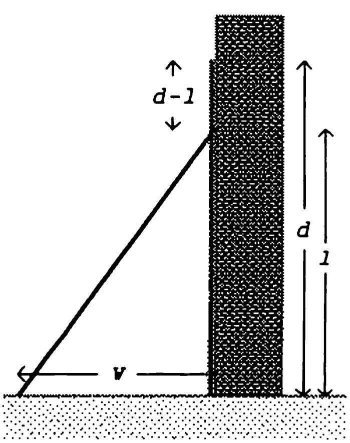


Figure 4. The reed leaned against the wall.

Problems treating of a reed or pole leaned against a wall are already present in an Old Babylonian anthology text BM 85196.[17] The situation is shown in Figure 4: A reed of length $d$ first stands vertically against a wall; afterwards, it is moved to a slanted position, in which the top descends to a

15 *Textes mathématiques de Suse* III,35 [Bruins & Rutten 1961: 26]. "Length and width" stands for the simplest configuration determined by a single length and a single width, that is, the rectangle.

16 Referred to in the *Liber mensurationum* [Busard 1968: 97] and in Leonardo's *Pratica* (ed. in [Boncompagni 1862: 70]).

17 Ed. Neugebauer in [*MKT* II: 44]. On the wide diffusion of this problem type, see [Sesiano 1987].

height $l$ (the descent thus being $d-l$) while the foot moves a distance $w$ away from the wall.

In the Old Babylonian versions that have come down to us, $d$ is given together with either $w$ or $d-l$. The solution thus requires nothing beyond simple application of the Pythagorean rule. We cannot know, of course, whether the problem type of the present tablet ($d-l$ and $w$ given) was also dealt with in Old Babylonian times; but if so, the solution would probably have taken advantage of the facts that $\square(w) = \square(d) - \square(l) = \sqsubset\!\sqsupset(2l+2w,(l-w)/2)$ — possibly expressed as $\sqsubset\!\sqsupset(l+w,\ l-w)$ —, and thus have found $d+l$ as $w^2/(d-l)$.[18]

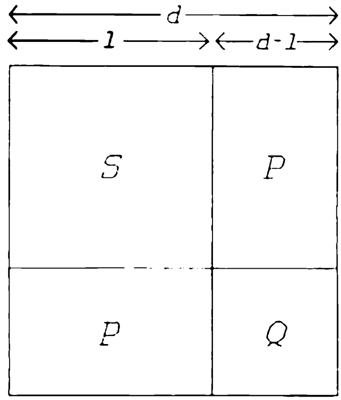

**Figure 5.** A geometric justification for the solution of BM 34568, no. 12.

This, as we see, is not done here. In algebraic symbolism, the solution follows from the observation that $(d-l)^2 + w^2 = d^2 + l^2 - 2dl + w^2 = 2d^2 - 2dl = 2d(d-l)$.

It can also be explained from the diagram of Figure 5: $(d-l)^2$ corresponds to the area $Q$, whereas $w^2$ corresponds to $\square(d) - \square(l) = \square(d) - S = 2P + Q$. The sum of the two hence equals $2P + 2Q = 2\sqsubset\!\sqsupset(d,\ d-l)$. In itself, this is nothing but a possible basis for the argument, though supported by the fact that halving precedes division by $d-l$, which makes best sense if the doubled rectangle is reduced first to a single rectangle; seen in the light of what follows imminently (and since halving *invariably* precedes division by the measure of a side in parallel cases), this or something very similar seems to have been the actual argument.

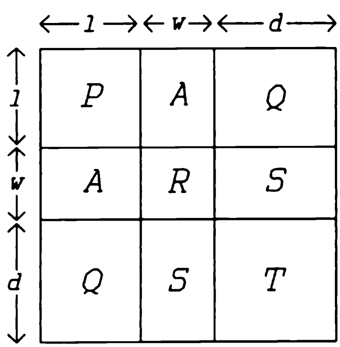

**Figure 6.** The geometric justification for the case $l+w+d=\alpha$, $w=\beta$.

Most in favour is obviously the type represented by (14), (17) and (18). The procedure can be explained from Figure 6: $(l+w+d)^2$ is represented by $P+R+T+2A+2Q+2S$; removing $2A$ and making use of the fact that $P+R=T$ we are left with $2Q+2S+2T = 2\sqsubset\!\sqsupset(d,\ l+w+d)$.

This proof is given by Leonardo Fibonacci in the *Pratica geometrie*, in a way which indicates that he has not invented it himself — at most he has inserted a diagonal (omitted here) in order to explain the construction of the diagram in Euclidean

18 Here and in the following, $\square(s)$ designates the (measured) square with side $s$, and $\sqsubset\!\sqsupset(l,w)$ the rectangle contained by $l$ and $w$.

manner.[19] In view of the faithfulness of both Leonardo and Abū Bakr in as far as the sequence of computational steps is concerned when they repeat the "new" procedures of the Seleucid style (see p. 21), it is a reasonable assumption that the geometric proofs go back to the same source.

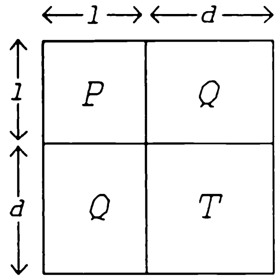


**Figure 7.** The probable geometric argument for the type $d + l = \alpha$, $w = \beta$.

Of the remaining problem types, two are quite new and one only new as far as the procedure is concerned. The type of (3), (4) and (11) can be regarded as a counterpart of (12) (the reed problem). The similarity to this type and to that of (14), (17) and (18) makes it reasonable to look for an analogous justification — see Figure 7: the whole square represents $(l + d)^2$; if we notice that $T = P + w^2$ and remove $w^2$, we are left with $2P + 2Q = 2\square(l, l + d)$.

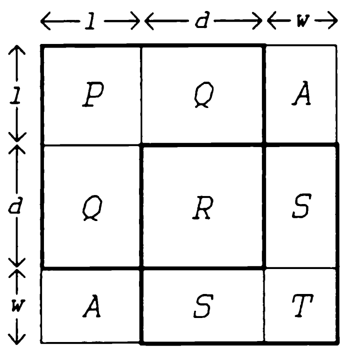


**Figure 8.** The probable geometric justification for the case $d + l = \alpha$, $d + w = \beta$.

The type of (13) and (19) is likely to be based on a slight variation of Figure 6, here shown in Figure 8 (which might equally well have served for the type $l + w + d = \alpha$, $A = \beta$, but which happens not to be the proof given by Leonardo). As we see, $(d + l)^2 + (d + w)^2 = (P + 2Q + R) + (R + 2S + T) = (l + w + d)^2 + (R - 2A)$. That $R - 2A$ equals $\square(l - w)$ was familiar knowledge since Old Babylonian times; it follows from Figure 3 if we draw the diagonals of the four rectangles $A$, as shown in Figure 9 (cf. presently); if we express $R$ as $\square(l) + \square(w)$ it can also be seen from the "naive" version of *Elements* II.7 used

19 Ed. [Boncompagni 1862: 68]. Firstly, it is evident from scattered remarks in the treatise that Leonardo renders what he has found; secondly, all steps of the procedure correspond to what is given in the *Liber mensurationum* (ed. [Busard 1968: 97]), but in the passage in question there is no trace of verbal agreement with Gherardo's translation (in other passages Leonardo follows Gherardo *verbatim*, correcting only the grammar). Moreover, Leonardo's statement runs as follows:

> *Si maius latus et minus addantur cum dyametro, et sint sicut medietas aree; et area sit 48,*

whereas Gherardo has:

> *aggregasti duo latera eius et diametrum ipsius et quod provenit, fuit medietas 48, et area est 48.*

Gherardo's "medietas 48" instead of "24" is obviously meaningless unless it is already presupposed that this 48 represents the area — in other words, that the sum of all four sides and both diagonals equals the area. We may therefore conclude that Leonardo had access to another version of Abū Bakr's work, in which, however, the proof was given, or to some closely related work.

in Figure 5, which is likely to have been in still longer use.

The last type is (10). In this exact form it is not found in earlier Babylonian texts, but the closely related problem $d = \alpha$, $A = \beta$ is found in the tablet $Db_2$-146.[20] There it is solved by subtracting $2A$ from $\square(d)$, which leaves $\square(l-w)$, cf. Figure 9. $l$ and $w$ are then determined from

$$\frac{l+w}{2} \quad \text{and} \quad \frac{l-w}{2},$$

that is, average $\mu$ and deviation $\delta$. This problem recurs with the same procedure in Savasorda's *Collection on Mensuration and Partition* (the *Liber embadorum*);[21] Abū Bakr and Leonardo[22] use the complementary method and add $2A$ to $\square(d)$, finding thus $l + w$. All three solve the rectangle problems that result by means of average and deviation.

Our Seleucid text starts by finding $2A$, namely as $\square(l+w) - \square(d)$, and next calculates $\square(l-w)$ as $\square(l+w) - 4A$.[23] The rest follows the asymmetric procedure of (9).

Two other cuneiform problem texts of Seleucid date are known. VAT 7848[24] contains geometric calculations of no interest in the present context. AO 6484 was already mentioned. It is a mixed anthology, which is relevant on three accounts:

1) One of its problems (obv. 12, statement only) is a rectangle problem of the type $l + w + d = \alpha$, $A = \beta$.
2) It is interested in the summation of series "from 1 to 10." In obv. 1–2, $1 + 2 + \ldots + 2^9$ is found; in obv. 3–4, $1 + 4 + \ldots + 10^2$ is determined. The latter follows from the formula $\sum_{i=1}^{n} i^2 = (1\cdot\frac{1}{3} + n\cdot\frac{2}{3})\cdot\sum_{i=1}^{n} i$ .
3) It contains a sequence of *igûm-igibûm* problems (see p. 12), and thus demonstrates that the tradition of second-degree algebra had not been totally interrupted within the environment that made use of the sexagesimal place value system with appurtenant tables of reciprocals. The problems differ from the Old Babylonian specimens by involving multi-place numbers — no doubt an innovation due to the environment of astronomer-priests where the tablet was produced, and in which multi-place computation was routine.[25]

20 Ed. [Baqir 1962].

21 Ed. [Curtze 1902: 48] (Plato of Tivoli's Latin translation); ed., trans. [Guttmann & Millàs i Vallicrosa 1931: 44] (Catalan translation from a somewhat different redaction of the Hebrew text).

22 Ed. [Busard 1968: 92] and [Boncompagni 1862: 64], respectively.

23 This seemingly roundabout procedure is the clearest evidence that the configuration of Figure 9 is indeed used. From a strictly algebraic point of view, it would be simpler to find $(l-w)^2$ as $2d^2 - (l+w)^2$.

24 Ed. Neugebauer & Sachs in [*MCT*: 141].

25 The colophon of the tablet tells that it was produced by Anu-aba-utēr, who identifies himself as "priest of [the astrological series] *Inūma Anu Enlil*," and who is known as a possessor and producer of astronomical tablets.

So far there is no particular reason to believe that the other innovations of which the Seleucid texts are evidence were also due to this environment — or *not* to believe it. Although BM 34568 is theoretically more coherent than AO 6484, it is still a secondary mixture — as shown by the presence of the alloying problem, which is certainly no invention of the astronomers.

## Demotic Evidence

At this point, two Demotic papyri turn out to be informative: P. Cairo J.E. 89127–30, 89137–43 and P. British Museum 10520, the first from the third century BCE, the second probably of early (?) Roman date.[26]

The latter begins by stating that "1 is filled up twice to 10," that is, by asking for the sums

$$T_{10} = \sum_{i=1}^{10} i \quad \text{and} \quad P_{10} = \sum_{i=1}^{10} T_i$$

and answering from the correct formulae

$$T_n = \frac{n^2 + n}{2}, \quad P_n = (\frac{n+3}{2}) \cdot (\frac{n^2 + n}{2})$$

— not overlapping with the series dealt with in AO 6484 but sufficiently close in style to be reckoned as members of a single cluster.[27]

The Cairo Papyrus is more substantial for our purpose. Firstly it contains no less than 7 problems about a pole first standing vertically and then leaned obliquely against a wall (cf. p. 15 f.). Three are of the easy type where $d$ and $w$ are

26 Both ed., trans. [Parker 1972].

27 All our evidence for this cluster is second-century BCE or later. Yet, since no other traces of possible Greek influence is found in these texts, the interest in square, triangular and pyramid numbers and in the sacred 10 of the Pythagoreans is intriguing.
Even more intriguing is the observation that the Demotic determination of $T_{10}$ as $(10^2 + 10)/2$ corresponds to an observation made by Iamblichos in his commentary to Nicomachos's *Eisagoge*, *viz* that $1 + 2 + ... + 9 + 10 + 9 + ... + 2 + 1 = 10 \cdot 10$ [Heath 1921, I: 114], whereas the Seleucid determination of $1^2 + 2^2 + ... + 10^2$ as

$$(1 \cdot \frac{1}{3} + 10 \cdot \frac{2}{3}) \cdot \sum_{i=1}^{10} i$$

turns up in the pseudo-Nicomachean *Theologumena arithmeticae* (X.64, ed. [de Falco 1975: 86], trans. [Waterfield 1988: 115]), in a quotation from Anatolios. More precisely, Anatolios gives the sum as "sevenfold" $\sum_{i=1}^{10} i$, that is, in a form from which the correct Seleucid formula cannot be derived — another hint that the Pythagorean knowledge of the formula was derivative.
All in all it thus appears that heuristic proofs made by means of *psēphoi* were borrowed by the Greeks from Near Eastern calculators; it also seems plausible that the summations of series "from 1 to 10" originated centuries before the Seleucid era.

given, and solved by simple application of the Pythagorean rule; three are of the equally simple type where $d$ and $d - l$ are given, and solved similarly (both, we remember, are treated in an Old Babylonian text). Two, finally, are of the more intricate type found in BM 34568 ($w$ and $d - l$ given) and solved as there.

Further on in the same papyrus, two problems about a rectangle with known diagonal and area are found. The solution is clearly related to the problems of BM 34568 (cf. also Figure 9): Addition of $2A$ to $\square(d)$ and subsequent taking of the square root gives $l + w$, whereas subtraction and taking of the root yields $l - w$. $l$ and $w$ are then found by the habitual asymmetric procedure.

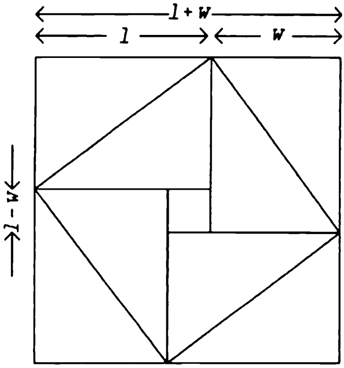


**Figure 9.** The diagram underlying $Db_2$-146 and BM 34568, no. 10.

These and other Demotic mathematical papyri also contain material that descends from the Pharaonic mathematical tradition as we know it from the Rhind and Moscow Papyri. What they share with the Seleucid tablets, however, has no known Egyptian antecedents (neither in actual content nor in style); in some way it represents an import (there is no reason to doubt the West Asian origin of the reed problem, nor the ultimate descent of the rectangle problems from surveyors' tradition). It is clear, however, that they represent the new stage at least as well as the Seleucid texts. This does not prove that Demotic Egypt was the place where the transformation occurred; but the appearance of the characteristic problems as already fully integrated components of Egyptian scribal mathematics in the third century BCE makes it implausible that the novelty be due to the contemporary environment of Seleucid astronomer-priests. Indeed, West Asian tax collectors and surveyors will certainly have followed the Assyrian and Achaemenid conquerors to Egypt; on the other hand, the methods of Babylonian mathematical astronomy, when they eventually reached Egypt, did so in a reduced version which shows them to have been carried by astrological "low" practitioners, not by the scholar-astronomers that had created them.

## Abū Bakr and Leonardo

The evidence offered by the *Liber mensurationum* and Leonardo's *Pratica* seems to speak more generally against a localization of the Seleucid-Demotic innovations within the core tradition of the surveyors (however this core may have looked in the late first millennium BCE — but the chain that transmitted not only problems but also standard phraseology can reasonably be regarded as a "core"). With slight variations, Abū Bakr has all, and Leonardo almost all, of the problems from BM 34568.[28] However, all problems that originated in earlier epochs (given $l + w$ and $l - w$, given $A$ and $l \pm w$, given $A$ and $d$, given $l + w$ and $d$) are solved traditionally, by means of average and deviation; asymmetric procedures occur only in connection with the definitely *new* problems. The general shift to asymmetry which we encounter in BM 34568 and the Demotic papyri was evidently not accepted in the core tradition, although it accepted the new problems, and did not attempt to reformulate these in more symmetric ways.[29] This speaks definitely against the emergence of these new problems within the core.

## Greco-Roman Problems

Relevant material from Greek and Latin sources is extremely scarce. Only two texts that I know of contain evidence of influence from the new "Seleucid" style.

One is the Latin *Liber podismi*,[30] the title of which betrays it to be a translation from a Greek original (or at least to be inspired by a Greek model). One of its problems deals with the rectangle (actually a right triangle, as preferred in all Greek sources) with given area and diagonal, and does so in the manner of the Demotic papyrus, with the small difference that $l$ is found first as $\frac{1}{2} \cdot (\langle l + w\rangle + \langle l - w\rangle)$, and $w$ next as $l - \langle l - w\rangle$. The next problem is overdetermined, giving $A$, $d$ and $\langle l + w\rangle$; it first finds $\langle l - w\rangle$ from $d^2$ and $A$, next $l$ as $\frac{1}{2} \cdot ([l + w] + \langle l - w\rangle)$, and finally $w$ as $(l + w) - l$. Obviously, the solutions of BM 34568, no. 10 and the Demotic diagonal-area problems are combined, but with the same variant of the asymmetric method as in the previous problem.

The other is Papyrus graecus genevensis 259,[31] probably from the second century CE. It contains three problems on right triangles:

$$\begin{array}{lrl} (1) & w = & 3,\ d = 15 \\ (2) & w + d = & 8,\ l = 4 \\ (3) & l + w = & 17,\ d = 13 \end{array}$$

The first tells us nothing. The second (identical with BM 34568, no. 4) makes use

28 The "reed problem" occurs in the regular rectangle version, the problem $d + l = \alpha$, $d + w = \beta$ as $d + w = \beta$, $l = w + \gamma$ (which explains the error in the formulation of BM 34568, where $(l - w)/2$ appears as an unexplained 1).

29 In another respect, however, even the new problem types as appearing with Abū Bakr and Leonardo give evidence of normalization, cf. above, note 19: the problem $l + w + d = 24$, $A = 48$ is formulated in a way which shows it beyond reasonable doubt to have been derived from a problem ${}_2l + {}_2w + {}_2d = A = 48$ (subscript "2" meaning "both"). This predilection for "the perimeter" in problems on rectangles is related to what circulated in Neopythagorean environments and to what we find with Mahāvīra; in contrast, all Babylonian sources, Old Babylonian as well as Seleucid, are interested in "both sides" (cf. also BM 34568, no. 17, $l + w + d = A = 12$).

30 Ed. [Bubnov 1899: 511 f.]. The two relevant problems are reproduced in [Sesiano 1998: 298 f.].

31 Ed. [Sesiano 1999].

of the fact[32] that $l^2 = d^2 - w^2 = (d+w)\cdot(d-w)$ and finds $[d-w]$ as $l^2/(d+w)$; $w$ is then found as $\frac{1}{2}\cdot([l+w]+[l-w])$, and finally $d$ as $(d+w)-w$. The third problem — identical with BM 34568, no. 10 apart from the numerical parameters — is solved in an algebraically more straightforward way (cf. note 23): first $\langle(l-w)^2\rangle$ is found as $2d^2 - (d+w)^2$, next $w$ as $\frac{1}{2}\cdot([l+w]-\langle l-w\rangle)$, and $l$ finally as $(l+w)-w$.

All in all, these Greek problems might belong in the periphery of a cluster where the Demotic and Seleucid texts represent something closer to the core:

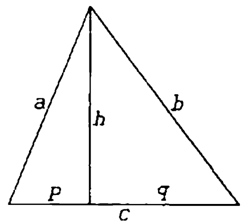

**Figure 10.** The height in the scalene triangle.

32 This trick, we notice, was *not* used in the Seleucid reed-problem. On the other hand, it appears to have been used in what must be a pre-Euclidean (in all probability, a pre-Greek) version of the computation of the height of a scalene triangle, and in problems about two concentrically located squares; it will thus have been quite familiar (cf. also the version presented by *Elements* II.8).
The pre-Euclidean computation of the height of the scalene triangle (see [Høyrup 1997a: 82], also for references) makes use of observation that

$$\square(b) - \square(a) = \{\square(q) + \square(h)\} - \{\square(p) + \square(h)\} = \square(q) - \square(p)$$

(see Figure 10). Since $\square(q) - \square(p)$ is the border between two squares,

$$\square(q) - \square(p) = \sqsubset\!\sqsupset\left(\frac{q+p}{2}, 2(q+p)\right) = \sqsubset\!\sqsupset\left(\frac{q+p}{2}, 2c\right),$$

$$\text{whence } \frac{q-p}{2} = \frac{b^2-a^2}{2} \div c \text{, while } \frac{q+p}{2} = \frac{c}{2}.$$

Using the principle of average and deviation we therefore have

$$q = \frac{c}{2} + \frac{b^2-a^2}{2} \div c, \quad p = \frac{c}{2} - \frac{b^2-a^2}{2} \div c,$$

from which $h$ can easily be derived by means of the Pythagorean theorem.
Alternatively, one might use the "extended Pythagorean theorem" as formulated in *Elements* II.13,

$$\square(a) + 2\sqsubset\!\sqsupset(q,c) = \square(b) + \square(c), \text{ whence } q = \frac{b^2+c^2-a^2}{2} \div c.$$

Hero indeed does so in the *Metrica*. In contrast, treatises on practical geometry from the Middle Ages invariably use the former, "quasi-algebraic" method at least for acute-angled trapezia (Savasorda, Leonardo Fibonacci), mostly also for acute-angled triangles (Abū Bakr, al-Karajī, Ibn Thabāt); at times the Euclidean formula (or a variant using *Elements* II.6) is given as an alternative; in the case of obtuse-angled triangles and trapezia, on the other hand, we invariably find a formula derived from *Elements* II.12. The only reasonable explanation seems to be that the practitioners' environment was in possession of the quasi-algebraic formula at an early moment, but only for "genuine," that is, inner heights (al-Khwārizmī, who is the only author who uses algebra proper, explains indeed that obtuse-angled triangles possess only one height). The Greek theoreticians generalized the concept of heights, reformulated the calculation as a theorem and found the counterpart corresponding to the case of the obtuse angle; finally, the practitioners took over what was new (as they took over the Archimedean $\pi$), but conserved the old formula in the cases for which it had originally been created.

they deviate slightly in their choice of actual procedures, but the general tenor is the same. They share the asymmetric approach, but may have replaced geometric visualization by more genuine algebraic manipulation. Further, they are formulated in terms of right triangles instead of rectangles.

## An Indian Witness: Mahāvīra

Mahāvīra's ninth-century *Gaṇita-sāra-saṅgraha* is in itself a late source — slightly later than al-Khwārizmī and roughly contemporary with Thābit ibn Qurra.[33] There are, however, no reasons to doubt Mahāvīra's assertion that he has taken advantage of "the help of the accomplished holy sages" when "glean[ing] from the great ocean of the knowledge of numbers a little of its essence [...] and giv[ing] out [...] the *Sāra-saṅgraha*, a small work on arithmetic,"[34] that is, that he presents what was since times immemorial part of the Jaina tradition; moreover, internal evidence also speaks in favour of the claim.[35] Given the history of the Jaina community, the adoption of material from the Near East or the Mediterranean region of which Mahāvīra's work bears witness is likely to have taken place in late pre-Christian or at most early Christian centuries.

As a matter of fact, we should probably rather speak of "adoptions" in the plural. Mahāvīra divides his chapter on geometry into four sections: "approximate measurement"; "minute accurate calculation"; "devilishly difficult problems"; and one on the "Janya operation", which does not concern the present argument. All groups encompass material that is not known from non-Indian sources, but all also contain rules that are familiar from the Near East and the Mediterranean.

In the section on "approximate" area measurement (pp. 187–197, stanzas VII.7–48) we find the "surveyors' formula"; the determination of the area of a circular ring as average circumference times width (indeed exact); the rule that the circular circumference is three times the diameter; and the problem of finding the separate value of circumference, diameter and area from their sum (in this order, the characteristic order of the pre-Old-Babylonian riddle tradition), together with a corresponding rule (based on the choice of the circumference as the basic parameter and on $\pi = 3$).

The section on "minutely accurate" area determination (pp. 197–208, stanzas VII. 49–89½) gives the "pre-Euclidean" method for finding the (inner) height in a scalene triangle, together with Hero's formula for triangular and quadrangular areas (for the latter set forth as if it were of general validity); gives the circular area as one fourth of the product of arc and diameter;[36] and states the circumference to be $\sqrt{10}$ times the diameter.

33 I am grateful to Yvonne Dold-Samplonius for having first directed my attention to this important source.

34 Ed., trans. [Raṅgācārya 1912: 3]. All subsequent page references are to this translation.

35 See the discussion in [Høyrup 2001], which I shall not repeat here.

36 This formula is already used in Old Babylonian material, but only for the semicircle where both arc and diameter are external measures.

Among the "devilishly difficult" problems (pp. 220–257, stanzas VII.112½–232½) we find several of the characteristic traditional surveyors' riddles (together with variants with "non-natural" coefficients corresponding to transformation of the tradition within an institutionalized school environment): area equal to perimeter (for squares and rectangles);[37] and later on, rectangular area and perimeter given (a slight variation of the problem $A = \alpha$, $l + w = \beta$); rectangular perimeter and diagonal given (equivalent to problem 10 of BM 34568 and problem 3 of the Geneva Papyrus); and rectangular area and diagonal given. The perimeter-and-diagonal problem has the same parameters as the Geneva version and, more strikingly, finds $(l - w)^2$ in the same way, *viz* as $2d^2 - (l + w)^2$; the area-and-diagonal problem finds both $l + w$ and $l - w$, as do the Cairo Papyrus and the *Liber podismi*. Even the area-and-perimeter problem is solved via $l + w$ and $l - w$, not via average and deviation. But in contradistinction to all the western versions, $l$ and $w$ are then found by a symmetric procedure with a technical name of its own (*saṅkramaṇa*), as average and deviation of $l + w$ and $l - w$. As in the Seleucid and Demotic problems, rectangles and not right triangles are concerned.

The riddles have thus been adopted in "Seleucid-Demotic," not "Old Babylonian" versions; in some respects, however, they echo the Greek rather than the Near Eastern forms. Those of the "minutely accurate" calculations that are not specifically Indian are related to developments that can only have taken place in or reached the Near East around the mid-first millennium BCE.[38] Whatever "minutely precise" was borrowed will thus have arrived when the Jaina school was already established; but its arrival may well have preceded that of the "devilishly difficult" problems. Those of the "approximate" rules that were adopted from elsewhere, on the other hand, may have arrived long before the appearance of Jainism.

Chapter VI of Mahāvīra's work contains a section on the summation of series (pp. 168–176, stanzas 290–317) which at first looks similar to the Seleucid-Demotic cluster: arithmetical series, geometrical series, and series of the form

37 Of these problems, only the rectangular variant $A = l + w$ is found in the Old Babylonian material. However, they certainly circulated in the Mediterranean region during the classical epoch: the *Theologumena arithmeticae* mentions repeatedly that the square □(4) is the only square that has its area equal to the perimeter (II.10, IV.23 ed. [de Falco 1975: 11, 29], trans. [Waterfield 1988: 44, 63]; the second passage cites Anatolios), doing so in a way that demonstrates this to be a traditional observation; Plutarch on his part refers to Pythagorean knowledge of the equality of area and perimeter in the rectangle ▭(3,6) (*Isis et Osiris* 42, ed., trans. [Froidefond 1988: 214 f.]).

38 The determination of the height of the scalene triangle must predate Euclid, since *Elements* II.13 reformulates it and II.12 extends it to the case of the outer height, see note 32; but the absence of the computation from the pre-Seleucid but still Late Babylonian tablets published in [Friberg, Hunger & al-Rawi 1990] and [Friberg 1997] shows that the invention cannot predate 500 BCE by much, if at all.
The whole tenor of the technique is so close to what was known in Western Asia since the early second millennium BCE and so different from what we know from early Indian mathematics that a diffusion from India to the Near East is much less plausible than a diffusion in the opposite direction.

$\sum_i N_i$, where $N_i$ are squares, cubes or triangular numbers and $i$ runs through an arithmetical progression (with any starting point, any number of members and any difference). What Mahāvīra offers is, however, so much more elaborate than what we find in the Near Eastern texts, and its formulas so different, that inspiration one way or the other becomes a gratuitous hypothesis.

Other Indian sources confirm that such series were a much more serious concern in India at least after ca. 500 CE than they seem to have been in the Seleucid-Demotic region; indeed, both Brahmagupta and Bhāskara II deal with the same types as Mahāvīra, and give rules whose formulation comes closer to the Seleucid-Demotic ones than Mahāvīra's.[39] This might still mean that the Indian calculators developed borrowed material to greater sophistication, or that the Demotic-Seleucid cluster was a borrowing from an earlier and less sophisticated Indian stage. Since the Bakhshālī manuscript (likely to be somewhat earlier than Brahmagupta) contains none of the more complex types (sums of squares, cubes, and triangular numbers) and invariably uses a formula for the arithmetical series which is wholly different in spirit,[40] independent development in the two areas followed by cross-fertilizations may be the most plausible explanation.

## *Nine Chapters on Arithmetic*

The Chinese *Nine Chapters on Arithmetic*[41] is roughly contemporary with our Seleucid and Demotic sources. Though in the main a witness of an independent tradition, this work does contain problems that seem to point to the Near East and the Mediterranean — not least a version of the "reed against the wall" in IX.8 (in which, however, a slanted beam is compared to the same beam in horizontal, not in vertical position).

On first inspection, the whole sequence IX.6–13 looks as if it were related to the Seleucid and Demotic rectangle problems, even though formulated in varying dress and mostly so as to deal with right triangles (the right triangle is indeed the topic of the chapter as a whole). On closer inspection, however, the evidence turns out to be inconclusive.

Translated into the usual *l-w-d* symbolism — that is, expressed in the way which will make kinship stand out as clearly as possible — the problems and their solutions are the following:

39 Trans. [Colebrooke 1817: 290–294] and [Colebrooke 1817: 51–57], respectively. Their rule for the sum of triangular numbers is identical with the Demotic rule; the rule for the sum of squares is but slightly different in formulation from the corresponding Seleucid rule.

40 Namely, $S = [(n-1) \cdot d/2 + a] \cdot n$, $n$ being the number of terms, $a$ the first term, and $d$ the difference (ed. [Hayashi 1995: 439 and *passim*]), termed the "Minus-One" method — that is, the average term multiplied by the number of terms. It thus seems to be based on a purely arithmetical consideration, whereas the Demotic-Seleucid-Pythagorean formula appears to have been derived from *psēphoi* considerations.

41 Ed., trans. [Vogel 1968].

(6) $d-l=1$, $w=5$ (the analogue of the Seleucid reed problem). $l$ is found as $\frac{w^2-[d-l]^2}{2(d-l)}$, whence $d$, where BM 34568, no. 12 finds $d$ as $\frac{½\cdot(w^2+[d-l]^2)}{d-l}$ and next $l$. We observe that the Chinese procedure does not halve before dividing by $(d-l)$, which suggests that a geometric justification, *if* once present, had been forgotten (cf. p. 16).[42]

(7) $d-l=3$, $w=8$. The structure of the problem is the same, but the solution proceeds differently:

$\langle d+l\rangle$ is found as $\frac{w^2}{d-l}=\frac{(d+l)\cdot(d-l)}{d-l}$, and $d$ as $½\cdot(\langle d+l\rangle+[d-l])$.

This is related to the second problem of the Geneva papyrus, but different from everything in the Seleucid and Demotic texts (cf. also note 32).

(8) $d-l=1$, $w=10$. Same problem type and same procedure as (7).

(9) Another variation of (7).

(10) Yet another variation of no. 7.

(11) $d=100$, $l-w=68$.

$\left\langle\frac{l+w}{2}\right\rangle$ is first found as $\sqrt{\frac{d^2-2\cdot(\frac{l-w}{2})^2}{2}}$, and $w$ then as $½\cdot\left(\left\langle\frac{l+w}{2}\right\rangle+\left\langle\frac{l-w}{2}\right\rangle\right)$

— certainly not Seleucid-Demotic in style with its use of average and deviation, nor however similar in detail to anything from the older Near Eastern tradition.

(12) $d-l=2$, $d-w=4$. The solution of this problem builds on the observation that $\square(d-[d-l]-[d-w])=2(d-l)\cdot(d-w)$. This can be argued from a diagram similar in style to Figure 8, which served for the problem type $d+l=\alpha$, $d+w=\beta$ — namely the one shown in Figure 11: the full square $\square(d)$ must equal the sum of the squares $\square(l)$ and $\square(w)$; therefore, the overlap $S=\square(d-[d-l]-[d-w])$ must equal the area which they do not cover, that is, $2R=2\sqsubset\!\sqsupset(d-l,d-w)$.[43]

42 As I am told by Karine Chemla, Liu Hui's commentaries to the *Nine Chapters* "describe diagrams that are very comparable to those which [are here suggested to] lie behind some Seleucid solutions." In the absence of familiarity with these commentaries I can form no reasonable opinion as to whether these diagrams are likely to derive from familiarity with an earlier tradition or to be Liu Hui's own constructions.

43 It should be observed that we are dealing here with the case of constant differences, $d-l=l-w$. The Chinese solution, of course, does not take advantage of that, and we might therefore claim that the closest kin in the western sources is the case of arbitrary differences. This case is not treated by Abū Bakr (nor in the ancient texts); it *is* treated by Leonardo (ed. [Boncompagni 1862: 71]), but he solves it by means of algebra, not by a rule or a diagram pointing toward our geometric tradition. Even though the Chinese solution may build on a geometric argument similar to the ones known from Leonardo (etc.), nothing allows us to connect the actual problem or procedure to Near Eastern or Mediterranean texts.

(13) $d+l=10$, $w=3$. Solved as no. 2 of the Geneva papyrus: $\langle d-l\rangle$ is found as $w^2/(d+l)$, and $l$ as $\frac{1}{2}\cdot([d+l]+[d-l])$.

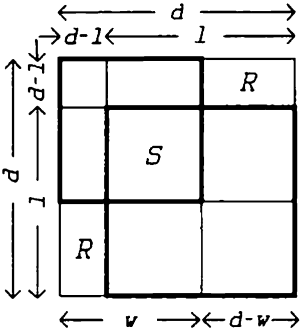


**Figure 11.** The possible geometric basis for problem IX.12 of *The Nine Chapters.*

Similarities are certainly present, and the appearance of a pole leaning against a wall makes it highly plausible that contact has played a role. But the similarities are always relative; *if* the whole interest in this problem type and the approach used in its solution is ultimately inspired from abroad (which does *not* follow from the pole-against-wall dress), then the use of average and deviation in problem (11) and the predominant use of the identity $w^2=(d+l)\cdot(d-l)$ would rather point to a contact preceding the Seleucid epoch and ensuing independent development — but even that is certainly no necessary consequence.

## Cautious Final Remarks

It is highly risky to base the construction of a stemma on the transformations undergone by a single phrase, in particular when crosswise contamination is possible. *If* we are none the less tempted to engage in a similar perilous game we may notice that those of our sources that make preferential use of the identity $w^2=(d+l)\cdot(d-l)$ are those that formulate problems in terms of right triangles, not quadrangles; if not allowing any positive conclusions the observation may at least remind us that the "Seleucid" innovations need not have emerged together, however much the scarcity of sources tempts us to see them as belonging together. This is of course particularly clear regarding the summations and the new approach to the rectangle, where positive though indirect evidence suggests rather different datings.

This observation may be generalized into a conclusion: A number of innovations turn up in Seleucid and contemporary mathematical sources which otherwise reflect the style of second-millennium mathematics. But the written sources that reflect diffusion of the particular "Seleucid" problem types outside the Near Eastern area are, like the sources *from* the Near East itself, too few and too diverse to allow any certain inferences concerning the details of the transmission pattern, or regarding the precise locations and dates where the various innovations were first introduced.

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# Some Ancient and Medieval Approximations to Irrational Numbers and Their Transmission

by J. LENNART BERGGREN

We investigate the historical roots and branches of a number of common approximations of some irrational quantities arising in ancient and medieval mathematics. Almost all of these values or methods were known from China to Western Europe, but in our investigations of their origins and diffusion we have taken into account the varying contexts in which they appear. The historical record of their diffusion seems to suggest, in at least one case, a single origin and diffusion from that center, but, in other cases, multiple origins and again diffusion from these.

## Introduction and Some Methodological Considerations

Could $\pi$ be equal to $\sqrt{10}$ ? Could the side of a square in a circle of radius 60 be equal to 84 307/360? Could 13/15 represent the height of an equilateral triangle of side length 1? A mathematician may find these values curious. But a historian must find it even curiouser (as Alice would say) that only relatively few values for these fundamental constants are found in one culture after another, ranging from China to North Africa and Europe, and over a span of time ranging from Hammurabi ca. 1800 B.C. to Lorenzo de Medici in the $15^{th}$ century A.D. And this curious fact raises the question of whether these values were independent discoveries, repeated time and again, or whether some or all of them may have been transmitted from one ancient or medieval civilization to another.

If, for example, one finds an accurate approximation to an irrational number in several cultures one is tempted to suspect that the value was transmitted from one to the other. So when Zu Chongzhi uses the highly accurate approximation 355/113 for $\pi$ in fifth-century China[1] and Nīlakaṇṭha uses it around 1500 in India[2] one suspects that the value was transmitted.

On the other hand, if two civilizations possess effective iterative algorithms for computing approximations to arbitrary accuracy, the coincidence of accurate values is less surprising. A case in point is the value that the astronomer and geographer, Claudius Ptolemy, used for the side of a square in a circle, a value accurate to six decimal places.[3] It is, in fact, based on the same value for $\sqrt{2}$ as one famous Old Babylonian approximation dating from the time of Hammurabi (1;24,51,10 ($\approx$ 1.4142130)). Did Ptolemy learn of his value from what were ulti-

1 [Lam & Ang 1986: 334]. All dates are A.D. unless otherwise specified.

2 [Gupta 1975: 3–4].

3 [Toomer 1984: 49–50]. Ptolemy's circle had radius 60 and the value he gave for the side of its inscribed square was 84;51,10.

mately Old Babylonian sources? Perhaps he did, since Hellenistic Greeks, after all, learned from Babylonian sources

- the sexagesimal number system in which Ptolemy expressed this value from the Babylonians, and
- astronomical parameters and arithmetic models of planetary motion.[4]

On the other hand, Ptolemy had complete control of the Babylonian arithmetic, and his writings show that he was able to calculate square roots to arbitrary accuracy. Thus it is equally likely that his value was the result of an independent computation by Ptolemy himself, or by an earlier Greek mathematician which had become traditional by Ptolemy's time.

Sometimes context is helpful. Thus the fact that the same approximation to $\sqrt{3}$ appears in Greek and Latin texts discussing the area of an equilateral triangle suggests that the Latin value is derived from the Greek texts.

The discussion as to how the Old Babylonian value for $\sqrt{2}$ could be obtained also raises a methodological point. One scholar has argued that no approximation that, like the Babylonian, is less than the true value can be obtained from any method of approximation by excess.[5] However, someone who was aware that a given method produced values in excess of the true might well look at such a value and drop a small fractional part, suspecting quite correctly that it would improve the approximation.[6]

Of course, algorithms for computing values may have been transmitted just as well as individual values, and we shall see some examples of this later on.

There is also the complication that approximations, such as the very accurate approximation for $\sqrt{2}$ discussed above, or the less accurate 17/12, can be derived by a variety of methods, a point we shall discuss later. Here it suffices to note that knowledge of previous practice within a culture or of a particular writer can assist us in deciding which of the possible methods might be the origin of a particular value.

However, it is not only the accurate approximations and algorithms that pose problems. The problems are no fewer when it comes to markedly crude (e.g., $\pi = 3$), or even bizarre ($\pi = \sqrt{10}$), values found in two different cultures. Although the use of $\pi = 3$ in a number of cultures is hardly evidence of transmission, the situation for $\pi = \sqrt{10}$ is more complicated. We shall go into this example more fully later, and our point here is simply that there are no general rules for deciding between transmission and independent discovery.

## Some Methods for Finding Square Roots

According to one definition, "A method is a trick you use twice." Here, then, are seven methods for approximating $\sqrt{N}$. (We assume N is not a perfect square.)

4 See [Jones 1997].
5 [Gupta 1985: 16].
6 [Fowler & Robson 1998: 374–375] suggest this.

*Method 1.* Find an integer a so that $a^2 < N < (a+1)^2$, and let $r = N - a^2$. By linear interpolation one obtains the approximation $\sqrt{N} = a + r/(2a+1)$. In the case of $\sqrt{2}$ and $\sqrt{3}$ (where a = 1) this yields the values 4/3 and $\sqrt{3} = 5/3$.[7]

Heron's approximation of $\sqrt{54} = 22/3$ in his *Stereometrica*[8] of the first century is the one value in Greek mathematics that can be derived from this method.[9] Similarly, there is indirect evidence of Brahmagupta's use of the method in the seventh century.[10] The method was, however, common in medieval Islam and appears, for example, in the works of three tenth-century writers and one in the 11th century. In the 13th century the famous Persian philosopher and mathematician Naṣīr al-Dīn al-Ṭūsī even calls 2a + 1 "the conventional denominator."[11]

Since Heron depended heavily on Babylonian material and the use of linear approximation is characteristic of Babylonian astronomical methods, Method 1 may have originated in ancient Babylon, even though it is not attested there. From there it went to Hellenistic Greek writers and to India. In neither culture did it gain much popularity, but it came into its own when it went to medieval Islam, through Islam's early adoption of Indian methods.

*Method 2.* A simple check by squaring shows that the approximations obtained by Method 1 are low, so it might be tempting for one using this method to decrease the denominator from 2a + 1 to 2a. This would increase slightly the value of the whole, and one would obtain the approximation, $\sqrt{N} = a + r/2a$.

One could also obtain this by solving $a^2 + r = (a+e)^2$ (approximately) for e by dropping $e^2$ in the expansion of $(a+e)^2$.[12] On solving the resulting linear equation, one obtains $\sqrt{N} = a + r/2a$. But, however it may have been derived, the approximation was quite common. In fact, all values of square roots in Old Babylonian texts can be derived, by iteration if necessary, from this formula.[13] Demotic papyri of the third century B.C. (see [Parker 1969]) provide evidence of the use of this formula,[14] and the papyrus P. Carlsberg 30, a century later, gives a good approximation to $\sqrt{200}$, namely 14 1/7, which fits the use of this formula.[15] The oldest Chinese algorithms for finding non-terminating square roots, which date from the first three centuries of our era, leave on the calculating board exactly the value

7 Note that this rule yields the same denominator of the fractional part for all numbers between two successive squares.

8 [Heath 1921, II: 319] points out that this is not an unaltered work of Heron's.

9 [Heath 1921, II: 51].

10 See [Gupta 1985: 14].

11 According to [Saidan 1978: 441].

12 [Gupta 1985: 15].

13 [Fowler & Robson 1998]. On p. 371 the authors cite an explicit use of the method from BM 96957 + VAT 6598, but the statement that all Old Babylonian values can be derived by this method seems to appear only in their abstract on p. 366.

14 [Parker 1969: 138] (cited in [Gupta 1985]).

15 This implies an approximation to $\sqrt{2}$ accurate to 5 decimal places, since $\sqrt{200}/10 = \sqrt{2}$. The same approximation is used at the end of the ninth century in Egypt by Abū Kāmil [Sesiano 1996: 4].

obtained by this approximation.[16] In India the rule was a favorite of the Jaina writers.[17] Although it never became as popular as the previous method in medieval Islam, one finds the rule used in al-Khwārizmī in the ninth century,[18] and then in writers in the $10^{th}$ and $14^{th}$ centuries.[19] It was likely from Arabic texts, such as the Latin translation of al-Khwārizmī, that it passed to Latin writers and then made its way to central Europe in the $15^{th}$ century.[20]

Unlike linear approximation, this approximation is evidently high, since its square exceeds N by $r^2/4a^2$. When $r > a$, the error in the square is more than 1/4, and as r gets close to $2a + 1$ the error becomes unacceptable. For example, Ibn Ṭāhir pointed out[21] that for $\sqrt{2}$ it yields 3/2, whose square exceeds 2 by 1/4, and for $\sqrt{3}$ it yields 2.

However, in these cases one can use for $a^2$ the first square larger than N, and then from $N = a^2 - r$ derive $\sqrt{N} = a - r/(2a)$ (*), again an approximation by excess, but a better one all the same. For N = 3 this yields $\sqrt{3} = 1\ 3/4$, the standard Old Babylonian value for this quantity.[22] Al-Qalaṣādī, from $15^{th}$-century Granada, recommends this method as the one to use when $r > a$.[23]

Iteration of Method 2 gives the approximation

$$\sqrt{N} = (a + r/2a) - \frac{(r/2a)^2}{2(a + r/2a)}$$

which, when applied to the approximation $\sqrt{2} = 3/2$, gives $\sqrt{2} = 3/2 - (1/4)/3 = 17/12$. Old Babylonian texts use this common ancient value and Heron uses it in *Metrica* I 23, in the context of determining the area of a regular octagon from the square of the length of its side. In the early third century, Marcus Cetius Faventinus, in his builders' manual, *De diversis fabricis architectonicae*, recommends forming a triangle of sides two feet, two feet, and two feet ten inches in order to make a carpenter's square.[24] In addition, both the Bakhshālī manuscript from India[25] and al-Qalaṣādī[26] give the rule for iterating the approximation $\sqrt{a^2 + r} = a + r/2a$.

16 [Chemla 1992: 138].
17 [Gupta 1975: 2].
18 [Folkerts 1997: 96–97].
19 See the excellent survey in [Saidan 1978: 441–445].
20 [Folkerts 1996: 109] points out that the method is used in Regiomontanus's Vienna *Rechenbuch*.
21 [Saidan 1978: 441].
22 See [Fowler & Robson 1998: 372–373].
23 He gives the formula (*) in the mathematically equivalent form $\sqrt{a^2 + r} = a + (r+1)/(2a+2)$. [Rebstock 1999–2000: 197–98] points out that not only was the technique known to the Moroccan al-Ḥaṣṣār in the $12^{th}$ century (in the subtractive form referred to earlier) but it was known in the form (*) two generations earlier to the specialist in inheritance law, al-Ṣardafī, in the Yemen.
24 Cited from [McCague 1993: 22].
25 [Gupta 1985: 17]. The date of the Bakhshālī manuscript is a subject of considerable dispute, and estimates range from the second to the twelfth centuries.
26 [Woepcke 1854: 384]. Thus Saidan's statement [Saidan 1978: 445] that "this formula has not been traced in Arabic texts" is mistaken.

It should be noted, however, that the approximation 17/12, can also be derived by continued fractions and by Theon of Smyrna's method (see below). Therefore, not every appearance of 17/12 is clear evidence for use of an iterated version of Method 2.

In any case, if one iterates Method 2 again, one obtains as an approximation to $\sqrt{2}$ the number 577/408 which, when expressed in unit fractions, yields $1 + 1/3 + (1/3 \cdot 4) - (1/3 \cdot 4)(1/34)$. Some scholars have argued that this is, in fact, how the ancient Indians derived this value, which was, it seems, first mentioned in the *Śulba Sūtras*[27] and became common in ancient India.

The evidence suggests that Method 2 and its iterates originated in Babylon and from there radiated outwards to China and India. From India it, like Method 1, went to medieval Islam and thence to the Latin West. Within this larger transmission of a method, or algorithm, was a minor tradition, within users of mathematics, such as builders, of the transmission of the particular value of $\sqrt{2} = 17/12$.

*Method 3.* An iterative method based on a formula equivalent to Method 2 is the following: If a is any approximation to $\sqrt{N}$ then N/a is an approximation in the other direction. Then, take the average of a and N/a, namely (a + N/a)/2 as a new approximation.[28]

Heron explicitly states this method in his *Metrica*, in the context of showing how to use the square root to compute the area of a triangle from the lengths of its sides[29]. The area of Heron's triangle is $\sqrt{720}$, and Heron, starting with the approximation 27, applies the method to obtain $\sqrt{720} \approx 26 + 5/6$. He then remarks that one can do the same thing again, this time starting with the last value obtained, if one wishes to find a closer approximation. Heron's is the first explicit description of such a method, but the 14th-century Byzantine mathematicians, Barlaam of Calabria and Nicolas Rhabdas, also describe the method,[30] as do Luca Pacioli (ca. 1500) and other Italian algebraists. We are not aware of evidence that Islamic writers used this iterative method, and it is possible that the Byzantines learned it from Heron. From Byzantium, it could then have gone to the Latin West. (The earliest reported Indian use of Method 3 is in a treatise of the early 16th century.[31])

As to the ultimate origins of Method 3, Neugebauer conjectured that they were Old Babylonian. Certainly the appearance of the method in Heron is consistent with this hypothesis, but we would not be surprised to find that the procedure originated in the Hellenistic world.

27 See the edition of the *Śulba Sūtras* in [Sen & Bag 1983], especially the discussion of this value and the various methods by which it may have been derived on pp. 164–168. See also [Gupta 1985: 16–17].

28 For possible roles of averaging in generating a number of medieval approximations of square roots see Gupta 1981.

29 [Høyrup 1997, note 32] has argued that this passage is part of a section (*Metrica* I.vii and viii) that is interpolated in the *Metrica*, but likely one by Heron himself or, at least, another ancient source.

30 According to [Heath 1921, II: 324].

31 According to [Gupta 1985: 16].

On the surface it appears, and some have argued,[32] that the Method 3 would be simpler to apply than the mathematically equivalent Method 2, where one has to calculate $N - a^2$ at each stage, and then divide by 2a. The argument is plausible until one actually tries the two methods. One quickly realizes that the problem is the calculation of N/a, which occurs in either method. In the absence of some uniform system for handling fractions, such as the sexagesimal or decimal system, the division and addition of the fractions obtained by successive iterations become tedious chores.

*Method 4.* Another iterative method for $\sqrt{2}$ found in Greek writings is that given in the *Expositio* of the second-century writer Theon of Smyrna. It involves calculating approximations (S and D) to the side and diagonal of a square according to the rule $S' = S + D$ and $D' = 2S + D$ $(= S + S')$. The quotients D/S, D′/S′, etc. are successive approximations to $\sqrt{2}$.

One may start with $S = D = 1$ to obtain the pairs 2,3; 5,7; 12,17; 29,41; 70,99; 169,239; 408,577. The quotients formed from these are in fact the successive values produced by the continued fraction expansion of $\sqrt{2}$, and 577/408 yields the Indian value from the *Śulba Sūtras*. Theon's is computationally much simpler than any of the methods we have discussed to this point, because it involves no computations with fractions. But it seems to have had no effect on later methods, nor are the origins of the method known. One can only observe, as above, that it yields a number of commonly used ancient values for $\sqrt{2}$ as well as some intermediate terms.[33] As we remarked earlier, however, a given approximation can often be derived by a variety of methods, and, in particular, it seems unlikely that the ancient Indian value came from that set out by Theon of Smyrna.

*Method 5.* Several cultures used the algorithm based on the expansion $(a + b)^2 = a^2 + 2ab + b^2$, explained by Theon of Alexandria,[34] and known from Chinese, Indian, and Arabic sources as well.

Theon, who wrote in the fourth century, explains the method in his commentary on Ptolemy's *Almagest*, and the many accurate approximations to square roots found in Ptolemy's work suggest that Ptolemy knew the algorithm in the first half of the second century. In China one finds it described, very tersely it is true, in the *Nine Chapters* in the first century,[35] and explained more fully in Liu Hui's third-century commentary on this classic work. In India it is known from the *Āryabhaṭīya*, written by Āryabhaṭa in 499. One meets an account in Arabic first in al-Khwārizmī.[36]

32 See, for example, [Fowler & Robson 1998: 376].

33 Thus 7/5, 17/12, and 577/408 (albeit in the Indian form) are all attested ancient values.

34 See [Thomas 1980, I: 52–61] for Theon's account of the process.

35 For the explanation in the *Nine Chapters*, Liu Hui's commentary thereto, and the authors' historical commentary see [Shen, Crossley & Lun 1999: 204–213]. Both the method and the geometric thought that seems to lie behind it seem to be very close to that of Theon.

36 Chapter 14. See [Folkerts 1997: 96–99].

This last fact argues for a transmission of the method from India to Islam, and the early appearance of the work in China (relative to the appearances in Greek and Indian literature) argues for independent invention there. On the other hand, Theon's polished discussion of the method for the example of $\sqrt{4500}$ is set so firmly in the traditional Greek doctrine of rectangular areas and gnomons that it seems likely that it, too, arose as part of Greek mathematics, albeit after the introduction of sexagesimal fractions in the second century B.C.

The difference between the Chinese and Greek accounts on the one hand and the Indian on the other is that the two former describe a procedure that they are aware can be continued to obtain successive fractional parts (decimal in the case of the Chinese and sexagesimal in the case of Theon). And although Theon never explains how to find the whole number part, 67, of the approximation to $\sqrt{4500}$, Āryabhaṭa seems to intend his procedure to extract that part only.[37] For this reason it is hard to imagine the Indian method being derived from the Greek or the Chinese.

*Method 6.* A convenient method for approximation, whatever system one has for fractions, is to multiply the radicand, N, by an integer square, $b^2$. Then one takes the integer part of the square root of the product, and divides the result by b.

The two examples of this usually cited from the works that have been ascribed to Heron, however, seem less a general method than a particular device that arises from the circumstances. For example, in the *Geometrica*,[38] he calculates the number $4\sqrt{8+1/4+1/8+1/16}$ as $\sqrt{135}$, by bringing the 4 inside the radical as 16. He then calculates $\sqrt{135}$ as 11 + 1/2 + 1/14 + 1/21.[39]

Relying, according to his own testimony, on Indian sources, al-Khwārizmī follows a procedure similar to that in the *Geometrica*. Thus, to find $\sqrt{2+1/3+1/13}$ he expresses the radicand as 94/39, multiplies numerator and denominator by 39, takes the root of the numerator, $94 \cdot 39$, by method 5, and divides the result by 39.

This much al-Khwārizmī says that he got from the Indians. He then goes on to explain[40] that one can also approximate the square root of a small number by adding an even number of zeros to its right. One then takes the square root of the number so augmented and divides that by 1 followed by half the number of zeros. He illustrates this by calculating $\sqrt{2}$ as $\sqrt{2{,}000{,}000}/1000$. This he claims to be his

37 Indeed, even if one has a convenient fractional system it can be simpler, for extracting the fractional part of the root, to use the algorithm to produce the integer part of the square root only, and then find a fractional part by another method.

38 *Geometrica* cited in [Heath 1921, II: 321]. The work has been traditionally assumed to be based on a work of Hero, but in Heiberg's judgement, it "was not made by Hero, nor can a Heronian work be reconstructed by removing a larger or smaller number of interpolations" (Høyrup's translation in [Høyrup 1997: 72]). For additional information see [Heath 1921, II: 319], who says only that "The *Geometria*...is not his [Heron's] work in its present form."

39 This is not one of the convergents of the continued fraction expansion for the root.

40 See Chapters 18 and 19, and the associated commentary in [Folkerts 1997]. This procedure is, as Folkerts points out, discussed more generally in the previous chapters, where al-Khwārizmī calculates $\sqrt{2+1/3+1/13}$ as $\sqrt{94/39}$ and then as $\sqrt{3666}/39$ to obtain 60/39 + a remainder.

own idea, but in fact in India this method was known practically contemporaneously with al-Khwārizmī in the work of Śrīdhara in the *Pāṭīgaṇita* as well as in other writers.[41] Although one supposes that al-Khwārizmī was telling the truth when he wrote that he did not learn of the device from Indian sources, it seems likely all the same that Indian writers knew the general rule as $\sqrt{N} = \sqrt{(Nb)^2}/b$. The method was ubiquitous among the Muslim authors,[42] and it was doubtless from an Arabic text that John of Seville learned the method in the first half of the 12[th] century.[43]

Maximos Planudes claimed the method as his own *ca.* 1300 in his *Great Treatise on Indian Reckoning*.[44] However, T. L. Heath [1921, I: 62–63] has argued convincingly that Ptolemy used the method to obtain the approximation to $\sqrt{3}$ implicit in his table of chords, 1;43,55,23, which is very much better than Heron's. Moreover, the method was known to other early Greek writers, and one finds evidence of it in the scholia to *Elements* X. Certainly, once the Greeks were aware of Method 5 operating on the sexagesimal system a proficient calculator like Ptolemy would not have needed anyone to suggest the idea of calculating, say, an approximation to $\sqrt{37}$ by calculating $\sqrt{37 \cdot 60^2}/60$.

*Method 7.* According to [Gupta 1994: 63], the value 7/4 was a common approximation to $\sqrt{3}$; however, Heron gives a much better value when in *Geometrica* 14 he expresses $\sqrt{3}/4$ as 1/3 + 1/10, which implies $\sqrt{3} = 26/15$. The same approximation is implicit in the value 43 1/3 for $\sqrt{1875}$, found in his *Metrica* 17, since $1875 = 25^2 \cdot 3$. In addition, Heron's contemporary, the Latin writer Columella, in his *De re rustica*, uses the same value for $\sqrt{3}$, and around the year 1000 Abbot Gerbert of Aurillac implicitly gives the same value for $\sqrt{3}$ in a letter to his pupil Adelbold of Liège in recommending the use of the ratio 30 : 26 for the ratio of the side of an equilateral triangle to its altitude. In all these writings the context in which $\sqrt{3}$ appears is that of finding the area of an equilateral triangle.[45]

None of the methods discussed above produce this excellent approximation to $\sqrt{3}$, which is the fifth convergent of the continued fraction expansion of that number. However, T. N. Thiele, suggested a simple method based on taking successive powers of $5 - 3\sqrt{3}$ ($< 0$) for obtaining not only Heron's approximation but Archimedes' bounds $265/153 < \sqrt{3} < 1351/780$ in the *Measurement of the Circle*.[46]

Dutka points to what he sees as a flaw in this explanation of the approximation, namely that the initial approximation $\sqrt{3} = 5/3$ is not attested in the ancient

41 [Gupta 1985: 15].
42 [Saidan 1978: 441–452].
43 For John's knowledge see [Gupta 1985: 15].
44 [Allard 1981: 157–166]. [Gupta 1985: 15] suggests that what Planudes saw as his contribution was the explicit combination of Methods 5 and 6.
45 As in the case of the approximation 17/12 for $\sqrt{2}$ it seems likely that these coincidences, within the tradition of a particular geometrical problem, involve transmission of the approximation rather than of the method for obtaining it. For Columella see [Dilke 1987: 32]. For Gerbert see [McCague 1993: 148–149].
46 [Dutka 1986: 23].

literature. However, this is not an insuperable difficulty, since it is quite credible that once methods were found to produce approximations to $\sqrt{3}$ better than 5/3 the approximation 5/3 would vanish without a trace.

*Summary of Methods for Square Roots.* Although we have given our views about the different methods at the end of the sections dealing with them, it may be useful here to summarize our views of the different kinds of transmission found with the different methods.

For example, evidence suggests that Methods 1 and 2 originated in Babylon, spread from there to India, and then to medieval Islam, where Method 1 became the more popular. But from there Method 2 alone spread to the West and earlier, from Babylon, spread to China and Hellenistic Egypt. A particular value of $\sqrt{2}$ obtainable from Method 2, 17/12, had a career of its own from Old Babylon, to Heron, and then to the Latin West. We have also stated our belief that Method 3 might have been a Greek invention that spread *via* Heron to Byzantium and the West. In contrast to Method 2, Method 5, although as widely distributed (China, Greece, India, Islam — and thence to the Latin West), may have originated independently in each of China, Greece, and India. Method 4 appears "out of nowhere" in a second-century Greek text purportedly written for students of philosophy and then goes, apparently, nowhere. (One might also argue whether a procedure that applies only to $\sqrt{2}$ qualifies as a method.) Method 6, finally, appears to be one that medieval Islam learned from India and, after developing it, transmitted it to the West. At the same time one can also trace a Greek and Byzantine tradition of its use, and, possibly, the Greeks developed it on the basis of an *ad hoc* method found at least as early as Heron.

## Approximations to $\pi$

We begin with the approximation $\sqrt{10}$, which we mentioned at the beginning of this paper. This value appears first in the work of the Chinese mathematician, Zhang Heng (ca. 100). According to the account of the third-century mathematician, Liu Hui,[47] Zhang knew the old value 4 : 3 for the ratio of the area of a square to the area of its inscribed circle, a value based on $\pi = 3$.[48] He also knew that the ancients had supposed that the volume of a cube to the volume of its inscribed sphere was the square of that, 16 : 9, so $V = 9D^3/16$, where D is the diameter of the sphere and V is its volume. However, Zhang realized that this rule provided a value less than the true value, whatever that value might be. He therefore corrected the formula by adding another $D^3/16$ to the value, to get $V = 10 \cdot D^3/16$. Hence the volume of a cube is to that of the inscribed sphere as 8 : 5. Zhang then applied this back to the square and inscribed circle and argued that the correct

47 See the account of the story in [Lam & Ang 1986], which is based on the account by Liu Hui in the third century of our era.

48 Of course this is the same as the ancient Babylonian value for this constant.

ratio for the corresponding plane figure should not be 4 : 3 but $\sqrt{8}:\sqrt{5}$. This, of course, implies that $\pi=\sqrt{10}$.

Despite its early appearance in China, the value $\sqrt{10}$ has been called the Jaina value for $\pi$ because it is often used in the works of writers adhering to that sect.[49] Dates of Jaina writings are controversial, but competent authorities agree in assigning to the first century a work by Umāsvatī, where the value $\pi=\sqrt{10}$ is explicitly stated in a rule for the circumference of a circle.[50] Since Zhang Heng probably wrote his work in the early part of the second century A.D. he could hardly have been Umāsvatī's source.

From that time onwards, however, the line of descent is clearer, and one finds the value $\pi=\sqrt{10}$ in the *Sūryasiddhānta*,[51] and cited by Brahmagupta as "the accurate value."[52]

In a similar vein, al-Khwārizmī says in his *Algebra*[53] that although the common man uses 3 1/7 as the multiplier to produce the circumference of a circle from its diameter, geometers take the circumference as $\sqrt{10\cdot d\cdot d}$. He was writing in Baghdad and depended on Indian sources. The value $\pi=\sqrt{10}$ also appears in 11th-century Andalusia,[54] and writers in both India and China continued to use it until at least the 17th and 18th centuries (respectively).[55]

In view of the deductive context in which it is situated in Chinese mathematics it is likely that the value there originated independently with Zhang Heng, but there is no such context for the Indian value. Smith suggests[56] that it may have arisen when a mathematician who knew the value $\pi = 3\ 1/7$ observed that the approximation $\sqrt{N} = a + r/(2a + 1)$ of Method 1 yields $\sqrt{10}=\sqrt{9+1}=3\ 1/7$. (We have it on the authority of al-Bīrūnī in his *India* that Brahmagupta knew the value $\pi = 3\ 1/7$, and a number of later Indian writers knew the value.[57]) As ingenious as Smith's suggestion is, it seems more likely that the value $\pi=\sqrt{10}$ was proposed by someone who believed it was different from 3 1/7, not by someone who believed it was the same.

Another widespread value for $\pi$ is 62,832/20,000, i.e., exactly 3.1416. A fragment in a Chinese commentary on the *Nine Chapters* gives the value of $\pi$ implicitly in stating that the area of a circle of radius 1 *chi* can be taken as 314 4/25 [square] *cun* (where 1 *cun* = 1/10 *chi*), which gives $\pi = 3.1416$.[58] This value is

49 [Gupta 1975: 2].

50 [Gupta 1975: 5].

51 *Sūryasiddhānta* I 59 [Burgess 1935: 43]. It is identified as the factor by which one multiplies the earth's diameter to obtain its circumference, and later as the factor to use in calculating planetary orbits.

52 [Gupta 1975: 3].

53 [Rosen 1831: 71].

54 [Bond 1921/22: 314] (cited in [Gupta 1975: 3]). The Andalusian writer is al-Zarqāllū.

55 According to [Gupta 1975: 3].

56 [Smith 1925, II: 254].

57 [Gupta 1975: 3].

58 More precisely, the area is given as 314 *cun* and 4/25 *cun*. Experts disagree as to whether Liu Hui (fl. ca. 265) or a later commentator wrote the fragment. See [Volkov 1994a: 139–145; Lam & Ang 1986: 333–334].

also found,[59] in the *Paulisasiddhānta*, a Sanskrit translation of the third or fourth century of a Greek astronomical work. One or two centuries later, Āryabhaṭa (b. 476) gave the value in the form of Rule 10: Add 4 to 100, multiply by 8, and add 62,000. The result is, he says, "approximately the circumference of a circle of diameter 20,000."[60] Then, in Baghdad around 780, Ya'qūb ibn Ṭāriq, who states that he is basing himself on an Indian informant, gives the same value when he states that the circumference of the outer circle of the zodiac is 125,664,000 farsakhs and its radius is twenty million.[61] In the next century al-Khwārizmī, in his *Algebra*, identifies the value as that of the astronomers, as opposed to the geometers' value, $\sqrt{10}$. And, again, one finds it in 11th-century Andalusia.[62]

Given the fame of Archimedes' approximation of 3 1/7, one might be surprised to find in a work translated from Greek a value as good as 3.1416. However, one need not be, for Heron cites a very good value for $\pi$ that he found in Archimedes' *On Plinthides and Cylinders*. The figures are garbled but there is reason to believe that 3.141596 is very close to the value Archimedes derived.[63] In any case, Ptolemy used his table of chords to find that the circumference of the circle is 3.14166... (3;8,30) times the diameter.[64] And Apollonius found bounds for the value of $\pi$ closer than those (3 10/71 and 3 10/70) that Archimedes gave.[65] Since, then, there is a Greek context for values of $\pi$ considerably better than the 3 1/7 attributed to Archimedes, a value equal to 3.1416 should not be regarded as an unlikely one in the Hellenistic world.

One of the most remarkable ancient values for $\pi$ is 355/113, derived, by a method we can only conjecture, by the Chinese Zu Chongzhi, who was perhaps an older contemporary of Āryabhaṭa. His value of $\pi$ is also found in India ca. 1500 in the works of the astronomer Nīlakaṇṭha.[66] In fact, Gupta points out that if, in the prescription $C = 3D + (16D + 16)/113$, which relates the circumference (C) of a circle to its diameter (D) and is found in a commentary of Vīrasena ca. 800, the "illogical constant" 16 were not added on, the formula would yield the Chinese value of $\pi$.[67] In addition the value appeared in European writers.[68]

59 According to al-Bīrūnī in his *India* [Sachau 1888, I: 168]. The value Liu Hui is known to have deduced is 3.14.

60 [Clark 1930: 28].4

61 Cited by al-Bīrūnī in a table of planetary distances which he says is taken from Ya'qūb's *Tarkīb al-aflāk* (see [Pingree 1968: 106]).

62 [Gupta 1973: 19].

63 According to Tannery, cited in [Heath 1921, I: 232–233]. The citation from Heron is from his *Metrica* (I 26).

64 *Almagest* VI 7. It is unclear why [Gupta 1973] ascribes Ptolemy's value to Apollonius.

65 [Heath 1921, I: 234].

66 In his *Tantra-saṅgraha* and in his *Golasāra*.

67 [Gupta 1975: 3]. The commentary is the *Dhavalā*.

68 [Smith 1925: 310]. Valentin Otho, in the 16th century, derived it from the Archimedian and Ptolemaic values (22/7 and 377/120, respectively) by forming (377–22)/(120–7), so there is no evidence of Chinese influence.

## Concluding Remarks

We have already summarized our conclusions and conjectures about the history of approximations to roots, and shall not repeat them here. What interests us here is whether there are any noticeable differences between the diffusion of $\pi$ and that of square roots. One striking difference is that during the historical period covered here one finds a number of algorithms for approximating square roots but only one algorithm for approximating $\pi$, discovered independently in Hellenistic Greece (from Archimedes to Ptolemy) and China. Despite this algorithm's potential for generating ever more accurate approximations the difficulties of doing the computations ensured that prior to the time of al-Kāshī mathematicians limited their efforts here to producing values that were arguably useful. In general, the criteria of utility and convenience were primary in the ancient approximations to irrational numbers. Calculation, especially involving fractions, was hard work and, for the most part, the ancients were quite content with a serviceable, but fairly rough, approximation.

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# A Reconstruction of Greek Multiplication Tables for Integers

by JACQUES SESIANO

Greek multiplication tables for integers have come down to us in fragmentary form only. But we know that the Copts and Armenians adopted the Greek system of numeration and what their own multiplication tables looked like. Now it appears from the Greek fragments that these tables were indeed the antique ones.

Nous ne connaissons les tables de multiplication grecques pour les entiers que par des fragments isolés. En revanche, nous connaissons les tables des Coptes et des Arméniens, qui avaient adopté le système grec de numération. Or il apparaît des fragments grecs conservés qu'elles correspondent aux tables de multiplication antiques.

Greek multiplication tables for integers have come down to us in fragmentary form only. An additional problem is that editors of such fragments generally transcribe them without any attempt to look at the structure of the tables and thus often without drawing parallels with other related fragments. The purpose of this study is to restore, using also non-Greek sources, the form of the original main Greek tables.

## Multiplication Tables

A fundamental step for beginners in mathematics is to know about multiplication. Thus people had either to know the multiplication tables or to have them readily at hand. Such tables took various forms, depending on the base of the numerical system and the symbols used for the numerals.

In theory, a complete multiplication table would be set out as follows. If there are $m$ symbols, they form the first row and column, and the product is read where the corresponding horizontal and vertical lines meet (Fig. 1). This is sufficient for multiplying numbers with several digits, since all that remains to do is to multiply the products of the single digits by the appropriate power of the base.

Now such a table can be simplified without any loss of information by taking into account the commutativity of the product. Since all the products below the diagonal are also found above it, we may disregard either of the parts thus separated. By doing so, we reduce the $m^2$ products of the original form to $\frac{1}{2}m(m+1)$ and there are far fewer products to write out (marked by * in Figure 1). We may also ignore the obvious multiplications, such as those by 1, or by 0 if the numerical system has a zero. This leaves $\frac{1}{2}(m-2)(m-1)$ products to know.

In the Indian system of numeration used today, with ten digits and base ten, this means knowing 36 products. In mediaeval times the corresponding multipli-

| | 0 | 1 | 2 | 3 | … | $m-3$ | $m-2$ | $m-1$ |
|---|---|---|---|---|---|---|---|---|
| 0 | * | * | * | * | … | * | * | * |
| 1 | • | * | * | * | … | * | * | * |
| 2 | • | • | * | * | … | * | * | * |
| 3 | • | • | • | * | … | * | * | * |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | … | ⋮ | ⋮ | ⋮ |
| $m-3$ | • | • | • | • | … | * | * | * |
| $m-2$ | • | • | • | • | … | • | * | * |
| $m-1$ | • | • | • | • | … | • | • | * |

**Figure 1**

cation tables, either complete or in the reduced, triangular form, were commonly found in manuscripts. But in Mesopotamia, where base sixty was used, and thus 1711 different products occurred, the tables most frequently found give, for one specific number, the results of its multiplication by the integers 1 to 20, then by 30, 40, 50 (as well as by some larger integers and inverses of integers).

The case of Greece is different, due to the peculiarity of its numerical system. Although the base is ten, there are symbols — actually letters — representing the nine units, the nine tens and the nine hundreds. For the thousands, the signs for the nine units are used again but with a stroke. A horizontal line above distinguishes numerals from letters (Fig. 2; note the unclassical signs for 6, 90, 900). This makes it possible to form the numerals from 1 to 9999 by means of at most four letters, while no symbol for zero is necessary. For larger numbers, these numerals are used again, with a sign placed before each such group indicating which power of $10^4$ it is the coefficient of. Therefore, a complete Greek multiplication table should have 36 entries on each side, and thus contain $36^2 = 1296$ products. In the triangular form, it would be reduced to 666 (630 without the multiplications by one). The products by 10000, the first unit of higher order, may also be included, which increases the number of products to 1369 and 703 respectively.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| α̅ | β̅ | γ̅ | δ̅ | ε̅ | ϛ̅ | ζ̅ | η̅ | ϑ̅ |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ι̅ | κ̅ | λ̅ | μ̅ | ν̅ | ξ̅ | ο̅ | π̅ | ϙ̅ |
| 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 |
| ρ̅ | σ̅ | τ̅ | υ̅ | φ̅ | χ̅ | ψ̅ | ω̅ | ϡ̅ |
| 100 | 200 | 300 | 400 | 500 | 600 | 700 | 800 | 900 |
| ,α̅ | ,β̅ | ,γ̅ | ,δ̅ | ,ε̅ | ,ϛ̅ | ,ζ̅ | ,η̅ | ,ϑ̅ |
| 1000 | 2000 | 3000 | 4000 | 5000 | 6000 | 7000 | 8000 | 9000 |

**Figure 2**

We have today only fragmentary evidence of Greek multiplication tables. These fragments have been conveniently listed by David Fowler.[1] But they would be insufficient to provide a complete view of multiplication tables had we not had the help of some non-Greek sources, from a much later period, which adopted and thus preserved the Greek system of numerals together with the way of computing with it. One of these sources is an account, in Arabic, of the Coptic system of numeration and its multiplication tables.[2] The other is by the sixth-century Armenian Anania Shirakatsí, who was trained in (Greek) mathematics in Trebizond and who has preserved a complete set of tables in his arithmetical treatise.[3] These two sources will enable us first to distinguish between different aspects of these tables. Then we shall see that Coptic, but also Greek, fragments do confirm the existence of such tables. (These fragments appear to differ considerably in the size of the extant parts and the material used, which may be papyri, wax tablets, ostraka, and even graffiti.)

## Simple Tables

In this, the simplest form of the tables, each of the thirty-six symbols from $n = 2$ to $n = 10000$ is multiplied by 1, 2, ..., 10. The whole table thus consists of thirty-six smaller tables of ten lines, each containing a triad of numbers, namely the two factors and their product (Fig. 3; we shall refer to it as Type I). After giving the thirty-seven Coptic signs for the numbers (the four sets of units and 10000), the (anonymous) Arabic author describes this first set of multiplication tables in the following way:

> They (the Copts) ordered the multiplications of these thirty-seven symbols in thirty-six tables, beginning with (the table of) two; they omitted the table for one, because all the tables begin with the multiplication by one. They called these (tables) *baqsāt*. They formed a table for each of these symbols, with (its multiplication) by one, then by two, then by three (and so on), then by ten. The disposition of the multiplications in each table of the *baqsāt* is thus such that

| | | |
|---|---|---|
| $n$ | 1 | $n$ |
| $n$ | 2 | $(2n)$ |
| $n$ | 3 | $(3n)$ |
| $n$ | 4 | $(4n)$ |
| ⋮ | ⋮ | ⋮ |
| $n$ | 9 | $(9n)$ |
| $n$ | 10 | $(10n)$ |

**Figure 3**

1 See [Fowler 1987[1], 1999[2]].
2 See [Sesiano 1989].
3 See [Abrahamyan 1944: 214–220]. I am grateful to Prof. Françoise Micheau, Paris, for providing me with a copy of this rare book.

| | | |
|---|---|---|
| $n$ | 1 | $n$ |
| 1 | $n$ | $n$ |
| $n$ | 2 | $(2n)$ |
| 2 | $n$ | $(2n)$ |
| ⋮ | ⋮ | ⋮ |
| $n$ | 10 | $(10n)$ |
| 10 | $n$ | $(10n)$ |

**Figure 4**

each symbol is multiplied by ten symbols. Each symbol and its multiplier are read in Coptic, as also the result of the multiplication of the two symbols [Sesiano 1989: 55, 61–62].

Note that the word *baqsāt* is an Arabic plural of a Coptic denomination which surely goes back to the Greek ἄβαξ.

The extant fragments show that some variations occur in these simple tables of Type I. First, the multiplication by 10 may be omitted (Type I*a*). Next, each product of two different factors may be followed by the same product with inversion of the factors, as if the commutativity of the product were not evident (Type I′, see Fig. 4, and Type I*a*′). Finally, some texts find it necessary to add the inverted products even when the factors are identical, so that $k \cdot k$ occurs twice (Type I″, possibly Type I*a*″).

Since this arrangement is said to have been used by the Copts, we shall first mention some instances of it in Coptic fragments, classifying them according to the above types. We shall then do the same for Greek fragments. In what follows, "P." stands for papyrus, "O." for ostrakon, "WT." for wax tablet, while "s." is used for "century."

*A. Coptic evidence*

(1) Type I

**(C.I.1)** P. Vindob. ACh. 4519, s. XI. From $2 \cdot 1 = 2$ to $5 \cdot 1 = 5$ (fragmentary). See [*NTK*, No. 310].

**(C.I.2)** P. Vindob. ACh. 15853, s. XI–XII. From $2 \cdot 1 = 2$ to $2 \cdot 10 = 20$. See [*NTK*, No. 308a].

**(C.I.3)** P. Vindob. ACh. 5070, s. X. From $3 \cdot 5 = 15$ to $10 \cdot 10 = 100$. Fourteen lines missing at the top (thus originally contained the products from $n = 2$ to $n = 10$). See [*NTK*, No. 312].

**(C.I.4)** O. Wadi Sarga 22. From $6 \cdot 1 = 6$ to $7 \cdot 10 = 70$. See [*NTK*, No. 316].

**(C.I.5)** P. Vindob. ACh. 4789, s. XI. From $50 \cdot 7 = 350$ (originally $50 \cdot 1 = 50$) to $100 \cdot 10 = 1000$. See [*NTK*, No. 320].

The existence of Coptic multiplication tables of Type I, mentioned by the Arabic text, is thus confirmed by extant Coptic sources, from $n = 2$ to $n = 100$ at least.

(2) Type I′

**(C.I′.6)** PP. Vindob. K 10261 & K 11206, s. XI. From $2 \cdot 1 = 2$, $2 \cdot 2 = 4$ to $100 \cdot 10 = 1000$, $10 \cdot 100 = 1000$ (thus from $n = 2$ to $n = 100$; but $n = 20$ omitted). Multiplication by 1 not inverted. Followed by a table of Type III (below, C.III.1). See [*NTK*, No. 309].
**(C.I′.7)** P. Vindob. K 10495, s. X–XI. From ⟨$2 \cdot 7 = 14$⟩, $7 \cdot 2 = 14$ to $2 \cdot 10 = 20$, $10 \cdot 2 = 20$. See [*NTK*, No. 308].
**(C.I′.8)** P. Vindob. K 6479, s. VIII. From $3 \cdot 1 = 3$, $1 \cdot 3 = 3$ to $3 \cdot 10 = 30$, $10 \cdot 3 = 30$. See [*NTK*, No. 307].

(3) Undefined types

**(C.*I*.9)** P. Vindob. ACh. 32367, s. XI–XII. From $2 \cdot 1 = 2$ to $3 \cdot 6 = 18$. Type I or I*a* (end of the series of multiplications not extant). See [*NTK*, No. 310a].
**(C.*I*.10)** P. Vindob. ACh. 26980, s. X–XI. From $30 \cdot 6 = 180$ to $30 \cdot 9 = 270$. Type I or I*a*. See [*NTK*, No. 317a].
**(C.*I*.11)** O. Wadi Sarga 23, palimpsest. From $7 \cdot 1 = 7$, $1 \cdot 7 = 7$ to $7 \cdot 8 = 56$, $8 \cdot 7 = 56$. Type I′ or I*a*′. See [*NTK*, No. 317].
**(C.*I*.12)** P. Vindob. ACh. 16845, s. XI. From $70 \cdot 1 = 70$ to $70 \cdot 8 = 560$. Type I or I*a*. See [*NTK*, No. 321].

*B. Greek evidence*

(1) Type I

**(G.I.1)** P. Michigan Inv. 5663a, s. II–III. From $2 \cdot 7 = 14$ to $10000 \cdot 4 = 40000$ (thus from $n = 2$ to $n = 10000$), in several fragments, but with relatively few parts of the text missing. See [Sijpesteijn 1982b: 4–5, No. 686].
**(G.I.2)** PP. Vindob. G 27748 & G 29371, s. II–III. From $7 \cdot 8 = 56$ to $10 \cdot 10 = 100$. Followed by a table of the roots and the squares of order units. See [*NTA*, No. 152].
**(G.I.3)** WT. Würzburg K 1014 & K 1015, s. VI–VII, schoolbook. From $20 \cdot 1 = 20$ to $5000 \cdot 6 = 30000$, not in order. Thus take the tablets K 1014 in the order: $2^A$, cols. 3–5 ($20 \cdot 1 = 20$ to $50 \cdot 10 = 500$); $1^B$ ($60 \cdot 1 = 60$ to $300 \cdot 10 = 3000$); $2^A$, cols. 1–2 ($400 \cdot 1 = 400$ to $600 \cdot 10 = 6000$); $3^B$ (fragmentary, from $700 \cdot 3 = 2100$ to $3000 \cdot 7 = 21000$); then take, of K 1015, tablet $2^B$, cols. 3–4 ($4000 \cdot 1 = 4000$ to $5000 \cdot 6 = 30000$, fragmentary). See [Brashear 1985: 19–22; 1986: 2 (with reproduction)].

The existence of multiplication tables of Type I in Greek times can thus be considered as attested.

(2) Type I′

**(G.I′.4)** P. Heid. Inv. 1, a–e (VBP IV, 64), s. VI–VII. From $2 \cdot 1 = 2$, $1 \cdot 2 = 2$ to $500 \cdot 10 = 5000$, $10 \cdot 500 = 5000$. Then follows another type of table (below, G.III″.2). See [Bilabel 1933, 2: 177–179, line 436].

**(G.I′.5)** WT. Moen 601, s. VI–VII. From $6 \cdot 1 = 6$, $1 \cdot 6 = 6$ to $9 \cdot 8 = 72$, $8 \cdot 9 = 72$. See [Sijpesteijn 1982a (with reproduction)].

**(G.I′.6)** WT. Leid. Pap. Inst. 4, s. III. From $40 \cdot 1 = 40$, $1 \cdot 40 = 40$ to $40 \cdot 10 = 400$, $10 \cdot 40 = 400$. See [Boswinkel 1975: 27].

(3) Types I & I′

**(G.I–I′.7)** P. Società Italiana I, 22–24 (PSI VIII, 958), s. IV. From $1 \cdot 1 = 1$, $2 \cdot 1 = 2$, but with numerous copyist's errors and omissions, particularly in the beginning; Type I′ for $n = 1$ (only: $1 \cdot 1 = 1$), 2, 20 (both incomplete), then from $n = 30$ to $n = 400$, ending with $400 \cdot 10 = 4000$, $10 \cdot 400 = 4000$; followed by Type I, from $n = 500$ to $n = 4000$ (ending with $4000 \cdot 10 = 40000$), except that the initial product is inverted as before (thus: μ α μ, ἅπαξ μ μ, meaning 40 times 1 is 40, once 40 is 40). See [*Papiri greci et latini*: 155–158, No. 958]. Reproduction of the parts $80 \cdot 6$, $6 \cdot 80$ to $100 \cdot 10$, $10 \cdot 100$ and $200 \cdot 1$, $1 \cdot 200$ to $600 \cdot 10$ in [Pintaudi 1983, tavv. XXXIX–XL].

(4) Type I*a*′

**(G.I*a*′.8)** WT. Moen 4, s. VII. From $7 \cdot 1 = 7 = 1 \cdot 7$ to $9 \cdot 9 = 81$. Followed by a table of halves of order units. See [*NTA*, No. 154].

(5) Types I & I″

**(G.I–I″.9)** WT. Louvre Inv. AF. 1196, 1–3, s. V–VI, school exercise. With reconstruction of some illegible parts, from $2 \cdot 1 = 2$ to $9000 \cdot 10 = 90000$ (thus from $n = 2$ to $n = 9000$). For we find the multiplications from $2 \cdot 1 = 2$, $1 \cdot 2 = 2$ to $8 \cdot 6 = 48$, $6 \cdot 8 = 48$, Type I″, on face *A* of the first tablet; from $8 \cdot 7 = 56$, $7 \cdot 8 = 56$ to $30 \cdot 10 = 300$, $10 \cdot 30 = 300$, certainly extended to $50 \cdot 6 = 300$, $6 \cdot 50 = 300$, Type I″, on face *A* of the second tablet; from $5000 \cdot 1 = 5000$ to $9000 \cdot 10 = 90000$, Type I, on face *A* of the third tablet. See [Boyaval 1973 (with reproduction); Cauderlier 1983 (with reproduction); Brashear 1984].

## Completed Tables

With the above tables, if we want to multiply two numbers having respectively $r_1$ and $r_2$ symbols (there is no zero, remember), we have to search for $r_1 \cdot r_2$ products and then calculate the appropriate powers of ten. Thus, $1924 \cdot 7340$ would involve multiplying 1000, 900, 20 by the digits 7, 3, 4 and then, before the final addition, replacing the results thus obtained with other symbols of higher orders, since the units of digits, tens, hundreds and thousands are all written in a different way. Incidentally, it should be noted that the purpose of Book II in Pappus's *Collectio* (edited in [Hultsch 1875–78]) is to facilitate such conversions.

| | | |
|---|---|---|
| $n$ | 1 | $n$ |
| $n$ | 2 | $(2n)$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $n$ | 10 | $(10n)$ |
| $n$ | 20 | $(20n)$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $n$ | 90 | $(90n)$ |
| $n$ | 100 | $(100n)$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $n$ | 900 | $(900n)$ |
| $n$ | 1000 | $(1000n)$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $n$ | 9000 | $(9000n)$ |
| $n$ | 10000 | $(10000n)$ |

**Figure 5**

The completed Greek tables overcome this difficulty by giving all the products, that is, the products of the thirty-six units of the four orders, from $n = 1$ to $n = 9000$, by the same units and 10000. The whole table thus consists of thirty-six smaller tables giving, for each number $n$, thirty-seven products (Type II, Fig. 5). Another form completes the tables by adding 10000 to the list of multiplicands $n$ (Type II′).

That such completed tables existed is seen from their presence in the arithmetical work of Anania Shirakatsí (Type II). We have further confirmation of their existence in a letter of the Byzantine Nicolas Rhabdas. The only difference is that Rhabdas's tables are of the Type II′ and are triangular in form (each time one product less from the top). This is certainly late, since Rhabdas belongs to the 14th century; but as he attributes these tables to Palamedes, they were then considered to be ancient in origin.[4]

Other evidence that such tables were used is, however, scarce. A Coptic fragment contains a few products from the beginning of one table (which clearly appears to be part of an extended table); and (possibly ninth-century) graffiti, Coptic or Greek, found on the ruins of the Phoebammon Monastery near Thebes may depict parts of such tables.

*A. Coptic evidence*

**(C.II.1)** P. Vindob. ACh. 11276, s. XI. From $2 \cdot 1 = 2$ to $6 \cdot 30 = 180$, not in order and with a few omissions. The multipliers do not go beyond 1000. See [*NTK*, No. 311].

4 Edition of this letter in [Tannery 1886].

*B. Greek or Coptic evidence*

**(G-C.II.2)** Graffito Phoebammon 10. From $3 \cdot 1 = 3$ to $3 \cdot 40 = 120$; seems to continue (traces). See [Rémondon *et al.* 1965, No. 5 and Pl. III (illegible); from which *NTK*, No. 313].

**(G-C.II.3)** Graffito Phoebammon 11. From $5 \cdot 1 = 5$ to $5 \cdot 10000 = 50000$. See the review [Schwartz 1967] of [Rémondon *et al.* 1965]; Graffito 11 was omitted from the latter work.

One of these graffiti is of an undefined type (it could be of Type I):

**(G-C.II.4)** Graffito Phoebammon 186. From $4 \cdot 2 = 8$ to $4 \cdot 10 = 40$. See [Rémondon *et al.* 1965, No. 153; from which *NTK*, No. 314].

## Decadic Tables

These tables provide exactly the same information as the tables of Type II, but are set out differently. For the symbols $n$ from $n = 2$ to $n = 9000$ are multiplied by ten tetrads, each consisting of the corresponding units of the four orders (Type III, Fig. 6). So this time the whole table comprises thirty-five smaller tables of forty products each (a table for 1, indeed superfluous, is no longer present, while three products are repeated in the last tetrad). The existence of these tables is reported by the Arabic text on Coptic reckoning, after the description of the first set, as follows:

> They (the Copts) replaced these thirty-six tables, the so-called *baqsāt*, by a second disposition of the multiplications, which (tables) they called *dhāqāt*. In this disposition they begin with the first table, namely the table of two, which is multiplied by one, then by ten, then by one hundred, then by one thousand, then by two, then by twenty, then by two hundred, then by two thousand; then they multiply by three, then by thirty, then by three hundred, then by three thousand, and so on, each symbol being multiplied by four symbols, those of the units and the corresponding units of higher order to the thousands; this is done up to, and no farther than, (the multiplication by) ten, one hundred, one thousand, ten thousand. Such is the disposition of the multiplications in each table of the *dhāqāt* [Sesiano 1989: 55, 62].

The word *dhāqāt* is an Arabic plural of a Coptic denomination obviously coming from the Greek word for ten, δέκα.

Since, as already mentioned, the last group in each series (multiplications by 10, 100, 1000, 10000) partly repeats the first one (multiplications by 1, 10, 100, 1000), it may consist of the multiplication by 10000 only (Type III'), and sometimes the whole tetrad is omitted altogether (Type III"). The existence of such decadic tables is documented by Coptic as well as Greek sources.

| | | |
|---|---|---|
| $n$ | 1 | $n$ |
| $n$ | 10 | $(10n)$ |
| $n$ | 100 | $(100n)$ |
| $n$ | 1000 | $(1000n)$ |

| | | |
|---|---|---|
| $n$ | 2 | $(2n)$ |
| $n$ | 20 | $(20n)$ |
| $n$ | 200 | $(200n)$ |
| $n$ | 2000 | $(2000n)$ |

| | | |
|---|---|---|
| $n$ | 3 | $(3n)$ |
| $n$ | 30 | $(30n)$ |
| ⋮ | ⋮ | ⋮ |

| | | |
|---|---|---|
| $n$ | 9 | $(9n)$ |
| $n$ | 90 | $(90n)$ |
| $n$ | 900 | $(900n)$ |
| $n$ | 9000 | $(9000n)$ |

| | | |
|---|---|---|
| $n$ | 10 | $(10n)$ |
| $n$ | 100 | $(100n)$ |
| $n$ | 1000 | $(1000n)$ |
| $n$ | 10000 | $(10000n)$ |

**Figure 6**

*A. Coptic evidence*

(1) Type III

**(C.III.1)** PP. Vindob. K 10261 & K 11206, s. XI. From (p. 6, line 11) $2 \cdot 1 = 2$, $2 \cdot 10 = 20$ to $4 \cdot 9000 = 36000$. Preceded by a table of Type I′ (above, C.I′.6). See [*NTK*, No. 309].

**(C.III.2)** P. Vindob. K 20796, s. X. From $5 \cdot 7 = 35$, $5 \cdot 70 = 350$ to $6 \cdot 40 = 240$ (fragmentary). See [*NTK*, No. 315].

**(C.III.3)** Codex Lond. M.B. Or. 5707, fols. $1^r$–$5^v$, palimpsest, s. X. From $7 \cdot 1 = 7$, $7 \cdot 10 = 70$ to $9000 \cdot 10000 = 90{,}000{,}000$ (thus from $n = 7$ to $n = 9000$, only fragmentary for $n = 10$). See [Crum 1905, No. 528; Drescher 1948/49 (with reproduction and transcription of fol. $2^r$)]. A full reproduction of the manuscript is in [*NTK*, No. 332], but in the transcription two folios are inverted.

**(C.III.4)** P. Vindob. ACh. 20817, s. XI. From $7 \cdot 600 = 4200$, $7 \cdot 6000 = 42000$ to $9 \cdot 4000 = 36000$. See [*NTK*, No. 319].

With these fragments, the multiplication table of Type III can be reconstituted from $2 \cdot 1$ to $4 \cdot 9000$, $5 \cdot 7$ to $6 \cdot 40$, $7 \cdot 1$ to $9000 \cdot 10000$, and is thus almost complete.

(2) Type III′

**(C.III′.5)** P. Vindob. K 20797, s. XI. From $8 \cdot 5 = 40$, $8 \cdot 50 = 400$ to $10 \cdot 2000 = 20000$ (fragmentary, but clearly III′, originally beginning with $8 \cdot 3 = 24$). See [*NTK*, No. 318].

**(C.III′.6)** Cambridge University Library, T.-S. Ar. 39: 380 (one sheet), fols. $1^r$–$2^v$. From $50 \cdot 500 = 25000$ to $90 \cdot 5000 = 450{,}000$ (fragmentary). See [Goldstein & Pingree 1981: 186–189 (with reproduction)].

*B. Greek evidence*

(1) Type III

**(G.III.1)** P. Vindob. G 42313, s. IX–X. From $2 \cdot 3 = 6$ to $3 \cdot 10000 = 30000$ (fragmentary). See [*NTA*, No. 157] (the presence of the last product can be guessed from Pl. 74).

This confirms the existence of Greek tables of Type III for $n = 2$ and $n = 3$; the next example, without the last tetrad, extends it to $n = 30$.

(2) Type III″

**(G.III″.2)** P. Heid. Inv. 1, a–e (VBP IV, 64), s. VI–VII. From $2 \cdot 1 = 2$, $1 \cdot 2 = 2$, $2 \cdot 10 = 20$, $2 \cdot 100 = 200$ to $30 \cdot 9000 = 270{,}000$, where the copyist stopped (thus from $n = 2$ to $n = 30$, with a few copyist's omissions). Each time the inverted product for the multiplication by 1 is given. See [Bilabel 1933: 179 (line 437)–181].

(3) Undefined types (III, III′, or III″)

**(G.*III*.3)** P. Vindob. G 40552, s. V–VI. From $9 \cdot 3 = 27$, $9 \cdot 30 = 270$, $9 \cdot 300 = 2700$, to $9 \cdot 800 = 7200$. The sequence of the triads is horizontal, from right to left. The far-left column, with the multiplications by thousands, is missing. See [*NTA*, No. 156].

**(G.*III*.4)** P. Vindob. G 24847, s. II. From $20 \cdot 500 = 10000$, $20 \cdot 5000 = 100{,}000$ to $20 \cdot 9000 = 180{,}000$. See [*NTA*, No. 153] (it is clearly seen from Pl. 71 that the next column is for 30).

## Conclusion

Both common sense and documentary evidence suggest that two types of tables must have been used in Greek times. First, elementary tables (our "simple tables"), which were sufficient for current and school use (a more elementary form of which did not go beyond the multiplicand 10). Second, extended tables, which were adopted for a more "professional" use (sometimes with the multipliers up to 100 or 1000 only); one of these two forms of extended tables (our "decadic tables") became more common than the other, and must have, whenever complete, taken the form of a *libretto* like those mediaeval traders carried with them for their various trade and currency transactions.

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*NTA* = [Harrauer & Sijpesteijn 1985].

*NTK* = [Hasitzka 1990].

*Papiri greci e latini* = *Papiri greci e latini (della Società italiana per la ricerca dei papiri greci e latini in Egitto)*, Vol. VIII. Florence: Le Monnier, 1927.

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# Problems of Pursuit: Recreational Mathematics or Astronomy?

by ANDREA BRÉARD

In a seventh-century commentary on an astronomical problem from the *Mathematical Classic in Continuatian af Antiquity* (*Ji gu suan jing*) a problem of pursuit is borrowed from the *Nine Chapters on Mathematical Procedures* (*Jiu zhang suan shu*). Similar couples of problems can be found in an Indian commentary to the *Āryabhaṭīya* from the seventh century, a fourteenth-century Chinese astronomical treatise by Zhao Youqin and in the *Algorismus Ratisbonensis*, a fifteenth-century European manual. Our paper will deal with the various types of problems of pursuit and meeting in these contexts by analyzing the modes of problem formulation, classification and interpretation in different cultural contexts, and by questioning the possibility of dissemination from China westwards.

## Of Dogs, Rabbits, Horses and Travelers

What traditional historiography classifies as problems of pursuit, in general involves animals and human beings. The most famous model is the one in the form of the hound pursuing a hare. [Smith 1958, II: 546–547] tells the story of the transmission of this problem from its (probably first) appearance in the *Propositiones ad acuendos iuvenes* attributed to Alcuin of York (ca. 735–804) to other medieval Latin texts onwards. In China, a problem of a hound pursuing a hare can originally be found in the *Nine Chapters on Mathematical Procedures* (*Jiu zhang suan shu* 九章算術, ca. first century A.D.) and in a different formulation in Wang Xiaotong's *Mathematical Classic in Continuation of Antiquity* (*Ji gu suan jing* 緝古算經, before 626).[1]

In parallel, problems of couriers — horses and men — were formulated as well in the Chinese as in the Indian and Latin traditions. As [Smith 1958, II: 547] remarks, these were "not, however, always ones of pursuit, since the couriers might be traveling either in the same direction or in opposite directions." Again, the situation of horses (an excellent one and a worn out old horse) from the *Nine Chapters on Mathematical Procedures* reappears in a different formulation a mil-

1 A complete German translation of the problems is given in [Bréard 1999: 333–337]. Papers on different aspects of the similarities between Problem 1 from Wang Xiaotong's treatise and the *Nine Chapters* have been presented at the conference "The History of Reading the Ancients in Mathematics" at the CIRM (Centre International de Rencontres Mathématiques), Marseille, France, September 1995 ("Wang Xiaotong (?) reads the *Nine Chapters af Mathematical Pracedures*"), and at the symposium "Ten Classics of Ancient Chinese Mathematics" of the 20th International Congress of History of Science, Liège, Belgium, July 1997 ("Re-Creation in Mathematics — What is a Problem in the *Ji gu suan jing*?"). I am grateful to the audiences for their valuable comments on these talks.

lennium later, in Zhao Youqin's 趙友欽 *New Writing on the Image of Alteration* (*Ge xiang xin shu* 革象新書, ca. 1324–1337).[2]

## Possible Mythological Origins of Animal Pursuit and Flight

Numerous hunting scenes on vases,[3] cups[4] and flasks[5] from the Archaic to the Roman Imperial Period depict a hound chasing a hare, the figuration of the hunter becoming more frequent on Attic or Philistine images.[6] The chase was, of course, a favorite pastime of Greek gentlemen of leisure. In his *Memorabilia*, Xenophon describes in a dialogue between the beauty Theodote and Socrates how hounds were used to catch hares "by hot pursuit" and suggests this as a trick for hunting friends, "the noblest game in the world."[7] Numerous instructions on how to set about netting a hare with hounds can equally be found in his essay *On Hunting* (*Cynegeticus*).[8] Furthermore, the metaphor of a hound hunting a hare is used in mythological writings: Odysseus pursuing Dolon,[9] Daphne fleeing from

2 See [Volkov 1996/97a: 39] for a tentative rendering of the Chinese title into English: The image of the hexagram "alteration" was traditionally related to the changes in numerical data of celestial phenomena and to the establishment of a new calendar. In [Bréard 1999: 337–338] we give a complete German translation of the passage concerning the pursuit based on the edition in [*Siku quanshu*, 786: 228–229], and we discuss its astronomical relevance as compared to other problems of pursuit in [Bréard 1999: 41–43].

3 E.g., on the shoulder of the vase Mississippi 1977.3.79 a hound chases a hare to the right. Collection: University Museums, University of Mississippi, Attic Black Figure Ware, 550 BC; see [Beazley 1971: 199, n° 3^bis]. Similarly, the *lekythos* PC 58 in [*Corpus vasorum*: 40–41]. A more complete list of *lekythoi* from "The Hound-and-Hare-Group" can be found in [Beazley 1956: 514–515], supplemented in [Beazley 1971: 229–231].

4 E.g., the red-figure *phyxides* in [Beazley 1968: 963, n° 98 (40)].

5 E.g., the red-figure *askos* in [Vickers 1981: 547, Fig. 8ab]. [Hoffmann 1977: 11] lists seven *askoi* depicting a hound-hare pursuit.

6 See [Schnapp 1997: 213–214]. [Åkerström 1951: 48–59] gives a historical analysis of the hare-hunting theme in art. It first appears during the early second millennium B.C. in the Near East and becomes very popular in Greece after the eighth century B.C.

7 See Xenophon, *Memorabilia*, Book 3, Sections XI.8–9:

> [8] Since hares feed by night, hounds specially adapted for night work are provided to hunt them; and since they run away at daybreak, another pack of hounds is obtained for tracking them by the scent along the run from the feeding ground to the form; and since they are so nimble that once they are off they actually escape in the open, yet a third pack of speedy hounds is formed to catch them by hot pursuit; and as some escape even so, nets are set up in the tracks where they escape, that they may be driven into them and stopped dead.
> [9] Then can I adapt this plan to the pursuit of friends? [Marchant 1923: 243–245]

8 See Xenophon, *Scripta Minora. On Hunting*, Sections III.8, IV.10, V.15–16, V.32, VI.12, VI.23–25, and VIII.2 in [Marchant 1925: 381, 387, 393, 399–401, 407, 411–413, 419].

9 See Homer, *Iliad*, Book X.360:

> And as when two sharp fanged hounds, skilled in the hunt, press hard on a doe or a hare in a wooded place, and it ever runneth screaming before them; even so did the son of Tydeus, and Odysseus, sacker of cities, cut Dolon off from the host and ever pursue hard after him [Murray 1993: 463].

Phœbus,[10] and Hector pursued by Achilles round the walls of Troy.[11] Vickers [1981: 548] mentions that hunting scenes painted on Attic red-figure *askoi*, are always paralleled with "figures involved in some kind of opposition to one another."[12] A very beautiful panel-amphora from sixth-century B.C. Athens, for example, shows two wrestling athletes on the panel and, in the small picture above, a hound pursuing a hare.[13] What such iconic messages may signify is a matter that has best been analyzed by anthropologists and found to be the theme of sacrifice, the "theology of life and death," or erotic scenes of athletic pursuit.[14] Hull [1964: 75] concludes his chapter Hare Hunting by underlining the fact that, although "hunting as Xenophon practiced it was still carried on, [...] a new sport was introduced into the ancient world by the Celts some time before the second century after Christ. Although it may have been called hunting, it was in fact almost exactly the same sport as what we call 'coursing,' and its purpose was not to catch the hare but to watch the hounds race her."

In the Chinese realm, the famous saying, "Once the smart hare is caught, the hunting dog is fried" (*jiao tu si zou gou peng* 狡兔死走狗烹), meaning that once the enemy's kingdom is destroyed, the commanders will have to die, goes back to the legalist philosopher Han Feizi (ca. 230 B.C.),[15] who describes the ingratitude of the King.[16] Several classics and Tang poems cite a phrase from the *Book of*

10 See P. Ovidius Naso, *Metamorphoses*, Book 1 (Invocation), *Daphne and Phœbus*:

> She seemed most lovely to his fancy in her flight; and mad with love he followed in her steps, and silent hastened his increasing speed. As when the greyhound sees the frightened hare flit o'er the plain: — With eager nose outstretched, impetuous, he rushes on his prey, and gains upon her till he treads her feet, and almost fastens in her side his fangs; but she, whilst dreading that her end is near; is suddenly delivered from her fright; so was it with the God and virgin: one with hope pursued, the other fled in fear; and he who followed, borne on wings of love, permitted her no rest and gained on her, until his warm breath mingled in her hair [More 1922, 1: 38].

11 See Homer, *Iliad*, Book XXII.131–177: they ran "as when single-hooved horses that are winners of prizes course swiftly about the turning-points" [Murray 1998: 467].

12 See, for example, the base of the Protocorinthian *aryballos* Boston 95.10: "In main frieze, Chimaera attacked by Bellerophon on flying Pegasos, a tiny lizard between; .... Below, hounds closing in on hare." Collection: Museum of Fine Arts, Boston, ca. 650–630 BC; see [Amyx 1988, 1: 37].

13 [von Bothmer 1985: 48, Fig. 47]. Collection unknown, the vase was once on the Basel market.

14 See [Hoffmann 1977: 8] and the section *Les chasses aux lièvres protocorinthiennes* in Chapter 5 of [Schnapp 1997: 177–181].

15 See *Han Feizi*, Book 10, Chapter 31, Inner Congeries of Sayings, Lower series, 6 *minutiae* (*Nei chu shuo xia 6 wei di 31* 內儲說下六微第三十一): "When wild hares are exhausted, tame dogs would be cooked; when enemy states are destroyed, state councilors would be ruined" (*jiao tu jin ze liang quan peng, di guo mie ze mou chen wang* 狡兔盡則良犬烹，敵國滅則謀臣亡) (translation in [Liao 1939–59, 2: 10], Chinese in [Han 1959: 184]).

16 The metaphor is also used, for example, in the famous Chinese novel *Journey to the West* (*Xi you ji* 西游記), Chapter 27, when the Monkey King feels that his services to the Chinese monk Xuanzang are treated with ingratitude: "Today, you simply send me away back home because your lucidity is obscured and thus your senses troubled (*meizhe xingxing shi hutu* 昧著惺惺使糊涂). This really is 'When the bird is beaten, the arms are put aside. When the hare is killed, the dog is fried!' (*niao jin gong cang, tu si gou peng* 鳥盡弓藏，兔死狗烹)"

*Odes* (*Shijing* 詩經)[17]: "the smart hare runs back and forth,[18] yet it happens that he gets caught by a hound" (*tì tì chán tu yu quan huo zhi* 躍躍毚兔遇犬獲之).[19]

These few examples show that since early times the image of the hound pursuing a hare was widely spread among different cultural productions in ancient Greece and China, and probably had its origins in actually applied hunting techniques. For the opposition of the "good horse" and the "limping horse," we shall merely mention the *Rites of Zhou* (*Zhou li* 周禮) that describes them among other horse types and their respective numbers as used in rituals.[20]

## Chinese Mathematical Texts and the Conjunction of Planets

Turning to the mathematical context, let us look at the embedding of problems of pursuit and meeting on two levels: first their embedding into an algorithmic context, and secondly their interpretation in an astronomical context. An appendix to the present paper provides 1) diagrams and 2) algorithms to represent schematically the relevant problems under discussion.

For the *Nine Chapters*[21] we can distinguish two groups of algorithms and types of situations: the problems of couriers and of the hare chasing the rabbit are discussed in Chapter 6, "Proportional Assignment" (*jun shu* 均輸), and are basically an application of the "Rule of Three" (*jin you shu* 今有術). Let us call such kind of problems, where two actors meet after having traveled in the same direction, problems of pursuit. Problems 7.10, 7.11, 7.12 and 7.19 for the melon and the gourd, two aquatic herbs (French *scirpe des lacs*, Latin *scirpeus lacustris*), and two rats and two horses are more complex, since they ask for the time of the meeting when the actors come from opposite directions or include accelerated and decelerated motions in geometric or arithmetic progression. In the latter, for example, the good and the limping horse start off from the same place, but the good horse turns around after reaching a certain destination and runs back by the same track to meet the limping horse. Furthermore, it involves accelerated motion for the good, and decelerated motion for the limping horse: the traveled distance in-

17 The *Shijing* is the earliest anthology of Chinese songs, poems, and hymns dating from the Zhou Dynasty (1027–771 B.C.) to the Spring & Autumn Period (770–476 B.C.). It is also called *Maoshi* (Mao *Poems*) because it was by the hand of *Mao Heng* of the *Han* Dynasty that the *Book of Odes* was passed down to the present time.

18 A running gloss to the text explains *tì tì chan tu* 躍躍毚兔 as: "This means that the smart hare runs back and forth several times in order to hide the footprints of his flight" (*wei jiao tu shù wanglai taó nì qi jì* 謂狡兔數往來逃匿其跡).

19 See: *Mao Poems* (*Mao shi zheng yi* 毛詩正義), scrolls 9–15, Minor Festal Odes (*Xiao ya* 小雅), Decade of Nanshan (*Jie Nanshan zhi shen* 節南山之什), *Qiao yan* 巧言 ("Sweet Words") in [Ruan 1815]. [Legge 1960, IV: 342] translates as "Swiftly runs the crafty hare, But it is caught by the hound."

20 See, for example, the *Rites of Zhou* (*Zhou Li* 周禮), scrolls 28–33, Summer Ministry with the Overseer of Military Affairs, part 4 (*Xiaguan sima di si* 夏官司馬第四) in [Ruan 1815].

21 For all the books included in the Tang collection *Ten Books of Mathematical Classics*, we base our textual analysis on the critical edition [Qian 1963].

creases or diminishes daily by a certain amount of *li*. The problem is solved by calculation of the traveled distances for two estimations of the duration in days, which then makes it possible to apply the procedure for "Excedent and Deficit" (*ying bu zu* 盈不足), also known under the name "Rule of Double False Position" in the West. Although there are no explicit references to an astronomical context of Problem 7.19, we can at least assume that the commentators Liu Hui (A.D. 263) and Li Chunfeng (A.D. 656) had one in mind.[22] In the other three of the four problems, the actors meet coming from opposite directions; in the following we shall denote such types of problems as problems of meeting.[23]

In *Zhang Qiujian's Mathematical Classic* (*Zhang Qiujian suan jing* 張邱建算經) we can find several problems of pursuit (and none of meeting) grouped together in the "middle scroll" (*juan zhong* 卷中) of this book. In terms of problem formulations they involve (apart from the "actors" hare and hound) all the aspects from the problems of pursuit and meeting of the *Nine Chapters*, yet these are analyzed and synthesized in a different way to form new problem types: we find oppositional pairs of horses and walkers, variable daily motion of a single horse, an unaccomplished pursuit of a horse thief by a man, and inverse formulations of unknown and known magnitudes.[24] Again, no explicit reference to astronomical origins is given. But we shall note that several problems stated involve measures of time and might indicate a relation to the calculation of planetary conjunctions. All the problems are solved by an application of the "Rule of Three."

Next, the *Mathematical Classic in Continuation of Antiquity* of the mathematician and astronomer Wang Xiaotong (end of the seventh century), which contains in all 20 problems (only 16 of them are complete), places an astronomical one at the beginning of the book. It has a very special position in the compilation, also because it does not lead to an equation of second, third or fourth degree as all the other problems. Furthermore, it is the only astronomical one and cites a problem of pursuit of hare and hound similar to the ones found in the preserved Ming edition of the *Nine Chapters*.[25]

The astronomical situation is a realistic problem in calendrical calculations: at the beginning of the astronomical year, sun and moon should ideally meet precisely at midnight. But what is observed is that the sun only meets the moon a half-day after midnight. Given the actual position of the sun at midnight, the prob-

22 For details see [Bréard 1999: 43–47].

23 [Tropfke 1980, Chapter 4.2.1.4] uses a similar distinction between problems of pursuit ("Verfolgung") and meeting ("Begegnung").

24 Problem 2.4 for example asks for the daily motion of a good and a limping horse, whereas the advance and the distance until meeting are given. This type of stating a problem of pursuit is thus inverse to Problem 6.12 in the *Nine Chapters*. Zhang Qiujian's work shows a more general interest in inverse problems. See [Bréard 1999: 90–94] for further examples.

25 The oldest extant version of the *Nine Chapters*, the *Mathematical Classic in Nine Chapters* (*Jiu zhang suan jing* 九章算經), printed during the Southern Song Dynasty in 1213, only preserves the first five chapters. Dai Zhen 戴震 (1724–1777), editor of the Qing edition of the *Nine Chapters* in the *Complete Library of the Four Divisions* (*Siku quanshu* 四庫全書), made a complete copy of these from the now lost *Great Encyclopedia of Yongle Reign* (*Yongle dadian* 永樂大典), compiled in 1408 at the beginning of the Ming Dynasty.

lem asks for the position of the moon — moving "faster" than the sun — at that time. This situation can of course be interpreted in terms of pursuit: for the observer the moon is "chasing" the sun, the sun has a certain advance at midnight and is chased by the moon until both meet half a day later. In the commentary, the ancient procedure of hare and hound from the *Nine Chapters* is not used to produce a next step in the list of operations, but to produce parallels of the astronomical and the "recreational" situation through rewriting the procedure for sun and moon which the commentator wants to "prove." By showing the resemblances to an ancient list of operations, the procedure for solving the astronomical problem seems to be justified. The meeting of sun and moon at winter solstice ideally at midnight of the beginning of the new calendar year thus algorithmically becomes a classical problem of pursuit. The diagram below shows, for example, how this rewriting was accomplished on the level of names of the parameters involved in the algorithm. The parameter names from the *Ji gu suan jing* are adapted by the commentator to the expressions used in the problem of hare and hound cited from the *Nine Chapters*. Even their grammatical structure is reminiscent of a couplet as in Chinese parallel prose: the phrase "dog runs 100 *bu*" (*quan zou yi bai bu* 犬走一百步) has an equivalent number of characters and the same order of grammatical categories as "moon travels 9000 parts" (*yue xing jiu qian fen* 月行九千分), i.e., a subject followed by the verb, and a measure consisting of a number and its unit.

| The normative power of the Classical model | *Nine Chapters* | *Ji gu suan jiing* |
|---|---|---|
| Names given in the problem | Dog runs 100 *bu*<br>犬走一百步<br><br>Rabbit runs 70 *bu*<br>兔走七十步 | Constant parts traveled by the moon 9000<br>月行定分九千<br><br>Year cycle constant 700<br>章歲七百 |
| Adaptation in the commentary | Moon travels 9000 *parts*<br>Sun travels 700 *parts* | 月行九千分<br>日行七百分 |

Until Zhao Youqin's text, the *New Writing on the Image of Alteration* (written after 1324 but no later than 1337),[26] we have not found any further references to problems of pursuit and meeting in the Chinese mathematical tradition.[27] This

26 For this dating of the work see [Volkov 1996/97a: 39].

27 We do not include in our study calculations of planetary conjunctions that use interpolation methods as developed in the tradition of calendrical astronomy. As an example of how uniformly accelerated motion of celestial bodies was calculated in the *Huangji Calendar* (A.D. 600), see [Liu 1994]. [Liu 1995: 333–334] interprets the solution procedure of the problem of the good and the limping horse from the *Nine Chapters* in terms of interpolation with equidifferences.

might be due to the fact that many Tang (618–907) and Song (960–1279) dynasty works were lost. Those rediscovered during the 19th century concern mainly the solution of problems involving polynomial equations by algebraic techniques applying the method of the "celestial element" (*tian yuan* 天元) for one to four unknowns.[28]

The peculiarity of Zhao Youqin's astronomical treatise is that it is not a collection of problems, but more like a collection of essays on traditional mathematical procedures applied to astronomy.[29] As in Wang Xiaotong's first problem, the image of pursuit is used as an illustration of an astronomical situation. Yet here, a good and a limping horse stand for the sun and the moon respectively.[30]

Zhao's idea is to give an explanation for the time elapsing between two conjunctions of sun and moon, i.e., the return of sun and moon to the same angular position in the sky. He therefore compares one revolution of sun and moon around the earth to one round of a good and a limping horse, both running on a circle in a limited territory. The length of one round corresponds precisely to the number of *li* that the good horse runs in one day, which again is equivalent to the number of revolutions that the sun makes around the earth in one year (= 365 1/4). Since both, moon and sun, are attached to the movements of the sky, the moon seems slower than the sun:

> The sky completes one revolution from morning till nighttime. The sun (*tai yang* 太陽) moves only one *du* forward in the sky, but the moon (*tai yin* 太陰) moves 13 *du* forward. The fractional parts (*you ji* 有奇), that [the moon] lacks behind, are derived by measurements. Therefore sun and moon (*ri yue* 日月) are parting from the path of heaven and are not connected to the sky. If such is the case, then the sun (*ri*) revolves around the earth once a day and altogether (*zong ji* 總計) makes 365 and 1/4 revolutions per year. The sky moves one *du* more, therefore it also daily revolves around the earth for 366 and 1/4 *du*. The sun (*tai yang*) does not attain the revolutions of the sky for one *du* per day. Moon (*tai yin*) and sun (*tai yang*) are parting for 12 and 7/19 *du* per day. Therefore the sun (*ri* 日) is fast (*su* 速), the moon (*yue* 月) is slow (*chi* 遲).[31]

28 This technique was originally applied to geometrical problems, and later, in Zhu Shijie's *Jade Mirror of Four Elements* (*Si yuan yu jian* 四元玉鑑, 1303), was presented as a unifying method applicable to all kinds of problems from the traditional nine topics of mathematics. See [Bréard 2000].

29 See [Volkov 1996/97b] for the mathematical content of the treatise.

30 As [Volkov 1996/97b: 169] points out:

> The originality of Zhao's model is that the distances involved are not linear but angular (modulo *C*, where *C* is the length of the circumference in degrees). The very idea of applying this class of methods to astronomical computations was for the first time suggested by Wang Xiaotong, yet, the latter considered the moon as moving faster (in his interpretation the moon, compared with a dog, was chasing the sun, represented by a rabbit). This apparent difference between the two representations obviously comes from the way of measuring the (angular) distance between the two moving objects according to the cosmological model adopted.

31 *Juan* 1, p. 11b in [*Siku quanshu*, 786: 228]. In our translation we do not base ourselves on the *Improved Reedition of the New Writing on the Image of Alteration* (*Chong xiu ge xiang xin*

## Indian Astronomy and "Worldly Computations"

Turning to the Indian peninsula, we found occurrences of problems of pursuit and meeting associated to problems from the astronomical realm in the seventh-century commentary of Bhāskara on the *Āryabhaṭīya*.[32] The mathematical formulations of these problem types are classified as "worldly computations" (*laukika-gaṇita*) by the commentator, which suggests that they were considered to be the less abstract and noble occupations of the mathematician.[33] Other treatises merely contain problems of pursuit and meeting without referring to astronomy, as the "Meeting of two travelers"[34] and the "Equations of journeys"[35] in the *Bakhshālī Manuscript* (fifth–seventh century A.D.),[36] and the chapter "Meeting of Travelers" in the *Pāṭīgaṇita* of Śrīdharācārya (A.D. 850–950). We shall not discuss problems which deal with the pursuit of two animals along the perimeter of right triangles since they appear for the first time in Indian texts and do not, according to the present state of available sources, seem to have their origins in China.[37]

Problems of the *Bakhshālī Manuscript* (Sutras 16, 17 and 19) contain arithmetical progressions for the distances traveled per day which are given, as in the Chinese Problem 7.19 from the *Nine Chapters*, by their motion on the first day and the daily increase. Although the situation described in the examples in Sutra 19 are close to the Chinese problem, their algorithmic solutions are different. The Indian text calculates the positive solution of an equation of second degree, whereas the Chinese text solves the problem by applying the Rule of False Double Position. After calculating the time of the meeting, Sutra 15 applies the Rule of Three "for verification" (i.e., to calculate the respective distances traveled by the two men, which turn out to be equal) to the simplest case, where:

> A certain man goes at rate $v_1$ per time-unit. When the time period $t_1$ elapsed since his departure, another man starts and travels $v_2$ per time-unit. How long ($t_2$)

*shu* 重修革象新書) from the Ming-Dynasty by Wang Yi 王禕 [*Siku quanshu*, 786: 293–294], since it differs significantly from the original version. Taoist terms, as for instance for sun (*tai yang* 太陽) and moon (*tai yin* 太陰), have constantly been replaced by *ri* 日 and *yue* 月, but also the expressions for the two horses (*liang nu* 良駑), which in the original version correspond to the expressions used in Problem 7.19 of the *Nine Chapters*, were amended to *jun nu* 駿駑.

32 For the Indian part, I am particularly indebted to the valuable help of Agathe Keller, who has been doing research on Sanskrit mathematics for several years and recently defended her Ph.D. thesis [Keller 2000].

33 See [Keller 2000: 349–351].

34 See [Hayashi 1995: 373] for the commentary on Sutra 24 and [Hayashi 1995: 416] for a conjecture of Sutra C3.

35 Sutras 15, 16, 17, 19 and N5. [Hayashi 1995: 367] notes that similar examples can be found in GSS miśraka 327 1/2 (the *Gaṇitasārasaṃgraha* by Mahāvīra, ca. 11th century), and PM A4 (*Patan Manuscript*).

36 In the *Bakhshālī Manuscript*, problems on the motion of sun, Jupiter and Saturn [Hayashi 1995: 352, 426] only consider the speed of one of the planets, but not their conjunctions.

37 See for example problems 126 and 150 in Bhāskara's commentary, translated in [Keller 2000].

does it take for the second man to travel the same distance as the first? [Hayashi 1995: 366]

The *Pāṭīgaṇita* contains rules for calculating the time at meeting when two travelers start simultaneously from the same place and then come back by the same track to "meet each other on the way, one going ahead and the other coming back" [Shukla 1959: 52]; thus a situation partly similar to the one found in Problem 7.19 of the *Nine Chapters* on the good and the limping horse. Interestingly enough, the text shows, like *Zhang Qiujian's Mathematical Classic*, an interest in inverse formulations of the problem (see the "sub-rules" 66(ii) and 67(i) in Appendix 2).

## Medieval Latin Texts

Alcuin, private tutor and counselor of Charlemagne (born ca. 732, in or near York, Yorkshire, England, died May 19, 804, in Tours, France), wrote the first book on recreational mathematics in Latin, the *Propositiones ad acuendos iuvenes*.[38] The shipping of a wolf, a goat and a cabbage might be the most famous problem from these "Problems to sharpen the young," and is considered as the origin of the until then unknown river crossing problems.[39] Another famous problem often cited in histories of recreational mathematics is Alcuin's Problem 26, the *Propositio de campo et cursu canis ac fuga leporis* on a hound hunting a hare. It presents the simplest case of a problem of pursuit solved by applying the Rule of Three.

The problem of hare and hound subsequently circulated in Europe and can be found in similar formulations in several manuscripts.[40] As [Smith 1958, II: 546] notes, it "is given in Petzensteiner's work of 1483, Calandri used it in 1491,[41] Pacioli has it in his *Summa* (1494), and most of the prominent writers on algebra or higher arithmetic inserted it in their books from that time on." We shall not discuss these references in detail but finally look at another medieval manual which presents the problem in connection with astronomical situations, an aspect already identified in Chinese and Indian texts.[42]

38 See [Folkerts 1993].

39 See [Pressman & Singmaster 1989] and R. Franci's contribution to this volume, pp. 289–306.

40 [Tropfke 1980: 597] lists three different categories of conditions for the hare-and-hound problem found in occcidental sources:

1. Die Sprünge der beiden Tiere sind gleich lang, ihre Anzahl pro Zeiteinheit ist jedoch verschieden.
2. Hase und Hund machen beide gleichviele Sprünge, aber die Sprünge haben verschiedene Länge.
3. Sowohl die Länge, als auch die Zahl der Sprünge von Hase und Hund sind verschieden.

We do not consider these differences as significant in the process of problem transmission, but consider them as minor variations.

41 The illustration of Calandri's hare-and-hound problem is reprinted in [Tropfke 1980: 597].

42 According to R. Franci (personal communication), such a link between a mathematical and an astronomical pursuit or meeting cannot be found in the Italian abacus tradition.

The *Algorismus Ratisbonensis* was compiled in a Benedictine monastery near Regensburg during the first half of the 15$^{th}$ century and was the earliest mathematical manual for merchants in Germany.[43] Its third part, the *Practica*, a vast collection of problems, was copied by the Benedictine monk Fridericus Gerhart (also Fridericus Amann) presumably in 1449 or 1450 and was extended during the second half of the 15$^{th}$ century.[44] Problems of unknown origin, partly in German, partly in Latin or some vernacular mixture, were added successively. In particular, we can find a problem of pursuit of a hare by a hound similar to the one given in the *Propositiones*.[45] A commentary to the problem refers to the application of the same solution procedure to certain astronomical problems.[46] Probably, the author had the general procedure in mind, which is given for Problem 148 in the *Practica* that deals with the meeting of planets.[47]

## Dissemination and Knowledge Transfer

Historically, several channels and periods of transmission between the Byzantine, European and Arabic world and the Far East existed.[48] In the sixth century, for example, direct contacts with the Byzantine kingdoms were related to the silk supply,[49] and controlled by the West Turks,[50] later by the Tang Dynasty. During

43 See [Zimmermann 1978] for its textual structure and history. I am particularly grateful to Menso Folkerts, who provided me with this article and drew my attention to the correct historical situation of the *Algorismus Ratisbonensis* and its *Practica*.

44 Edited in [Vogel 1954].

45 As [Smith 1958, II: 546] points out, the problem of the hound pursuing the hare "was looked upon as one of the necessary questions of European mathematics, appearing in various later medieval manuscripts" after its appearance in the *Propositiones*.

46 This commentary on the movement of two planets is missing from one of the extant manuscripts. See [Vogel 1954: 72]:

> Item simile est de planetis ut scito motu duarum planetarum et scita distancia inter illas duas faciendum est ut supra.

47 See [Vogel 1954: 72] for Problem 148 of the *Practica*:

> Item: Wildu wissen, wenn 2 planeten sein, welichs sein zesam komen nach den mittenlauf, so mustu vor wissen, in welichen zaichen vnd grad sy sein; auch mustu wissen eines ydlichen täglich gank. Darnach mustu wissen, wie weit von einem zu den andern seij vnd dije selbigt weit mutiplicir durch den ganck lanksamers planeten. Darnach subtrahir des lancksameres ganck von dem gank des snellers. Darnach |d|, was da pleibt, diuidir durch dy distancz, dy du vor hast multiplicirt, so kumpt dir dy distancz, wie uil der lancksamer gangen ist, vncz bis in der sneller erfert, es sein grad oder minuten.

48 See, for instance, [*Lexikon des Mittelalters*, II: 1827–1829; Jaouiche 1998; Hirth 1885].

49 Several Byzantine golden coins were excavated along the Silk Road and close to the Yellow River in China. See [Koenig 1982].

50 The Byzantine historian Theophylactus Simocatta (sixth–seventh centuries A.D.) inserts a paragraph on China into the seventh book of his *Historiae*. [Boodberg 1938] assumes that the Byzantines had obtained precise information on the Middle Kingdom from the Turkish khan, who expanded his power in Central Asia.

the same period, Nestorian Christians from Syria came as missionaries to China (ca. 631).[51] Later, during the eighth century until the Mongol invasion, the Islamic influence expanded to Central Asia and Arab empires became the intermediary in mercantile exchange between Christian Europe and the Far East.[52] During the Yuan Dynasty (1279–1368), several papal legations, mostly Franciscans and Dominicans, traveled to China to gain the Mongols for an alliance against the Islamic world and to organize the Catholic mission (most importantly, Giovanni de Montecorvino in 1291/92). Soon after, merchants from the Polo and other Venetian and Genoese families traveled by land to China until the 1340s,[53] when the Mongol Empire began to decline.

Surveying problems of pursuit of different cultures over a period of approximately 1000 years, we are quite aware that the traits that seem common at first sight are not universal but general characteristics linked to the cultural and mathematical context in which the problems appeared. Yet, almost all of their various aspects reappear in nearly all the cultures.[54]

As [Folkerts & Gericke 1993: 294] point out, "Indian and Islamic mathematicians merely played a role in the transmission of some problems; their own achievements were not taken into account in the *Propositiones*." This does partly confirm what we can observe here: the more complex situations found in Chinese texts are developed and analyzed,[55] and their algorithmic solutions modified in Indian texts, but the problems themselves do not reappear in the same complexity in early medieval manuals. Only the simplest cases of pursuit and meeting solved by the Rule of Three are discussed in the latter.

"Since rules of false double position have not yet been found in Sanskrit texts," [Chemla 1997: 113] suggests for this rule, that there might be "a direct transmission from China to the Arabic-speaking world" bypassing India, and that *Zhang Qiujian's Mathematical Classic* "or a text close to it, played a specific role in the direct transmission of knowledge from China to the West." The cases of specific procedures of root-extraction and the hundred fowls problem support her argument for a transmission of one Chinese mathematical tradition (linked to Zhang Qiujian's text) directly to the West, and another one to India. For the Rule of False Double Position, [Chemla 1997: 113] further concludes that the direct transmis-

51 On the discovery of a Nestorian Monument in the 17th century see [Mungello 1985: 164–172].

52 See for example the "Imports of Iraq" from China during the mid-ninth century. Translated from a text ascribed to al-Jāḥiẓ in [Lopez & Raymond 1955: 28].

53 See for example the "Advice about the Journey to Cathay by the Road through Tana [Azov], [for merchants] going and returning with wares" by Francesco di Balduccio Pegolotti. Translated from the Italian in [Lopez & Raymond 1955: 355–358].

54 The only situation, which we have not yet found in other texts, is the one of interrupted pursuit, as in the hare-and-hound Problem 6.14 in the *Nine Chapters* and Zhang Qiujian's Problem 2.2.

55 In particular the situation of Problem 7.19 from the *Nine Chapters* can be found decomposed into 1) problems of pursuit containing arithmetical progressions for the daily speed of the actors (*Bakhshālī Manuscript*), and 2) a problem of meeting, where the faster actor goes to a certain destination and back again to meet the slower one (*Pāṭīgaṇita*).

sion, possibly linked to Zhang Qiujian, only concerns problems of one kind,[56] and not those of a second kind to which would belong the problems of meeting 7.10, 7.12 and 7.19 from the *Nine Chapters* discussed above.[57] All this would possibly fit into the following picture: Problems of meeting and pursuit have both been transmitted to India (possibly also in an astronomical context); only problems of pursuit have been transmitted directly from China to the West.

Unfortunately, in our study we are not able to analyze references to occurrences of problems of pursuit and meeting in the Arabic world.[58] Other historians have insisted upon the absence of problems of pursuit and meeting from Greek mathematical texts, but found various occurrences in Byzantine literature.[59] The two oldest collections of mathematical problems from the 14th and 15th century contain in particular the problem of hare and hound. The latter text seems to have been written when Byzantium was already under Turkish influence. [Hunger & Vogel 1963: 100; Vogel 1968: 154] assume that these manuals were compiled under Greek, Indo-Arabic and also under occidental influence, in particular because of the occurrence of the problem of hare and hound in Alcuin's *Propositiones*, and its absence in the Islamic tradition.

If we now assume that *Zhang Qiujian's Mathematical Classic* and not the *Nine Chapters* has played the above mentioned role in the dissemination of problems of pursuit directly to the West, then how do we explain the fact that hound and hare problems reappear in Western contexts? There are two possibilities: 1) Either Wang Xiaotong's treatise was also transmitted to the West,[60] since it refers to a simpler situation of a hound pursuing a hare than the one in the *Nine Chapters*, or 2) there was only the transmission of Zhang Qiujian's text, and, having in mind the mythological image of a hound pursuing a hare, it was a cul-

56 Namely, problems where two unknowns ($x$ and $y$) are linked by the suppositions $a$ and $a'$ and the respective excedents or deficits $b$ and $b'$ ($ax = y + b$, $a'x = y - b'$).

57 Namely, problems where suppositions $a$ and $a'$ are made on the unknown $x$ and the result produced by an algorithm $A$ is compared to the expected outcome $y$ ($A(a) = y + b$, $A(a') = y - b'$).

58 [Tropfke 1980: 593] for example mentions al-Karajī, al-Kāshī and the treatise *Key of Transactions* of the Persian Ṭabarī, that contains a group of four problems of meeting and pursuit. [Rebstock 1992: 259, n. 21] refers to three Islamic authors that deal with problems of motion in a last chapter on "rarities" (*nawādir*): Abū Manṣūr 'Abd al-Qāhir Ibn Ṭāhir al-Baghdādī, Abū 'l-Qāsim Ibn al-Samḥ al-Gharnāṭī, and Abū 'l-Ṭāhir Ismā'īl al-Māridīnī (Ibn Fallūs).

59 [Tropfke 1980: 594] has no doubt that problems of motion could be solved in Greece, since institutions in the Byzantine Empire were under Hellenistic influence.

60 With respect to the algorithms for the solution of quadratic and cubic equations, Wang Xiaotong's procedures seem more closely connected to those for the extraction of the square and the cube roots suggested by Zhang Qiujian, than to those from the *Nine Chapters*. See Chapter III.4 ("La résolution des équations cubiques dans le *Jigu suanjing*") in [Sentz 1996]:

> Nous venons donc de voir qu'il est possible de résoudre les équations cubiques présentes dans le *Jigu suanjing* avec l'algorithme d'extraction de racine cubique de Zhang Qiujian, moyennant une légère modification d'une étape, .... Auparavant, nous avions pu utiliser l'algorithme d'extraction de racine carrée du *Traité mathématique de Zhang Qiujian* pour résoudre les équations quadratiques du même ouvrage, en remarquant que cet algorithme offre une représentation positionnelle implicite de tous les coefficients d'une équation quadratique [p. 54].

tural association common to both China and Europe that led the authors to embed a certain transmitted form of the problem of pursuit into one with hare and hound. Given the present state of our research we cannot answer this question, and hope that further comparative studies will bring more arguments to light to clarify the process of transmission and the steady modification and reembedding of the elements involved in problems of pursuit and meeting.

## Mathematics Based on "Puzzle Type Problems"?

To conclude, let us briefly comment on the historiographical question of how the types of problems discussed above were interpreted. In historical accounts, considerations of problems of pursuit were generally classified as mathematical puzzles, games or recreations.[61] Although most authors admit that these led to the study of many areas of mathematics, or were used "as a way of making serious mathematics understandable or palatable" [Singmaster 1992], it seems that in the case of problems of pursuit and meeting, the situation is fairly different: these are paralleled in Chinese, Indian and medieval Latin texts with astronomical questions concerning the conjunction of planets.[62]

There is nothing wrong with applying the English phrase "recreational mathematics" to label thought about those aspects of mathematics that are seemingly entertaining from the modern or medieval[63] point of view. But if we wish to know whether the Chinese needed a word for such a class of problems, or a concept with the same boundaries, the answer is no. Problems of pursuit and meeting appeared in different algorithmic and theoretical contexts. They were partly put into relation with astronomical situations, but have never been regarded as puzzle type problems in Chinese traditional historiography.

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61 [Shen 1993] compares problems of pursuit and meeting with uniform or unequal motion found for example in Alcuin's *Propositiones*, Pacioli's *Summa* and the *Āryabhaṭīya* to those found in the *Nine Chapters* and Liu Hui's commentary. He classifies these as problems of kinematics to show that in China, "splendid historical achievements are not inferior to the West, but surpass the West" [p. 387], without discussing questions of transmission.

62 [Tropfke 1980: 590] only associates the type of problem to astronomical considerations, where the pursuit is along a circular path.

63 Michael Stifel (1487–1567) even looks down upon useless problems of the "hare-and-hound" type: "Solliche spötliche Exempla wöllen offt mehr wort haben denn die nutzliche" (cited in [Tropfke 1980: 597]).

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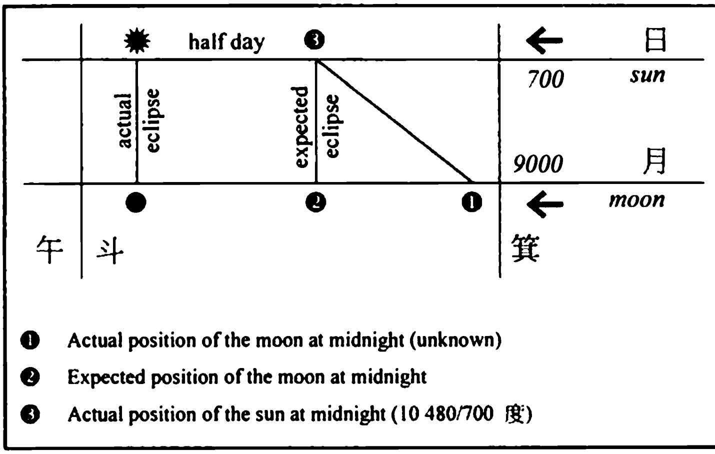
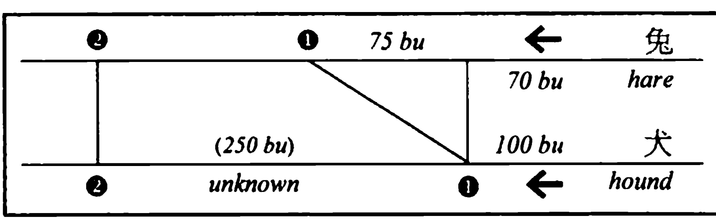
## Appendix 1. Diagrams

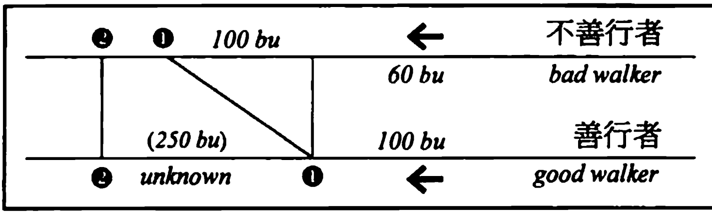
Wang Xiaotong 王孝通: *Mathematical Classic in Continuation of Antiquity*
(*Ji gu suan jing* 緝古算經, before A.D. 626)



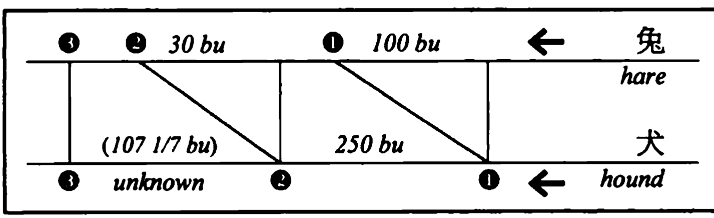




Problem 1

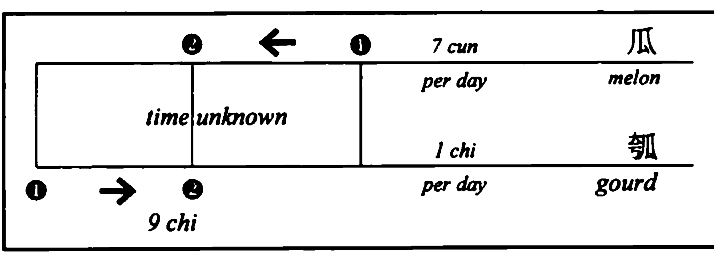
*Nine Chapters on Mathematical Procedures*
(*Jiu zhang suan shu* 九章算術, ca. first century A.D.)



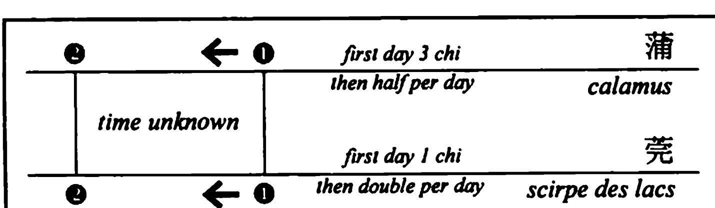

Problem 6.12



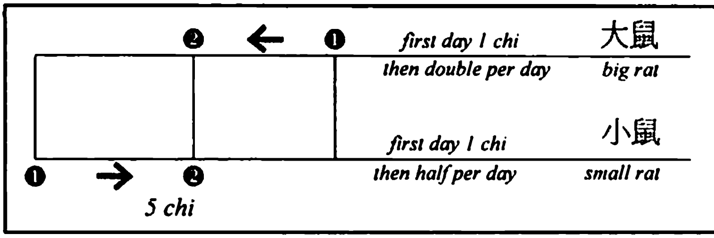

Problem 6.14



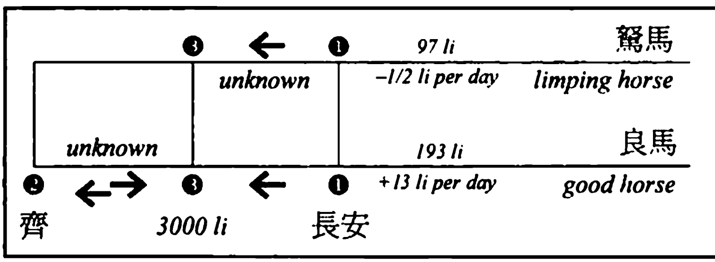

Problem 7.10




Problem 7.11

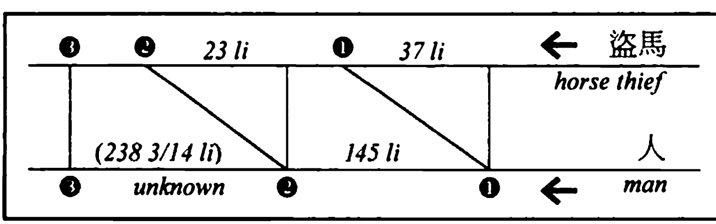



Problem 7.12

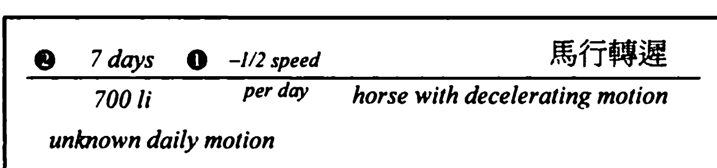



Problem 7.19

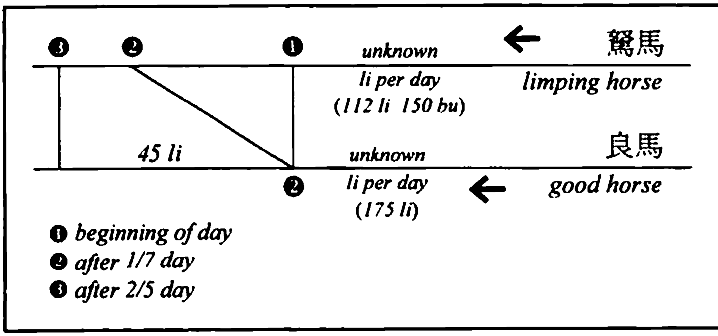
*Zhang Qiujian's Mathematical Classic*
(*Zhang Qiujian suan jing* 張邱建算經, ca. 466–485)



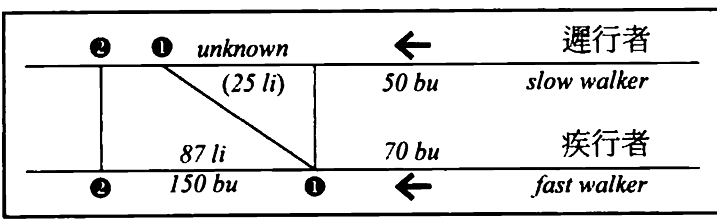

Problem 2.2



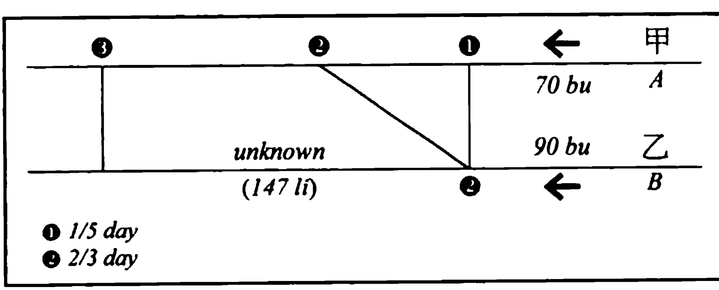

Problem 2.3




Problem 2.4

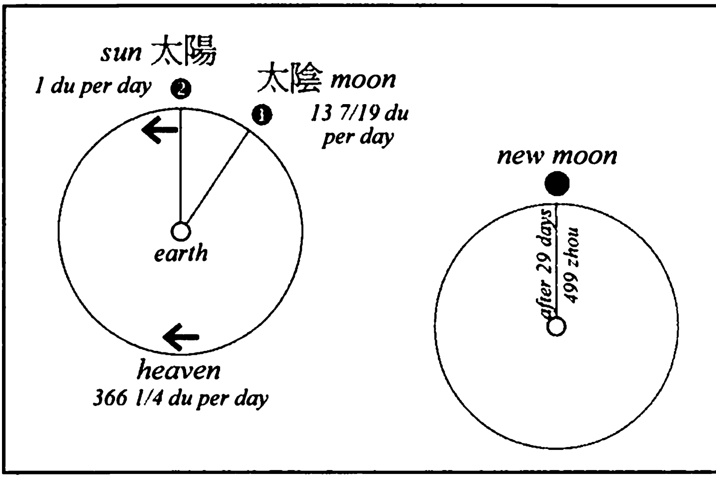


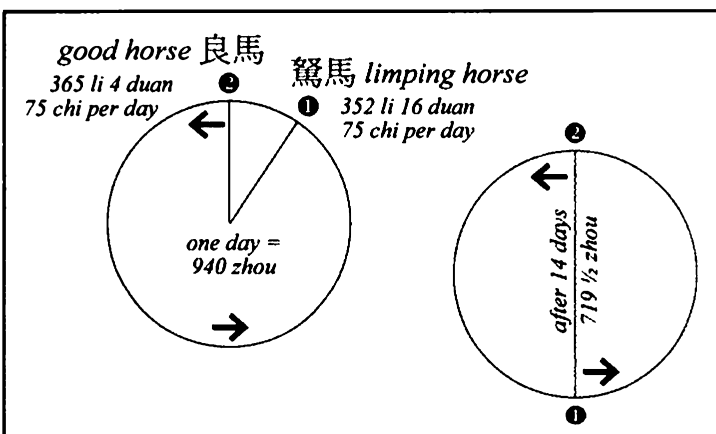

Problem 2.5



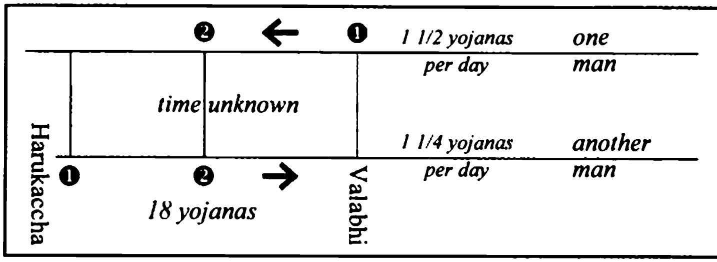

Problem 2.6

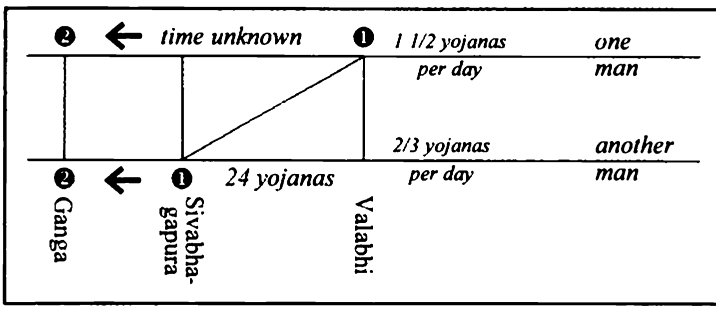
Zhao Youqin 趙友欽 (1271–1335?):
*New Writing on the Image of Alteration* (*Ge xiang xin shu* 革象新書)



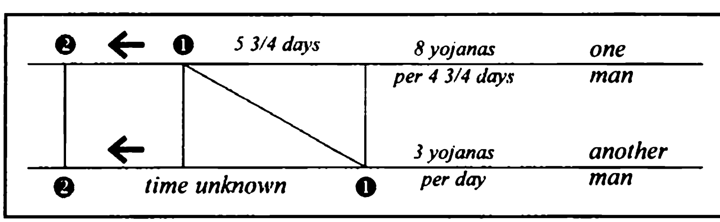



Bhāskara's commentary on the *gaṇita-pada* (mathematical chapter) of the *Āryabhaṭīya* (A.D. 629)

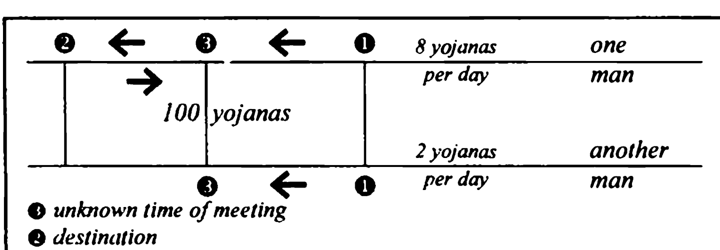



The *Pāṭīgaṇita* of Śrīdharācārya (850–950)

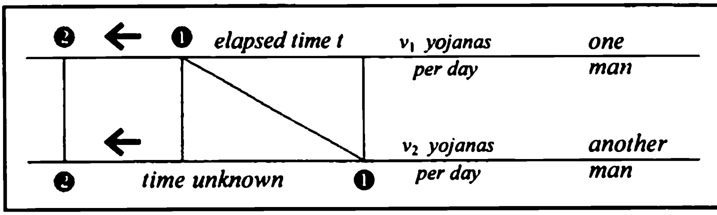



Example 81–82

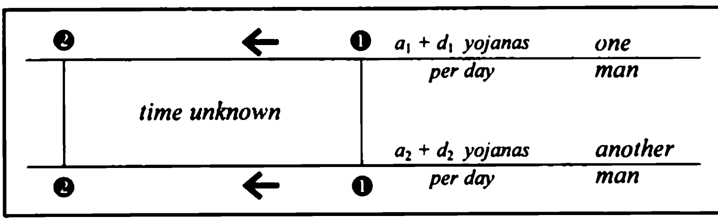



Example 83

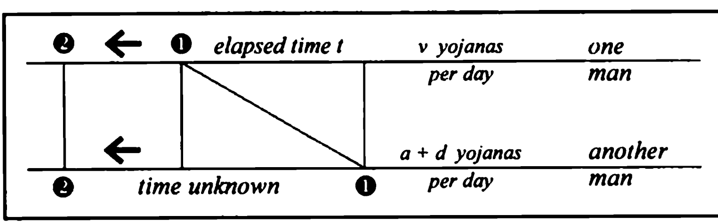
*The Bakhshālī Manuscript* (5th–7th century)



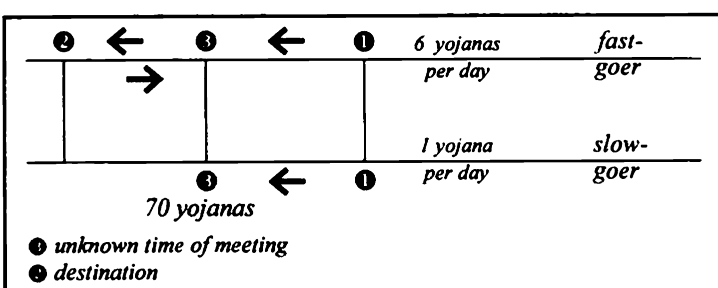

Sutra 15




Sutra 16

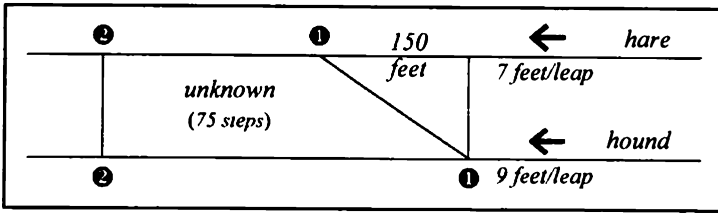



Sutra 19



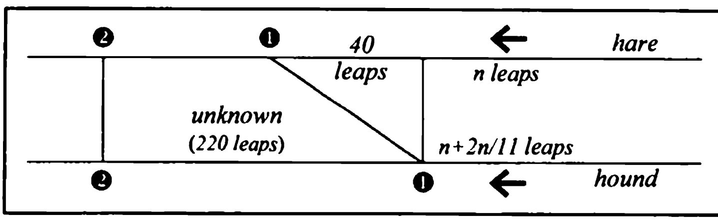

Sutra 24

*Propositiones ad acuendos iuvenes* (attributed to Alcuin, ca. 735–804)



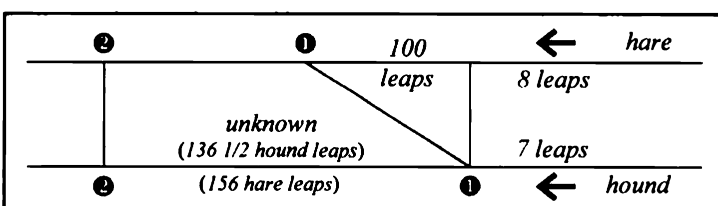

Problem 26: *Propositio de campo et cursu canis ac fuga leporis*

*An Early Fourteenth-Century Byzantine Manual*
(edited and translated in [Vogel 1968])




Problem 87

*A Fifteenth-Century Byzantine Manual*
(edited and translated in [Hunger & Vogel 1963])[64]

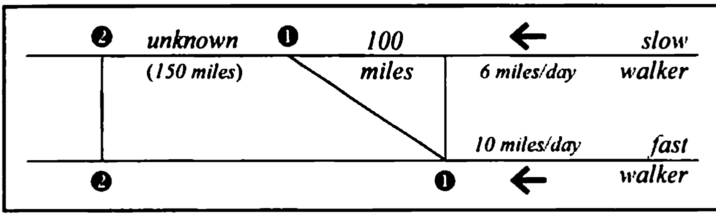



Problem 81

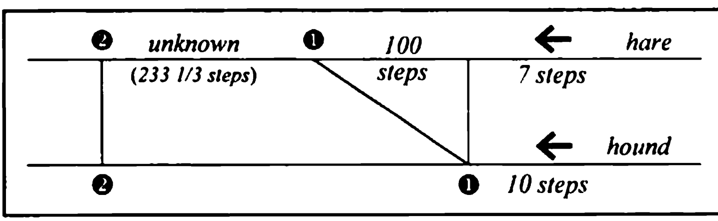
64 [Hunger & Vogel 1963: 65] points out a problem in the formulation of the following problem. Obviously the seven and eight leaps of hound and hare are equal in distance but do not coincide, and the answer is only correct if we assume that the hound makes 39 leaps while the hare makes 16:

$$100 + x = x \cdot \frac{8}{7} \cdot \frac{39}{16}$$
$$x = 56 \text{ leaps}$$

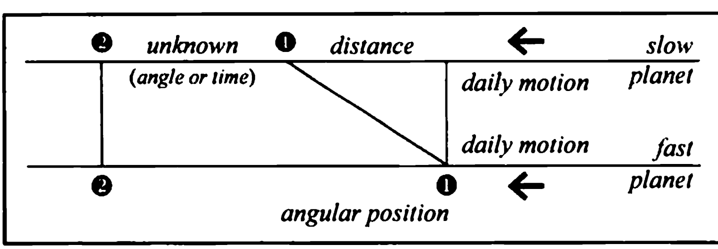
*Algorismus Ratisbonensis* (mid-15th century)




Problem 32




Problem 33




Problem 148

## Appendix 2. Symbolic Transcription of Algorithms

### Wang Xiaotong 王孝通: *Mathematical Classic in Continuation of Antiquity* (*Ji gu suan jing* 緝古算經, before A.D. 626)

$$\left[\left(\underbrace{9000}_{\substack{\textit{parts traveled}\\ \textit{by the moon}}} - \underbrace{700}_{\substack{\textit{year cycle}\\ \textit{constant}}}\right) \cdot \underbrace{10000}_{\substack{\textit{true deficient}\\ \textit{day remainder}}}\right] \div \underbrace{20000}_{\substack{\textit{day}\\ \textit{divisor}}} = \underbrace{4150}_{\substack{\textit{parts traveled}\\ \textit{in advance}}}$$

$$\underbrace{10480/700\ \text{du}}_{\substack{\textit{position of sun at}\\ \textit{midnight of the first}\\ \textit{day of the month}}} - \left(\underbrace{4150}_{\substack{\textit{parts traveled}\\ \textit{in advance}}} \div \underbrace{700}_{\substack{\textit{day}\\ \textit{divisor}}}\right) = \underbrace{4530/700\ \text{du}}_{\substack{\textit{position of the moon at}\\ \textit{midnight of the first}\\ \textit{day of the month}}}$$

$$\underbrace{100}_{\substack{\textit{what the}\\ \textit{dog runs}}} \cdot \underbrace{75}_{\substack{\textit{what the hare}\\ \textit{runs in advance}}} \div \left(\underbrace{100}_{\substack{\textit{what the}\\ \textit{dog runs}}} - \underbrace{70}_{\substack{\textit{what the}\\ \textit{hare runs}}}\right) = \underbrace{250}_{\substack{\textit{bu until pur-}\\ \textit{suit is over}}}$$

Problem 1

### *Nine Chapters on Mathematical Procedures* (*Jiu zhang suan shu* 九章算術, ca. first century A.D.)

$$\underbrace{100}_{\substack{\textit{bu of the}\\ \textit{good walker}}} \cdot \underbrace{100}_{\substack{\textit{bu that the bad}\\ \textit{walker walks in}\\ \textit{advance}}} \div \left(\underbrace{100}_{\substack{\textit{bu of the}\\ \textit{good walker}}} - \underbrace{60}_{\substack{\textit{bu of the}\\ \textit{bad walker}}}\right) = \underbrace{250}_{\substack{\textit{bu until}\\ \textit{reaching him}}}$$

Problem 6.12

$$\underbrace{250}_{\substack{\textit{bu that the hound}\\ \textit{pursues the hare}}} \cdot \underbrace{30}_{\substack{\textit{bu that the hound}\\ \textit{does not reach}\\ \textit{the hare}}} \div \left(\underbrace{100}_{\substack{\textit{bu that the hare}\\ \textit{runs in advance}}} - \underbrace{30}_{\substack{\textit{bu that the hound}\\ \textit{does not reach}\\ \textit{the hare}}}\right) = \underbrace{107\ 1/7}_{\substack{\textit{bu that the hound would}\\ \textit{still have had to run until}\\ \textit{reaching the hare}}}$$

Problem 6.14

$$\boxed{\begin{aligned} A(n-1) &= 9\ chi - b_1 \\ A(n) &= 9\ chi + b_2 \end{aligned}} \qquad \boxed{t = [(n-1)\cdot b_2 + n\cdot b_1] \div (b_1 + b_2)}$$

Problem 7.10

$$\boxed{\begin{aligned} A(n-1) &= length - b_1 \\ A(n) &= length + b_2 \end{aligned}} \qquad \boxed{t = [(n-1)\cdot b_2 + n\cdot b_1] \div (b_1 + b_2)}$$

Problem 7.11

$$\boxed{\begin{aligned} A(n-1) &= 5\,chi - b_1 \\ A(n) &= 5\,chi + b_2 \end{aligned}} \qquad \boxed{t = [(n-1)\cdot b_2 + n\cdot b_1] \div (b_1 + b_2)}$$

Problem 7.12

$$\boxed{\begin{matrix} A_l(n-1)=y_l & A_l(n)=z_l \\ A_g(n-1)=y_g & A_g(n)=z_g \end{matrix}} \quad \boxed{\begin{matrix} y_g-3000\,li+y_l=3000\,li-d_1 \\ z_g-3000\,li+z_l=3000\,li+d_2 \end{matrix}}$$

$$\boxed{t=[(n-1)\cdot d_2+n\cdot d_1]\div(d_1+d_2)}$$

Problem 7.19

## *Zhang Qiujian's Mathematical Classic*
## (*Zhang Qiujian suan jing* 張邱建算經, ca. 466–485)

$$\underbrace{145}_{\textit{li that the owner of the horse pursues}} \cdot \underbrace{23}_{\textit{li that are not reached}} \div (\underbrace{37}_{\textit{li that were already run}} - \underbrace{23}_{\textit{li that are not reached}}) = \underbrace{238\tfrac{3}{14}\,[li]}_{\textit{li until reaching it}}$$

Problem 2.2

| 64 | 32 | 16 | 8 | 4 | 2 | 1 | $\sum = 127$ |
|---|---|---|---|---|---|---|---|
| 44800 | 22400 | 11200 | 5600 | 2800 | 1400 | 700 | $\cdot \underbrace{700}_{\textit{li traveled}}$ |
| $\underbrace{352\tfrac{96}{127}}_{\textit{li traveled the first day}}$ | $\underbrace{176\tfrac{48}{127}}_{\textit{li traveled the next day}}$ | $\underbrace{88\tfrac{24}{127}}_{\textit{li traveled the next day}}$ | $\underbrace{44\tfrac{12}{127}}_{\textit{li traveled the next day}}$ | $\underbrace{22\tfrac{6}{127}}_{\textit{li traveled the next day}}$ | $\underbrace{11\tfrac{3}{127}}_{\textit{li traveled the next day}}$ | $\underbrace{5\tfrac{65}{127}}_{\textit{li traveled the next day}}$ | $\div 127$ |

Problem 2.3

$$\underbrace{45}_{\textit{li until reaching}} \div \underbrace{(\tfrac{2}{5}-\tfrac{1}{7})}_{\textit{lü of the good horse's motion}} = \underbrace{175\,[li]}_{\textit{li traveled per day}}$$

$$\underbrace{45}_{\textit{li until reaching}} \div \underbrace{\tfrac{2}{5}}_{\textit{lü of the limping horse's motion}} = \underbrace{112\,[li]\,150\,[bu]}_{\textit{li traveled per day}}$$

Problem 2.4

$$(\underbrace{70\,[bu]}_{\textit{bu of the fast walker}} - \underbrace{50\,[bu]}_{\textit{bu of the slow walker}}) \cdot \underbrace{87\tfrac{1}{2}\,[li]}_{\textit{bu until reaching}} \div \underbrace{70\,[bu]}_{\textit{bu of the fast walker}} = \underbrace{25\,[li]}_{\textit{li that the slow walker walks in advance}}$$

Problem 2.5

$$(\tfrac{2}{3}-\tfrac{1}{5}) \cdot \underbrace{70\,[li]}_{\textit{li of A's daily motion}} \cdot \underbrace{90\,[li]}_{\textit{li of B's daily motion}} \div (\underbrace{90\,[li]}_{\textit{li of B's daily motion}} - \underbrace{70\,[li]}_{\textit{li of A's daily motion}}) = \underbrace{147\,[li]}_{\textit{li that B walks until reaching A}}$$

Problem 2.6

Zhao Youqin 趙友欽 (1271–1335?):
*New Writing on the Image of Alteration* (*Ge xiang xin shu* 革象新書)

$$\underbrace{365\,\textit{li}\ \ 4\,\textit{duan}\ \ 75\,\textit{chi}}_{\substack{\textit{one tour, daily motion of the good}\\ \textit{horse (1 li=19 duan, 1 day=940 zhou)}}} - \underbrace{352\,\textit{li}\ \ 16\,\textit{duan}\ \ 75\,\textit{chi}}_{\textit{daily motion of the limping horse}} = \underbrace{12\,\textit{li}\ \ 7\,\textit{duan}}_{\substack{\textit{what the limping horse}\\ \textit{falls behind in one day}}}$$

$$= 235\,\textit{duan} = 23500\,\textit{chi} = 25\,\textit{chi / zhou}$$

$$365\,\textit{li}\ \ 4\,\textit{duan}\ \ 75\,\textit{chi} \div (25\,\textit{chi / zhou}) = 27759\,\textit{zhou} = \underbrace{29\,\textit{days}\ \ 499\,\textit{zhou}}_{\substack{\textit{length of cycle for meeting}\\ \textit{at the starting point}}}$$

Bhāskara I (A.D. 629), *Mahābhāskarīya*, Mbh. 6.49–6.51 and 4.64

Two planets with opposite movements:

*Difference of longitudes ÷ Sum of daily motions = Conjunction time in terms of days*

Two planets moving in the same direction:

*Difference of longitudes ÷ Difference of daily motions = Conjunction time in terms of days*

For ⟨the meeting of⟩ the sun and the moon:

*Unelapsed part of the tithi or elapsed part of the (next) tithi* · *Daily motions of the sun and the moon* ÷ *Difference between the (true) daily motions*

Bhāskara's commentary on the *gaṇita-pada* (mathematical chapter) of the *Āryabhaṭīya* (A.D. 629)

$$\underbrace{18}_{\substack{\textit{distance for two men}\\ \textit{in different motions}}} \div \underbrace{\left(\frac{3}{2}+\frac{5}{4}\right)}_{\substack{\textit{sum of the daily}\\ \textit{motions of the two men}}} = \underbrace{6\frac{6}{11}\ \text{days}}_{\substack{\textit{time in which the meeting}\\ \textit{of the two men takes place}}}$$

$$\underbrace{24}_{\substack{\textit{distance of two men}\\ \textit{in the same direction}}} \div \underbrace{\left(\frac{3}{2}-\frac{2}{3}\right)}_{\substack{\textit{difference of the daily}\\ \textit{motions of these two men}}} = \underbrace{28\frac{4}{5}\ \text{days}}_{\substack{\textit{time in which the two men}\\ \textit{that went along one and}\\ \textit{the same path meet}}}$$

## The *Pāṭīgaṇita* of Śrīdharācārya (850–950)

$$\boxed{\textit{time of meeting} = \frac{\textit{distance already traveled by the slow traveler}}{\textit{difference between the speed per day}}}$$

Sutra 65

"Rule for finding the time in which the fast traveler, who starts traveling on the same track after the slow traveler has already covered a specified distance, would overtake the slow traveler" [Shukla 1959: 52].

$$\boxed{\textit{time of meeting} = \frac{\textit{length of the track}}{\frac{1}{2}\cdot \textit{sum of the speeds per day}}}$$

Sutra 66 (i)

"Rule for finding the time when two travelers, one fast and the other slow, who start simultaneously from the same place, are destined to go to a specified distance and then to come back by the same track, will meet each other on the way, one going ahead and the other coming back" [Shukla 1959: 52].

$$\boxed{\textit{length of the track} = \textit{time of meeting} \cdot (\tfrac{1}{2} \cdot \textit{sum of the speeds per day})}$$

Sutra 66 (ii)

"Sub-rule for finding the length of the track" [Shukla 1959: 53].

$$\boxed{\frac{\textit{length of the track}}{\textit{time of meeting}} \cdot 2 - \textit{speed of one of the two travelers} = \textit{speed of the other traveler}}$$

Sutra 67 (i)

"Sub-rule for finding the speed of one of the two travelers" [Shukla 1959: 53].

## *The Bakhshālī Manuscript* (5th–7th century)

$$\boxed{\textit{unknown time} = \frac{v_1 \cdot t}{v_2 - v_1}}$$ Sutra 15

$$\boxed{\textit{number of days} = 2 \cdot \frac{a_1 - a_2}{d_2 - d_1} + 1}$$ Sutra 16

$$\boxed{time = \frac{\sqrt{\{2\cdot(v-a)+d\}^2+8\cdot v\cdot t\cdot d}+2\cdot(v-a)+d}{2\cdot d}}$$ Sutra 19

$$\boxed{time\ of\ meeting = 2\cdot\frac{distance}{sum\ of\ the\ speeds\ per\ day}}$$ Sutra 24

*Propositiones ad acuendos iuvenes* (attributed to Alcuin, ca. 735–804)

$$\underbrace{9}_{\text{feet that the hound makes per leap}} \cdot \underbrace{150}_{\text{length of the field}} \div \underbrace{2}_{\text{half of it}} = \underbrace{675}_{\text{feet that the hound covers until capturing the hare with his teeth}}$$

$$\underbrace{7}_{\text{feet that the hare makes per leap}} \cdot \underbrace{150}_{\text{length of the field}} \div \underbrace{2}_{\text{half of it}} = \underbrace{525}_{\text{feet that the hare covers until being captured}}$$

Problem 26: *Propositio de campo et cursu canis ac fuga leporis*

*Algorismus Ratisbonensis* (mid-15th century)

$$\boxed{100\cdot 6\div(10-6)=150}$$

Problem 32

$$100 \cdot \underbrace{7\div(10-7)}_{\text{for each 7 steps that the hound runs he makes up 3 steps advance}} = \underbrace{233\tfrac{1}{3}}_{\text{feet that the hare runs before being captured}}$$

Problem 33

*Motion of the slower planet · Distance ÷ (Motion of the faster – Motion of the slower) = Distance of the moticn of the slower, until the faster catches up with him*

Problem 148

# The Sanskrit *karaṇī*s and the Chinese *mian*

by KARINE CHEMLA and AGATHE KELLER

In this article we describe, despite noted differences, a cluster of similar features in the conception and use of quadratic irrationals in seventh-century India and in first- to third-century China. In both contexts, in contrast to what can be found in Greek texts, quadratic irrationals are introduced as such. They are considered as quantities, and arithmetical operations are performed with them. Furthermore, this article not only raises the question of the circulation of mathematical concepts and practices between India and China, but also underlines striking similarities between Akkadian, Greek and Sanskrit concepts on the one hand and Chinese, Indian and Medieval Arab practices on the other.

Dans cet article nous décrivons, malgré certaines différences, des caractéristiques communes à l'Inde du VIIème siècle et à la Chine des premier aux troisième siècles concernant la conception et l'utilisation d'irrationnels quadratiques. Dans ces deux contextes, contrairement à ce qui se trouve dans les textes grecs, les irrationnels quadratiques sont introduits en tant que tels. Conçus comme des quantités, ils entrent dans des opérations arithmétiques grâce à la mise au point de nouveaux algorithmes. Cet article non seulement pose la question de la circulation des pratiques mathématiques entre la Chine et l'Inde, mais encore note des similarités entre des concepts akkadien, grec et sanskrit d'une part, entre des pratiques mathématiques chinoise, indienne et arabes médiévales de l'autre.

The oldest Chinese and Indian mathematical texts that have come down to us both introduce a concept of quadratic irrationals and describe computations with such entities. Hence, dealing with irrationality was not a characteristic unique to Greek mathematics of Antiquity, even though the approach to such phenomena may have been different in the various mathematical traditions. By presenting ancient Indian and Chinese texts describing computations with quadratic irrationals, we would like to highlight some similarities that these texts present, in contrast to what is attested in extant ancient Greek sources. Despite the fact that differences between them can also be noted, this raises the question of whether transmission occurred, and, in case it did, how and when it took place.

Lacking so far evidence to answer these questions, we shall suggest a line of inquiry that could shed some light on the situation, and we shall provide some preliminary results in this direction. In particular, within the framework of this paper, we shall concentrate on the kind of computations which were developed both in China and India to handle such numbers.

As far as Indian texts are concerned, the word with which such irrationals were designated, *karaṇī*, seems to have had quite a long and complex history, the scheme of which is still lacking despite some first attempts.[1] What exactly was

1 See for instance [Hayashi 1977; Datta & Singh 1993; Jain 1995: 371 ff.; Hayashi 1995: 60–64].

meant by this term in different Indian writings, from the first mathematical texts that we possess in the Indian subcontinent, the *Śulbasūtras* [Sen & Bag 1983], to the sixteenth-century commentaries of the *Bījagaṇita*, is difficult to pin down. In the first part of our paper, we shall present the use of *karaṇī*s in the first systematic text dealing with them, i.e., Bhāskara's seventh-century commentary (abbreviated from now on as *BAB*) of the mathematical chapter (*gaṇitapāda*) of the *Āryabhaṭīya*, traditionally believed to have been composed in 499.[2] While analyzing the first Indian source displaying computations with quadratic irrationals, we shall meet with the problem of determining the meaning of *karaṇī*. Restricting ourselves, here, to simply indicating the problem, we shall leave the discussion of the concept *karaṇī* for another publication.

In a second part, we shall turn to the Chinese sources: the Canon dating from the beginning of the common era, *The Nine Chapters on Mathematical Procedures*, and the third-century commentary on it by Liu Hui. We shall examine how they attest to the fact that quadratic irrationals were introduced in ancient China, and that computations were carried out with them.

We shall conclude by comparing these sources in this respect, discussing the reason why we believe they raise a problem of transmission, and by outlining a program for further research on this topic that, as we shall see, seems to indicate that the question of transmission may become much more complex.

## 1. Computations with Quadratic Irrationals in a Seventh-Century Indian Commentary and Another Contemporary Text

As explained above, we shall restrict ourselves here to examining computations with quadratic irrationals in Bhāskara's commentary of the mathematical chapter of the *Āryabhaṭīya* (*BAB*). We think it useful to start with the *BAB* because, being a commentary — the first extant of its kind —, it is especially profuse in a tradition where treatises tend to be as concise as possible. However, for the sake of the comparison we plan to outline in our conclusion, we shall also confront what is found in this commentary with an astronomical treatise of the same time: Brahmagupta's *Brāhmasphuṭasiddhānta*, written in 628 (abbreviated as *BSS*).[3]

We will consider below the computations in Bhāskara's commentary, in which numbers designated as *karaṇī*s are involved. Some of the computations with such numbers are described in general terms, others on the basis of numerical examples. Before looking closely at these computations, let us briefly examine the contexts in which quadratic irrationals are manipulated in this text.

*1.1 When are quadratic irrationals manipulated?* Quadratic irrationals appear in the commentaries of the geometrical verses of the mathematical chapter of the

2 We have used the only existing edition, [Shukla 1976]. The reader can find a translation of the text in [Keller 2000, II].

3 We have used Dvivedi's edition [1902].

*Āryabhaṭīya*.[4] In most of the cases, they emerge when a computation using the procedure corresponding to the "Pythagorean Theorem" produces a square whose root cannot be extracted. However, it is the length of the segment, and not its square that is needed to solve a problem. Thus quadratic irrationals appear in the computation of the area of trilaterals and the volume of an equilateral pyramid, when the "Pythagorean Theorem" is used to derive the heights of such figures. In a similar manner, quadratic irrationals emerge when computing the volume of a sphere. This computation, as we will see below, involves using the square root of the area of a great circle. Bhāskara does not dwell on the links that such quantities bear with the procedure to extract square roots.[5] Indeed, a procedure to extract square roots is given in verse 4 of the mathematical chapter of the *Āryabhaṭīya*.[6] Bhāskara calls the result of a root extraction *vargamūla* (literally, "the root of a square"). In contrast, if one needs the square root of a number $N$ that is not a perfect square, the quantity is called $N$ *karaṇīs*.[7] This expression refers to what we will call "the square root number of $N$" ($N$ being any positive rational): $\sqrt{N}$. When such quantities are manipulated, as will become clearer below, the value of $N$ is what is used in computations, but $\sqrt{N}$ is in fact the number considered. Once the expression *karaṇī* is introduced in Bhāskara's commentary, any number $N$ whose "square root number" is manipulated is said to be *karaṇī*-ized (*karaṇī-gata*).[8]

Bhāskara sometimes gives approximate values of the roots he cannot extract. But when this happens, they are never elaborated explicitly.[9] Thus, while deriving geometrically a table of sines in a base circle with radius $R$ by means of the "Pythagorean Theorem,"[10] he does not consider the quadratic irrationals that arise. Without stating it, he in fact produces approximate square roots.[11]

So Bhāskara extracts square roots, sometimes approximately. But he also manipulates numbers like $\sqrt{N}$, some of which may be quadratic irrationals.

4 Specifically in [*BAB* 2.6ab, 2.6cd, 2.7cd, 2.9ab, 2.10].

5 Twice only he makes off-hand remarks on the subject. We will examine them below.

6 For a description of this procedure, see [Shukla & Sarma 1976: 36–37; Keller 2000, II: 58–62].

7 It is difficult to decide if *karaṇī*, when used to qualify the state of a quantity, abstractly, should be used in the singular or plural voice. This is one of the problems we hope to tackle in our forthcoming article.

8 In fact the word *karaṇī* may take many different meanings. We do not intend to discuss here the problems that arise when examining the different significations that the word can take. We have analyzed the problem in [Keller 2000, I: 199–207], and hope to clarify it in a forthcoming article. We will, however, give an interpretation of the word before each quotation of a Sanskrit text where it is used.

9 Bhāskara occasionally gives approximate results explicitly, but not for square roots in this text. For instance in his commentary of verse 12 of the mathematical chapter of the *Āryabhaṭīya*, as noted by [Hayashi 1997: 401–402].

10 In [*BAB* 2.11]. See [Keller 2000, II: 156–184].

11 Note that one of the uses of the word *karaṇī* in the *Bakhshālī Manuscript* covers exactly this status: an approximate square-root derived when one cannot extract an exact square-root. The *Bakhshālī Manuscript* is a mathematical text roughly datable to the seventh or eighth century. It was edited and translated in [Hayashi 1995]. See pp. 148–149 where the author suggests the seventh century as a probable period of composition.

Although we will not dwell on this aspect of *karaṇī* in the present article, it will be useful for us to evoke here the geometrical interpretation that Bhāskara gives of the word. A *karaṇī*-operation, geometrically, is the construction of a square knowing one of its sides. To be more specific, Bhāskara explains that a *karaṇī*-operation is what "makes" the hypotenuse equal to the other sides. This is an etymological pun: the word *karaṇī* is derived from the verbal root *kṛ-* (to make). Thus, when directly associated with the "Pythagorean theorem," *karaṇī*s seem to represent both a numerical entity (a quadratic irrational itself representing a length) and a geometrical one (the operation producing the square of which a given side is known). This already sheds some light on the versatility of the word. However, as we will see below, *karaṇī*s representing the values of areas and volumes are manipulated as well. The use of quadratic irrationals seems therefore not to have been restricted to expressing the values of lengths. *Karaṇī*, although always manipulated in geometrical contexts, may have been gaining, in Bhāskara's commentary, an independent numerical status.

Thus, within this text, quadratic irrationals appear in a partially explicit relation to the procedure of square-root extraction. But, a specific quadratic irrational is used seemingly outside of this context, in relation to the circumference of a circle. That is the value $\sqrt{10}$. It is called ten *karaṇī*s (alternatively *daśa karaṇya* or in a compound *daśakaraṇī*). It is explicitly considered by our author as a bad approximation of the circumference of a circle having 1 for diameter.

In conclusion, quadratic irrationals called *karaṇī*s appear in a geometrical context. And specific arithmetical rules are derived in order to enable one to carry out computations with these quantities. This is what we will turn to now.

*1.2 Multiplication of Quadratic Irrationals.* Bhāskara gives rules on how to multiply quadratic irrationals with each other and on how to multiply and divide a quadratic irrational by an ordinary quantity. These quadratic irrationals appear indifferently as the square roots of integers, fractions, or integers increased by a fraction (fractionary numbers).

For instance, in his commentary on the first half of verse 6,[12] Bhāskara carries out a computation that may be formalized as follows:

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$
$$a\sqrt{b} = \sqrt{a^2} \cdot \sqrt{b} = \sqrt{a^2 \cdot b},$$

where $a$ and $b$ may be either integers or fractions.

He does so when computing the area of an equilateral triangle of side 7.[13] Its area is computed as half the base $(3+1/2)$ multiplied by its height. The height is evaluated with the "Pythagorean Theorem" and found to be equal to $\sqrt{36+3/4}$. Bhāskara first makes a "square root number" out of $3+1/2$, $\sqrt{12+1/4}$. He then

12 Unless stated otherwise, from now on, all the verses referred to belong to the mathematical chapter of the *Āryabhaṭīya*.

13 Geometrical examples seldom use measure units, a number is just associated with a segment of a figure.

multiplies $\sqrt{36+3/4}$ and $\sqrt{12+1/4}$ to obtain the area, $\sqrt{450+3/16}$. As we shall see later, he probably transformed these two first fractionary numbers (i.e., quantities of the form $a \pm b/c$) into fractions (i.e., quantities of the form $(ac + b) / c$) before multiplying them. The result was presumably obtained as the quotient of the numerator and the denominator. In the end, it is presented in the form of the square root of an integer increased by a fraction.[14]

Let us examine closely how the computation is expressed in the text. In the following sentences the word *karaṇī*, used in the plural form as a sort of measure unit for the number it modifies, should be understood as indicating that it is the square-root of the stated number that is considered:

*tena, buhjāvarge karṇavargācchuddhe śeṣaṃ samadalakoṭīvargaḥ*
*36*
*3*
*4,*
*samadalakoṭī karaṇyaḥ*
*36*
*3*
*4*
*iti* /

Therefore, when the square of the base is subtracted from the square of the hypotenuse, the remainder is the square of the height
36
3
4,
the height is
36
3 *karaṇīs* [Shukla 1976: 56].[15]
4

This excerpt presents, in fact, the very first time quadratic irrationals are introduced in Bhāskara's text. The commentator underlines how the numerical value of "the square of the height" (*samadalakoṭīvarga*) is equal to the value of the height expressed as a number of *karaṇīs* (*samadalakoṭī karaṇyaḥ*). The difference is in the perspective in which such a value is taken into account. The first time the square of the length is both the number and the value considered. The second time, the quantity considered is the square root of the previous square, even though it is expressed by the value of its square. In other words, $\sqrt{36+3/4}$ is the number taken into account. But because one cannot express its value, the quantity is referred to by its square, $36+3/4$. And this is indicated in the text by using the expression: $36+3/4$ *karaṇīs*.

14 These manipulations of fractions in Bhāskara's commentary are described in [Keller 2000: 155–160].

15 We would like to thank T. Hayashi and J. Bronkhorst for their suggestions concerning all translations of Sanskrit texts.

In order to compute the area of the figure, Bhāskara needs to homogenize all the quantities. He therefore computes the square of half the base, to express its value as a number of *karaṇīs*. Notice that an ordinary number thus "becomes a *karaṇī*" when it is squared. The text goes on:

> *bhujārdham api karaṇyaḥ*
> *12*
> *1 /*
> *4*
> *tena, karaṇyoḥ saṃvargo 'stīti labdhaṃ kṣetraphalaṃ "samadalakoṭībhujārdhasaṃvargaḥ" iti karaṇyaḥ*
> *450*
> *3 /*
> *16*
>
> Half the base also is
> 12
> 1 *karaṇīs*.
> 4
>
> Therefore, since there is a product for two *karaṇīs*, the area obtained (with the rule) "the product of half the base and the height" is:
> 450
> 3 *karaṇīs* [Shukla 1976: 56].
> 16

So it is only incidentally that Bhāskara indicates that one can only multiply two *karaṇīs* (and not a *karaṇī* and a non-*karaṇī*). In other words, we learn that when a computation involves one quadratic irrational, this implies that other quantities entering the computations have to be transformed into *karaṇīs*.

Bhāskara carries out a similar computation when evaluating the volume of a pyramid. In the second half of verse 6, the *Āryabhaṭīya* gives the following rule to compute the volume of an equilateral triangular based pyramid:

> *ūrdhvabhujātatsaṃvargārdhaṃ*
> *sa ghanaḥ ṣaḍaśrir iti //*
>
> Half the product of that (area of the isosceles trilateral which forms its base) with the upright side, that is (the volume of) a solid called "six-edged" [Shukla 1976: 58].[16]

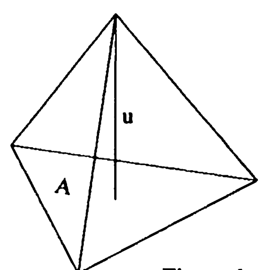



**Figure 1**

In other words, if $A$ is the area of the base, and $u$ the "upright-side" (cf. Figure 1), then the volume $V$ is given as:

$$V = (A \cdot u)/2 .$$

16 This rule is incorrect.

In Bhāskara's commentary of this verse, an example is given which underlines again how quantities are made into "square root numbers" in order to enter computations with quadratic irrationals. In the following example where the base of the pyramid has an area $\sqrt{3888}$ and a height of $\sqrt{96}$, this computation is carried out:

> *kṣetraphalaṃ [karaṇyaḥ] 3888 etāsāṃ kṣetraphalakaraṇīnām ūrdhvabhujākaraṇīnāṃ ca saṃvargārdhaṃ ghano bhavati / ardham ity atra karaṇitvād dvayoḥ karaṇībhiś caturbhir bhāgo hriyate / labdhaṃ ghanaphalaṃ karaṇyaḥ 93312 /*
>
> The area of the figure[17] is 3888 [*karaṇīs*]. Half the product of these *karaṇīs* of the area of the figure and of the *karaṇīs* of the upright side (previously obtained as 96 *karaṇīs*) is the volume. ⟨The rule uses the expression⟩ "half," but because two ⟨should be in⟩ *karaṇīs*, here, ⟨the product⟩ is divided by four *karaṇīs*. What is obtained is the volume 93312 *karaṇīs* [Shukla 1976: 59].[18]

Notice that the word *karaṇī* is used in this paragraph in the same way as we have described above, indicating, as some sort of a measure unit, a square-root number. In the third sentence, it seems to name a bit more than that. Indeed it refers to the *quality* that the quantity, here 2, should assume so that it can be manipulated with quadratic irrationals. Thus, two different uses of the word *karaṇī* appear here.

The computation carried out can be formalized as follows:

$$(\sqrt{a}\sqrt{u})/2 = \sqrt{(au)/4}\,.$$

Implicitly, to divide (a product of) *karaṇīs* by an ordinary number one squares the latter in order to make it into a *karaṇī*. The result of the division is then obtained as a *karaṇī*.

Bhāskara states in all its generality a rule to multiply a square root by its square when commenting on Āryabhaṭa's rule, in the second half of verse 7, to compute the volume of a sphere. Āryabhaṭa's rule runs as follows:

> *tan nijamūlena hataṃ ghanagolaphalaṃ niravaśeṣam //*
>
> That (i.e., the area of the circle) multiplied by its own root is the volume of a circular solid, without remainder [Shukla 1976: 61].[19]

Bhāskara supposes that the area of the diametrical section may not have an extractable root. In his commentary to this verse, he therefore writes the following sentence, where the word *karaṇī* means a square-root number in general:

> *tat punaḥ kṣetraphalaṃ mūlakriyamāṇaṃ karaṇitvaṃ pratipadyate*
>
> On the other hand that area becomes a *karaṇī* when being made into a root [Shukla 1976: 61].

17 The word *kṣetra* literally means "field," although in Bhāskara's commentary, as in other mathematical texts, it always refers to an abstract geometrical figure such as a triangle, a circle, etc.

18 Another interpretation and translation of this paragraph can be found in [Hayashi 1995: 61].

19 Once again, the rule is incorrect.

This is the only context where a quadratic irrational is clearly stated to appear when the extraction of a square root is undertaken. Implicitly, the commentator seems to consider that when the extraction is carried out an impossibility may arise. Bhāskara consequently decides to change the status of the area, thereafter considered as a *karaṇī*.[20] Note that the geometrical entity that becomes a *karaṇī* is not a segment, but an area. This modification of the status of the area induces a change in the procedure followed. Bhāskara indicates this in the following sentence. At the end of this sentence a verb is used, which was created from the word *karaṇī*. Its ambiguity, indicated by a slash (is squared/is made into a *karaṇi*), relates to the polysemy of the word *karaṇī*, to which we will return later.

> *tataḥ punar api karaṇīnām akaraṇībhiḥ saṃvargo nāstīti kṣetraphalaṃ karaṇyate* /
>
> Then, however, since there is no product of *karaṇīs* by non-*karaṇīs*, the area is squared/is made into a *karaṇī*. [Shukla 1976: 61]

Ultimately, this ambiguity exists because transforming an ordinary number into a *karaṇī* is, as we have noticed before, to square it.[21] Bhāskara indeed indicates in the next sentence that he understands the verse as follows:

> *evam ayam artho 'rthād avasīyate kṣetraphalavargaḥ kṣetraphalena guṇita iti* /
>
> Consequently, the following meaning is in fact understood: the square of the area of the figure is multiplied by the area of the figure [Shukla 1976: 61].

In other words, if $A$ is the area:

$$A\sqrt{A} = \sqrt{A^2 \cdot A}\,,$$

where $A$ may be an integer but also a fraction or a fractionary number.

For instance, in an example from the commentary on the latter half of verse 7, the area of the diametrical section is computed as $(3+177)/1250$. Bhāskara then adds a sentence where the irrationality of its square root seems to be stated. This is the second place where the appearance of quadratic irrationals is linked to a square-root extraction:

> *pūrvābhihitagaṇitakarmaṇā [dvi]viṣkambhakṣetrasya yat phalam āyātaṃ*
>
> *3*
> *177*
> *1250*
>
> *tasya mūlam etad eva karaṇīgatam aśuddhakṛtitvāt pratipattavyam* /
>
> With the previously told mathematical computation, the area of the figure having two for diameter is reached,
>
> 3
> 177
> 1250

20 The expression conveys square-rootness as a quality of the number: it is the *karaṇiness* (*karaṇitva*) of the area that is produced. But it also emphasizes the process, since this *karaṇiness* is produced (*pratipadyate*).

21 Bhāskara includes the word *karaṇī* in a list of synonyms of *varga*, square.

> This very (value), which has become a *karaṇī*, should (therefore) be considered to be its (square)-root, because it is an impure square [Shukla 1976: 62].

Here it is very clearly stated that $N$ should be considered as a *karaṇī*, because one cannot extract its root. Thus a *karaṇī* is characterized as a "square root number," in a case when the algorithm cannot yield the result. The irrationality of $\sqrt{N}$ is additionally spelled out as the fact that $N$ is "an impure square" (*aśuddhakṛti*). So we see how square-root numbers appear when one has to deal with quadratic irrationals of the type $\sqrt{3+177/1250}$.

Then, Bhāskara transforms the fractionary number $3+177/1250$ into what we call a fraction, $3927/1250$, before carrying out the procedure. This latter operation is called "to put into a same category" (*savarṇita*):

> *tac ca savarṇitaṃ jātam*
> *3927*
> *1250*
> *etat kṣetraphalavargeṇa guṇitaṃ jātaṃ ghanaphalaṃ karaṇyaḥ 31,*
> *karaṇībhāgāś ca*
> *12683983*
> *1953125000 /*
>
> And that is made into a same category (i.e., it is made into a fraction),
> what has been produced is
> 3927
> 1250.
> This is multiplied by the square of the area,
> what results is the volume 31 *karaṇīs* and
> 12683983
> 1953125000 parts of a *karaṇī* [Shukla 1976: 62].

Bhāskara deals with fractionary *karaṇīs* as he deals with ordinary fractions. He transforms the result of a computation which has the form of a numerator on a denominator into a fractionary number, i.e., as an integer increased by a fraction smaller then one. Again, the word *karaṇī* escorts these values as some kind of measure unit.

*1.3 A Rule of Three with Quadratic Irrationals.* Bhāskara also applies other common arithmetical operations such as the Rule of Three to quadratic irrationals. In the *BAB*, a standard verbal formulation (*vāco yukti*) is used when the Rule of Three is to be applied.[22] If the ratio of $A$ to $B$ is the same as the ratio of $C$ to $D$, the verbal formulation of the Rule of Three is expressed in the following way:

> If with $A$, $B$ is obtained, with $C$, what is obtained? $D$ is obtained.

This formulation corresponds to a computation, namely, $D=(B\cdot C)/A$.

22 For a general presentation and analysis of the Rule of Three in this text see [Keller 2000, I.1.5.b: 75–78; I.2.4.b: 164–166; II. *BAB* 2.26–2.27cd: 303–324].

Bhāskara uses such a rule when computing the hypotenuse or "ear" (*karṇa*) of a triangle in an equilateral triangular based pyramid in a case when some of the quantities involved are quadratic irrationals. In the following example, two similar triangles are considered ($B'HC$ and $BB'C$, as represented in Figure 2). Bhāskara states that $BB' : CB = CB' : CH$ by formulating the question which defines a Rule of Three. In this case, $BB'$ is a quadratic irrational, $\sqrt{108}$, whereas both $B'C$ and $BC$ are integers (respectively 6 and 12). They are, however, "made into" *karaṇīs*,[23] to be compared to the length of the first segment:

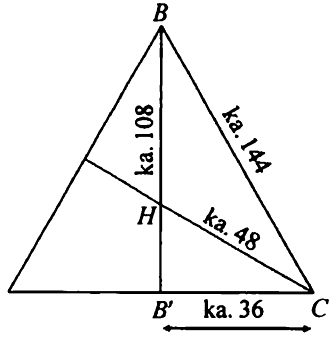



**Figure 2**

> *karaṇam-yady aṣṭottaraśatakaraṇikena [avalambakena] catuścatvāriṃśaduttaraśatakaraṇikaḥ karṇo labhyate, tadā ṣaṭtriṃśatkaraṇikenāvalambakena kiyān karṇa iti* /
>
> If an hypotenuse which is the *karaṇīs* of a hundred increased by forty-four is obtained with a perpendicular whose *karaṇīs* are a hundred increased by eight, then with a perpendicular whose *karaṇīs* are thirty-six, how much is the hypotenuse obtained? [Shukla 1976: 59]

The value of $CH$ is deduced from this:

$$CH = (CB \cdot CB')/BB'.$$

The result obtained is a quadratic irrational. Bhāskara only states:

> *labdho 'ntaḥkarṇaḥ [karaṇyaḥ] 48*
>
> The interior hypotenuse ($CH$) obtained is 48 *karaṇīs* [Shukla 1976: 59].

This example is important because it shows that Bhāskara holds that one can let quadratic irrationals express ratios, under the condition that all quantities are expressed as "square root numbers."

*1.4 Adding Quadratic Irrationals.* In a circumstance where he meets with the problem of adding quadratic irrationals, Bhāskara also gives a rule to add *karaṇīs*. This rule can be used in cases when the numbers considered are quadratic irrationals, but a division by 10 provides two numbers for which a square root extraction is possible. However, as Bhāskara underlines it in his commentary, this procedure does not always work.

23 Numbers are expressed in *bahuvrīhi* compounds ending with the irregularly formed (because one would expect a long "i") *karaṇika*. These compounds mean: the number whose *karaṇīs* are/whose *karaṇī* is …

This rule is given in a *prākṛta* verse in Bhāskara's commentary to verse 10, for which K. S. Shukla proposes a Sanskrit translation:

> *auvaṭṭi a dassakeṇa i mūlasamāsas samotthavat /*
> *auvaṭṭaṇāya guṇiyaṃ karaṇisamāsaṃ tu ṇāavvam //*
> *[apavartya ca daśakena hi mūlasamāsaḥ samotthaṃ yat /*
> *apavartanāṅkaguṇitaṃ karaṇisamāsaṃ tu jñatavyam //]*
>
> When one has reduced (the squares of the two to be summed) by ten, then, the sum of the roots (of the results is taken). That which arises from the same (sum; i.e., the sum is squared) is /
> multiplied by the digits of the reducer (i.e., ten), (the result is a *karaṇī*, in this way) the sum of (two) *karaṇīs* should be known // [Shukla 1976: 74][24]

Note that its expression is general. It can be formalized as follows:

$$\sqrt{a}+\sqrt{b}=\sqrt{10[\sqrt{a/10}+\sqrt{b/10}]^2},$$

where $a/10$ and $b/10$ are perfect squares.

Thereafter, Bhāskara gives an example where this rule may apply but where it also meets with its limits, which Bhāskara then spells out [Shukla 1976: 73–74]. He considers a set of figures inside a circle as represented in Figure 3, where a segment formed by a chord and the arc it subtends is called a "bow-figure" (*dhanuḥkṣetra*). In the following paragraph, we see the commentator trying to add the areas of each of the figures, all of which were obtained as quadratic irrationals. The numbers are preceded in Sanskrit by the abbreviated form of the word *karaṇī*, *ka*.

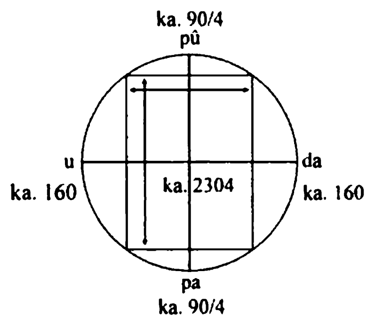



**Figure 3**

> *anayā gāthayā pūrvāparadhanuḥkṣetraphale*
> *ka. 90*
> *4*
> *ka. 90*
> *4*
> *ete kṣetraphale karaṇiprakṣepavidhānena prakṣeptavye / (...) tathā kṛtvā labdhaṃ ka. 90 / dakṣiṇottaradhanuṣor api tathaiva phale ka. 160, ka. 160 / [samāsaśca] ka. 640 / samas tayoḥ punaḥ samāsaḥ ka. 1210 / madhyasthāyatacaturaśra-kṣetraphalaṃ karaṇyaḥ 2304 / dhanuḥkṣetraphalasamāsarāśer asya ca karaṇī-samāsakriyayā samasyamāne rāśyor asaṃkṣepatā /*

24 This verse is difficult to understand. The formula we propose conforms both with our reading of the Sanskrit translation and the computations that Bhāskara carries out as we explain below. Another interpretation, given by A. N. Singh, is described in [Shukla 1972].

> The two areas of the eastern and western bow-figures are, according to this verse (*gāthā*),
>
> 90 *ka.*
> 4
> 90 *ka.*
> 4
>
> These two areas should be added using a method to sum *karaṇīs*. (...) Once it is done, what has been obtained is 90 *ka*. And then for the southern and northern bows, in exactly the same way, the areas are 160 *ka.*, 160 *ka*. [And the sum] is 640 *ka*. The sum of the two sums is indeed 1210 *ka*. The area of the rectangular figure standing in the middle is 2304 *karaṇīs*.
> When considering the sum of the two quantities, the quantity which is the sum of the areas of the bow figures and this (i.e., the area of the rectangle), with the ⟨previous⟩ procedure for summing *karaṇīs*, there is no addition [Shukla 1976: 74].

Whereas 90/4, 90, 160, 640, and 1210 when divided by ten all produce perfect squares, this is not so for 2304. The commentator cannot add all the areas of the figures in order to obtain the area of the circle as one single "square root number." As is shown here, this rule to add *karaṇīs* does not always provide results. Notice that in the last sentence *karaṇīs* are referred to as quantities (*rāśi*). This point will prove very important.

To summarize, we have so far observed that quadratic irrationals could have the form of the square root of an integer, the square root of a fraction or of an integer increased by a fraction. We have seen that when quadratic irrationals appear, then other quantities that have to be manipulated with them are, so to say, put under the square root or "made into *karaṇīs*."

We have remarked above that the product of two "square root numbers" was a "square root number." This is not stated in all its generality by our commentator, who notes the existence of such a product and produces a result that is a "square root number." Similarly, one can divide "square root numbers" by ordinary numbers, under the condition that these ordinary numbers be transformed into "square root numbers." The result of the division is then a "square root number." Bhāskara also knows of a method that sometimes can provide a sum of "square root numbers." Finally, he can manipulate ratios involving quadratic irrationals under the condition that all the quantities of the ratios become "square root numbers."

All of this, together with the above reference to quadratic irrationals as *rāśis*, shows us that *karaṇīs* are conceived of as a special kind of quantities.

*1.5 Quadratic Irrationals and Approximations.* In order to elucidate how Bhāskara conceives of such quantities, let us look closely at what he tells us both about the ratio of the circumference of a circle to its diameter, and about the value of 10 *karaṇīs* that is involved in its expression. Verse 10 of the mathematical chapter of the *Āryabhaṭīya* associates a diameter measuring 20 000 units with an approximate circumference, 62 832. Āryabhaṭa explicitly states that the circumference is an approximation, he calls it: *āsanno vṛttopariṇāho*. And Bhāskara, commenting on this part of the verse, raises a question:

(Question)
*athāsannaparidhiḥ kasmād ucyate, na punaḥ sphuṭaparidhir evocyate?*
Now why is the approximate circumference mentioned and not indeed the exact circumference?
(Answer)
*evaṃ manyante — sa upāya eva nāsti yena sūkṣmaparidhir ānīyate* /
They believe the following: there is no such method by which the exact circumference is computed [Shukla 1976: 72].

This answer given by Bhāskara may very well be his way of expressing the irrationality of the ratio of the diameter of a circle to its circumference. Note that this ratio is approximated by what can be seen as a non-reduced fraction (62 832 / 20 000). The plural form "they" is probably a reference to elder teachers, a tradition which Bhāskara claims to belong to.

The dialogue continues. It develops as a series of objections to Bhāskara's answers, objections that Bhāskara in turn attempts to refute. The ten *karaṇīs* are thus introduced as an objection to the approximate result presented by Āryabhaṭa. They are presented as the exact circumference of a circle whose diameter is one. Bhāskara first asks for a proof (*upapatti*) of this assertion. And the one who gives the objection answers as follows:

(Objection)
*atha manyate pratyakṣeṇaiva pramīyamāṇa rūpaviṣkambhakṣetrasya paridhiḥ daśakaraṇya iti* /
Now, some think that the circumference of (a circle) having one for diameter when measured directly, is ten *karaṇīs*.
(Answer)
*na etat, aparibhāṣitapramāṇatvāt karaṇīnām* /
This is not so because *karaṇīs* do not have a statable size [Shukla 1976: 72].

This answer seems to evoke very precisely the irrationality of $\sqrt{10}$. We can note that its status as being un-measurable is not related to the impossibility of stating an exact circumference. It seems, as a matter of fact, that these two problems are distinguished.

An example from this same part of Bhāskara's commentary seems to draw a parallel between two different kinds of irrationality, that of the ratio of the circumference of a circle to its diameter on the one hand, and the sides of a rectangle to its irrational diagonal. The relation here springs from the value $\sqrt{10}$. In any case this is within a dialogue, and the objection does not satisfy Bhāskara:

(Objection)
*ekatrivistārāyāmāyatacaturaśrakṣetrakarṇena daśakaraṇikenaiva tadviṣkambhaparidhir veṣṭyamāṇaḥ sa tatpramāṇo bhavatīti*

The circumference ⟨of the figure⟩ with that ⟨unity⟩ diameter, when enclosed by the diagonal, which measures precisely ten *karaṇīs*, of a rectangular figure whose width and length are respectively one and three, that ⟨circumference⟩ has that size (i.e., it measures ten *karaṇīs*) [Shukla 1976: 72].

These excerpts show that it was debated among mathematicians of Bhāskara's time, whether both the ratio of the circumference of a circle to its diameter on the one hand, and certain irrational values such as $\sqrt{10}$ on the other hand, could be expressed exactly or not.

*1.6 Rules of Computation in Brahmagupta's Work.* Brahmagupta's treatise is contemporary with Bhāskara's commentary. By examining briefly the computation on *karaṇīs* that he proposes, we would like to underline the continuity of the kind of work that was done in India on quadratic irrationals. However, despite elements of similarity which seem to indicate that seventh-century authors, as well as later Indian mathematicians, inherit from a common tradition of conception and manipulation of the *karaṇīs*, differences between them may betray the fact that the work on such procedures still went on for centuries and kept deriving new procedures. The continuity of this tradition will be of a striking contrast with what can be seen of the history of the manipulation of quadratic irrationals in China.

It is noteworthy that Brahmagupta, composing a treatise, systematically provides a set of rules for arithmetical computations with *karaṇīs*. Bhāskara, on the other hand, writing a commentary, only sporadically provides such rules when he needs them. Consequently, some of the differences that we will emphasize below may very well have their origin in the different natures of the two texts at hand. In the following we will not list all the similarities and differences between Brahmagupta's manipulation of quadratic irrationals and Bhāskara's, but just underline some striking features that are useful for us here. We will not discuss the interpretations given for Brahmagupta's verses. These rest upon those proposed by Datta, Singh and later K. S. Shukla, as can be seen in [Datta & Singh 1993], and those proposed by T. Hayashi [1977].

As in Bhāskara's commentary, quadratic irrationals are called *karaṇīs*. They are contrasted with the result of a square-root extraction that bears the name *pada*. Brahmagupta thus gives a rule to compute the sum or the difference of quadratic irrationals. In this specific case, if $a$ is not a perfect square, $a$ is called *karaṇī*. The expression thus does not refer to $\sqrt{a}$, but to the argument of the square-root which is implicit here:

> *iṣṭoddhṛtakaraṇīpadayutikṛtīṣṭaguṇitā 'ntarakṛtir vā* /
>
> The square of the sum of the square roots (*pada*) of the *karaṇīs* divided by a desired ⟨quantity⟩ is multiplied by that desired ⟨quantity, this gives the sum⟩, and the square of the difference ⟨of the square roots of the quotients being so treated will give the difference⟩ [*BSS*, viii, 38/39 ab].

This may be formalized as follows:

One would like to compute $\sqrt{a} \pm \sqrt{b}$. Let $a$ and $b$ be two non-perfect squares (*karaṇīs*); $c$ is the "desired quantity" (*iṣṭa*). First both $a/c$, $b/c$ are considered ("the *karaṇīs* divided by a desired ⟨quantity⟩"). These are probably meant to be perfect squares, since their square-roots (*pada*) are extracted. Then "the square of the sum of the square roots (*pada*)" is computed, that is: $[\sqrt{a/c} \pm \sqrt{b/c}]^2$. This last quantity is "multiplied by that desired (quantity)": $c[\sqrt{a/c} \pm \sqrt{b/c}]^2$.

The last quarter of the verse, "and the square of the difference ⟨of the square roots of the quotients being so treated will give the difference⟩," would concern the case where the difference of two *karaṇīs* is considered and not the sum.

Implicitly, it seems, operations are here carried out on the argument of the square root rather than on the square root itself. This is the usage in Bhāskara's text as well, when he considers the sum of *karaṇīs*. Therefore, the procedure described by Brahmagupta could be rendered by the following formula:

$$\sqrt{a} \pm \sqrt{b} = \sqrt{c[\sqrt{a/c} \pm \sqrt{b/c}]^2}\ .$$

The rule given by Brahmagupta appears as more general than the one that we have read in Bhāskara's commentary.

In contrast to Bhāskara, Brahmagupta gives also a rule to compute the product of such expressions, where the quantities are *karaṇīs*. This rule is different from those that were given by Bhāskara:

> *guṇyas tiryag adho 'dho guṇakasamas tadguṇaḥ sahitaḥ* //
>
> ⟨Put down⟩ the multiplicand horizontally below itself as many times as there are terms in the multiplier; and their products are added together [*BSS*, viii, 38/39 cd].[25]

This may be formalized as follows:

$$\sqrt{a}(\sqrt{c} + \sqrt{d}) = \sqrt{ac} + \sqrt{ad}.$$

Furthermore, Brahmagupta gives a rule "to divide" such expressions with *karaṇīs*:

> *sveṣṭarṇacchedaguṇau bhājyacchedau pṛthag yujāv asakṛt* /
> *chedaikagatahṛto vā bhājyo* //
>
> Multiply the dividend and divisor separately by the divisor after making an optional term of it negative, then add up the terms. Repeat (the same until the divisor is reduced to a single term). Then divide the (modified) dividend by the divisor reduced to a single term [*BSS*, viii, 39/40].

This may be understood as follows:

$$\begin{aligned}(\sqrt{a} + \sqrt{b})/(\sqrt{c} + \sqrt{d}) &= (\sqrt{a} + \sqrt{b})(\sqrt{c} - \sqrt{d})/[(\sqrt{c} + \sqrt{d})(\sqrt{c} - \sqrt{d})] \\ &= (\sqrt{a} + \sqrt{b})(\sqrt{c} - \sqrt{d})/(c-d)\end{aligned}$$

[Datta & Singh 1993: 256]. Such is the conclusion that might be drawn, when examining the arithmetical rules given by Brahmagupta in the *BSS*. Some, for adding *karaṇīs*, are more general than the ones given by Bhāskara. Others, as far as we know, appear there for the first time.

*1.7 Conclusion.* Let us summarize what we have seen in Bhāskara's text. Quadratic irrationals always appear in a geometrical context. They are referred to as the

25 We therefore disagree with the claim made by K. S. Shukla in his introduction [1976: xxii], where he writes that Bhāskara "knew" a rule to multiply the sum of "square root numbers."

non-extractable square roots of integers, fractions or fractionary numbers. They are called *karaṇīs*. When other quantities have to be manipulated with quadratic irrationals, they in turn are made into *karaṇīs*, that is into "square root numbers." This allows *karaṇīs* to enter a certain number of computations, such as a multiplication, a division, or a sum. Other rules pertaining to these computations were given by another seventh-century author, Brahmagupta. Therefore, quadratic irrationals and more generally "square root numbers" seem to be conceived as special kinds of quantities.

We have noted several uses of the word *karaṇī*, which put together may seem puzzling, and are certainly problematic. We list them here:

- *karaṇī* in the nominative plural modifies numbers as **some sort of measure unit**. This indicates that the value appearing in the expression is the square of the quantity referred to (e.g., it is a "square root number").
- *karaṇī* in compounds and as the root of verbs or derived nouns can name in all their generality "**square root numbers**." This expression is used when indicating that there is a product of two square-root numbers, or in several other situations that we analyzed. As such, it can also designate **quadratic irrationals**.
- *karaṇī* can designate **the square of a number**.
- the *karaṇī* operation, related to the "Pythagorean Theorem," is the **construction of a square from a given side**.

We will briefly come back to this constellation of meanings, when comparing this with the use of *dynamis* in Greek sources. Let us for now turn to one of the oldest Chinese sources available to show that the same type of quantities are introduced and involved in similar computations.

## 2. Computations with Quadratic Irrationals in Ancient China

Quadratic irrationals were introduced in the oldest Chinese mathematical treatise that has been handed down by the written tradition, *The Nine Chapters on Mathematical Procedures* (*Jiuzhang suanshu* 九章算術), below abbreviated to *The Nine Chapters*.[26] A compilation of knowledge available in China roughly up to the beginning of the common era, this book was composed in the first century either

26 The first paper to have put forward this interpretation is [Volkov 1985]. See also [Li Jimin 1990; Chemla 1992]. We refer the reader to Qian's [1963] edition of *The Nine Chapters*. Since 1984, K. Chemla has been working with Guo Shuchun (Academia Sinica, Beijing) on a new critical edition and a French translation of *The Nine Chapters*, including the commentaries on the treatise until the seventh century — a book soon to appear. In 1984, a mathematical text entitled *Book on mathematical procedures*, *Suanshushu* 算數書, was found in a tomb at Zhangjiashan in Hubei province; several editions of this text have recently been published. However, at first sight, it does not seem to introduce such quantities, even though there is an algorithm for root extraction at the end of which an interest appears in restoring the original number by applying an inverse operation.

before or after the common era and was soon to be considered a classic. This is how it was conceived of by its first commentator, Liu Hui 劉徽. His commentary, completed in A.D. 263, was handed down together with the text itself and is now part of all the extant editions.

### *2.1 The Context for Introducing, and Computing with, Irrationals.*

*The Result of a Root Extraction.* In Chapter 4, "Small width" (*Shaoguang* 少廣), of *The Nine Chapters*, an algorithm for the extraction of the square root of an integer $N$ is described [Qian 1963: 150]. The algorithm makes use of the representation of $N$ with a decimal-place value system. The square root of $N$ is thereby determined digit by digit, and in case the number is not exhausted when the digit of the units has been determined, the algorithm of *The Nine Chapters* prescribes:

> If by extraction, (the number) is not used up, this means that one *cannot* extract (its root). You *must* then call it (the number) with "side (*mian* 面)" [our emphasis].

Until recently, historians did not understand the exact meaning of this sentence. It was only in the last decades that an interpretation could be established for it. It is now understood as follows: in such cases, *The Nine Chapters* states the algorithm *cannot* provide the result, since the root *cannot* be extracted, and indicates that the result must be given in another way, in the form of "side of $N$" ($N$ *zhi mian*, $N$之面), or, $\sqrt{N}$. Hence, from the beginning of the common era, one finds quadratic irrationals introduced in ancient China in a way similar to what is found in our Indian sources, that is, as the results of square root extraction, in cases where the algorithm cannot exhaust the number whose root is sought. More precisely, the root of a number $N$ is obtained either as the result of the algorithm described by *The Nine Chapters*, or, when the algorithm does not exhaust the number, as a quantity "side of $N$."

*The Nine Chapters* does not mention such numbers elsewhere, which accounts for the fact that it took so long before such an interpretation of this sentence was put forward. It was only by relying on Liu Hui's commentary that one could find elements that led to this understanding. Liu Hui both describes computations with such numbers and comments on their nature and the reasons for introducing them.

*The Sphere and the Cube.* Let us first examine the computations to which he submits numbers of the type "side of $N$," so as to compare them with what we found in Indian texts, especially in Bhāskara's commentary discussed above. In the third-century commentary on the "extraction of the spherical root" (an algorithm to determine the diameter of a sphere whose volume is given), Liu Hui deals with the relationship between the sphere and the inscribed and circumscribed cubes. It is in this context that he presents computations that will require the introduction of such numbers. We can hence observe how they were manipulated in ancient China [Qian 1963: 156]. We shall follow Liu Hui step by step in this commentary and analyze the computations involved. In the first part of the passage we are interested in, Liu Hui considers the relationship between the sides and the areas of one face of, respectively, the inscribed and the circumscribed cubes (see Figure 4). He puts forward an algorithm which manifests the relationship between the sides:

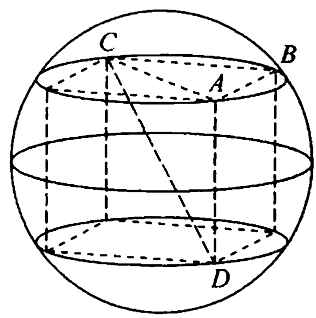



Figure 4

> Let the diameter of the ball (i.e., the sphere, [Trans.]) be multiplied by itself, divide by 3, extract the (square) root of this, hence the side of the cube (*lifang* 立方[27]) inscribed within the ball [Qian 1963: 156].

Next, Liu Hui sets out to establish the relationship between the areas that underlies the algorithm quoted. Hence, after an argument that we shall examine below, he expresses the proportion between areas corresponding to the algorithm just stated:

> This is why the square (*mi* 冪) that the side of the inscribed cube multiplied by itself makes is to the square (*mi*) that the diameter of the ball multiplied by itself makes as one third [Qian 1963: 156].

Note that the expression of the ratio between the two areas is given in the terms of a fraction: 1/3 (*san fen zhi yi* 三分之一), which is usual in Liu Hui's commentary. As is regularly the case, Liu Hui's reasoning to reach this conclusion is expressed with respect to a particular example:[28]

> Suppose the side of the cube inscribed within the ball is 5 *chi* 尺. One takes 5 *chi* as the base (of a right-angled triangle, *gou* 句).[29] The square (*mi*) that the base multiplied by itself makes is 25 *chi*.[30] If one doubles this, one gets 50 *chi*, which is taken as the square (*mi*) of the hypotenuse (*AC*), which is called the hypote-

27 We must stress that the word used to designate the side of a cube in this sentence differs from the word used in the expression "*side of N*." However, the word *mian* is sometimes used elsewhere in the commentary to refer to the side of a polygon, as well as the side of a cube, or even the face of a cube or of another polyhedron (see especially Liu Hui's commentary on the algorithm for cube root extraction [Qian 1963: 153–154]). As becomes clear below, Liu Hui knows that the side of the circumscribed cube is equal to the diameter of the sphere.

28 On the way in which examples are used in both *The Nine Chapters* and in Liu Hui's commentary, the reader is referred to [Chemla 1997a]. Geometrical examples often involve dimensions of the order of magnitude of the *chi*, and use integral values appropriate for the problem considered.

29 Naming the side of the inscribed cube as the "base" of a right-angled triangle is the canonical way in which ancient Chinese sources introduce the right-angled triangle *ABC*; see Figure 4. A similar practice may be seen in Bhāskara's text [Keller 2000, I: 228–230]. No figure is mentioned in this passage of the Chinese text. Let us stress, however, the difference between these two modes of designating triangles, the ancient Chinese one and ours: the Chinese text may refer to any face of the cube, whereas our drawing specifies one particular face. For the convenience of the reader, we shall insert between brackets the names of the segments with respect to the figure.

30 No distinction is made between units for measuring lines or surfaces. The passage from Plato's *Theaetetus* mentioned below manifests the same use.

> nuse corresponding to the side of 5 *chi* of the square that is the flat base (*pingmianfang* 平面方).[31] If one takes this hypotenuse as height (of a right-angled triangle, *gu* 股),[32] if one also takes 5 *chi* as base (of the right-angled triangle, *gou*, *AD*), adding the squares (*mi*) of the base and height, one obtains 75 *chi*, which makes the square (*mi*) of the great hypotenuse (*CD*). Extract the root of this, then one can know the great hypotenuse. The great hypotenuse is the longest diagonal of the inscribed cube, and this diagonal gives the diameter of the ball.

If the side of the inscribed cube, *AB*, is 5 *chi*, applying twice the right-angled triangle procedure (*gougushu* 句股術, with which the name of Pythagoras is nowadays commonly associated), Liu Hui determines that the *square* of its largest diagonal, *CD*, is 75 *chi*. *CD* is also the diameter of the sphere and, hence, the side of the circumscribed cube. The area of one face of the inscribed cube is known: it is 25 *chi*. The area of one face of the circumscribed cube is also known: its value was just determined as 75 *chi*. Liu Hui can thus conclude, "This is why the square (*mi*) that the side of the inscribed cube multiplied by itself makes is to the square (*mi*) that the diameter of the ball multiplied by itself makes as one third."

This concludes the part of our passage devoted to comparing the areas of the faces. Note that the hypotenuse *CD* was said to be known as the result of the root extraction applied to 75 *chi*, without the result being made explicit at this point. When, thereafter, Liu Hui turns to examining the relationship between the volumes of the two cubes, he will express the result explicitly.

Note also that the conclusion concerning the relationship between the *areas* of the faces of the two cubes ends a development that aimed at establishing the correctness of an algorithm, stated at the beginning of the passage and relating to the *sides* of the cubes:

> Let the diameter of the ball be multiplied by itself, divide by 3, extract the (square) root of this, hence the side of the cube (*lifang* 立方) inscribed within the ball.

In this case, Liu Hui avoids directly stating a ratio between the sides that would require quadratic irrationals and expresses, instead, a ratio between their squares. However, later on, it will appear that there is no such restriction holding in general.

31 The expression *pingmianfang* can be understood as "the side of the square (*fang* 方) of the flat base (*mian* 面)," in which "base" refers to a face of the cube. On this passage, we have two editorial remarks: first, all ancient sources agree in having "the square (*mi*) of the height" instead of "the square (*mi*) of the hypotenuse" above. This may well be a way of indicating that the next function of the same segment is going to be the "height" of another triangle (see below). If this were the case, there would be no need to change the text as it is given by all the ancient sources. Second, the expression "which is called the hypotenuse corresponding to the square of the base the side of which is 5 *chi*" is unusual from a syntactic point of view. One would rather expect "which is called the square (*mi*) of the hypotenuse corresponding to the square of the base the side of which is 5 *chi*." A few characters below, all ancient editions agree in having a superfluous *mi*, which may well be a copyist's displacement of the character that is missing here.

32 Again, the triangle *ACD* is introduced by designating a segment as its height.

Regularly, Liu Hui's commentary meshes finding the ratio between areas or volumes with establishing the correctness of algorithms evaluating geometrical magnitudes, or, in particular, accounting for the coefficients occurring in the statement of the algorithms. This is probably what is at stake in considering the ratio between the volumes of the inscribed and the circumscribed cubes, a problem to which Liu Hui turns next.

*2.2 Multiplying with Quadratic Irrationals.* In the following sentences, Liu Hui determines the volume of the circumscribed cube. He first prescribes the algorithm to be used:

> If one makes the great hypotenuse further multiply its square, this gives the volume of the cube circumscribed to the ball.

He then sets out to compute the actual result within the framework of his particular example, and this is where we first encounter computing with quadratic irrationals. He writes:

> Since, by extracting (the root of) the square (*mi* 冪) of the great hypotenuse, one does not use it up ...

One recognizes the formulation by which *The Nine Chapters* started the conclusive prescription of the algorithm for square root extraction. One must then turn to the other way of obtaining the result. Here, Liu Hui goes on:

> Making its square multiply twice itself and naming (the result) with "side," one gets, for the volume of the circumscribed cube, the side of 421875 *chi*.[33]

Namely, having to multiply $\sqrt{75}$ by 75, Liu Hui computes $75^3$, and gives the result as $\sqrt{75^3}$, that is $\sqrt{421875}$ or "side of 421 875." Symbolically, one may represent the procedure as follows:

$$A \cdot \sqrt{A} = \sqrt{A^3}.$$

Several points are worth noting here. First, the fact that the value of the *volume* of the circumscribed cube is expressed as "side of 421 875 *chi*" shows that "side of *N*" is *not restricted to representing a length*. Rather, it expresses a quantity the square of which is *N*. And, in case the interpretation of the result of a sequence of operations shows that *N* corresponds to the square of the value sought, then this value can be found by applying a root extraction algorithm to *N* and may therefore be expressed as "side of *N*," irrespective of the actual object it refers to. It is not the number *N as such* that may or may not be qualified as a square; it is the object, the value of which it represents, which may be shown to be the square of some other magnitude. The qualification of a magnitude as the square of another expresses in this case, with full generality, an *operational* relation between the two. Such quantities as "side of *N*" do not pertain to a given geometrical meaning. In other words, the operation of taking the square root appears to be conceived of,

33 Note that again the same word designates the unit for the volumes.

*not* as determining the length of the side of a given square, but as the inverse of the operation of squaring, and this "operational meaning" is conveyed to such quantities as "side of $N$"; they are qualified as the quantities the square of which is $N$. More on this point below.

Another point worth stressing is that, once quantities such as "side of $N$" have been introduced, they enter into computations — here in a multiplication — as, and with, any other quantities. And new procedures are hence described to perform the arithmetical computations with such numbers. These procedures appear subject to a constraint: the result of an operation must be given in the form "side of $N$" and cannot satisfactorily be expressed, as we would do, as the product of a rational by a quadratic irrational. Now the quantities such as "side of $N$" are not only produced as results of square root extractions, but also as results of other algorithms bearing on them.

It is also worth pointing out that all these facts equally hold for Bhāskara's text, where, as was shown above, the same procedure is described to carry out a similar multiplication involving similar quantities ($A$ and $\sqrt{A}$ ).

*2.3 Quadratic Irrationals as Terms of Ratios — Simplification, Rule of Three.* Let us return to the part of Liu Hui's commentary that we were reading. Our commentator next turns to computing the volume of the inscribed cube. His aim is to find the ratio between the two volumes. He writes:

> If, in addition to this, we make the multiplication of 5 *chi*, the side of the inscribed cube, by itself and again multiply this (the result) by the side of the cube, we obtain for the volume 125 *chi*.

As he needs to compare this volume to the one obtained previously as "side of 421 875 *chi*," he next transforms this result to put it in the same form:

> Multiplying 125 *chi* by itself and naming (the result) with side, one obtains, for the volume, the side of 15625 *chi*.

$$5^3 chi = 125\, chi = \sqrt{125^2\, chi} = \sqrt{15625\, chi}$$

or, symbolically,

$$a^3 = \sqrt{(a^3)^2} \quad \text{or} \quad A = \sqrt{A^2}.$$

The same remarks as those outlined above hold here too. The property of the expression "side of $N$" only to designate a quantity whose square is $N$, irrespective of what $N$ measures, appears here with even greater clarity. Moreover, one must stress that when it comes to using numbers and quantities of the type "side of $N$" in the context of the same operation, a new use of that expression is introduced: it not only stands for the root of numbers whose root could not be obtained as an integer, but it can also express integers in a new way, as "sides of their squares." This happens every time that a ratio is considered, one term of which is of the type "side of $N$," and this is also the case when performing computations where quadratic irrationals occur. This is indeed what happens next in the passage, where the ratio between the volumes of the inscribed and the circumscribed cubes is examined:

> One simplifies both by 625: the volume of the circumscribed cube is the side of 675 *chi* when the volume of the inscribed cube is the side of 25 *chi*.

In other words, the ratio between the two volumes was shown to be the one between $\sqrt{75^3 chi}$ and $\sqrt{5^6 chi}$. Liu simplifies both arguments by $625 = 5^4$ and obtains the ratio between $\sqrt{675\ chi}$ and $\sqrt{25\ chi}$. Symbolically, the ratio between $\sqrt{Ab}$ and $\sqrt{Bb}$ is given as equal to the one between $\sqrt{A}$ and $\sqrt{B}$.

This calls for two remarks. First, if we consider the computations involved, we can see that another arithmetical operation is applied to quadratic irrationals, namely, a simplification between them when they occur as terms of a ratio. And the way in which this operation is carried out requires some attention: confronted with a situation when one term of the ratio has the form $\sqrt{Aa^3}$, whereas the other term could have been given the form $Ca$, Liu Hui does not simplify the argument of the former by $a^2$, whereas he simplifies the latter by $a$. He rather puts all terms in the form "side of $N$" and simplifies the arguments in parallel. Such an algorithm therefore extends to cases when the number by which one simplifies is not a square, as is done later on in the passage (see below). Moreover, Liu Hui states the operation as a simplification by $a^2$, and not by "side of $a^2$." He thus operates directly on the argument of the square root.

This first remark relates to the second remark: we see that such quantities as "side of $N$" can thus express a ratio between entities. However, they do so under the condition that all terms entering the ratio have the same form, and this is where the possibility of expressing integers as "sides of their squares" enters into play. All numbers are transformed as quantities of the type "side of $N$" to operate, and the operations also produce quantities in the same form. This constraint on the expression of ratios will appear to hold throughout the text. It evokes what we previously described in Bhāskara's text.

Usually, in *The Nine Chapters* or in the commentaries, the numbers expressing the ratio between two entities are put into play in two related ways. They permit the formulation of an algorithm to compute one of the terms of the ratio, when an algorithm to compute the other term is known. We already met with an example of this fact above: a proportion between areas made it possible to establish an algorithm to produce the side of the inscribed cube, when the side of the circumscribed cube, i.e., the diameter of the sphere, was known. In the second case above, we could therefore think of an algorithm similar to this one, to determine the volume of the inscribed cube, since we know how to compute the volume of the circumscribed cube with the help of the diameter.

But the numbers expressing the ratio between two entities also permit — which is another way of expressing the same fact — the use of a Rule of Three to determine the volume of one of the cubes, when the volume of the other is known. Incidentally, this highlights how there is here no loss of generality when Liu Hui develops his reasoning on the basis of a particular example. Having ratios expressed with terms of the type "side of $N$" would thus imply involving quadratic irrationals in a Rule of Three, which is interesting in two ways. First, the quantities such as "side of $N$" would enter in yet another arithmetical operation: the Rule of Three, which would be appropriately adapted for dealing with them. Secondly, we would meet yet again with one of the algorithms which Bhāskara applied to *karaṇīs*. However, as we shall now see, this is not as clearly developed here as it is in the following sentences of our passage.

Before we set out to analyze this further, let us stress a last feature of the former computation: it may come as a surprise that Liu Hui does not simplify as much as he could have done: the ratio between $\sqrt{675\ chi}$ and $\sqrt{25\ chi}$ could have been given as the one between $\sqrt{27\ chi}$ and $\sqrt{1\ chi}$. Again we may have a hint of the reason why he does so in the topic to which he turns next.

*2.4 Reporting on Zhang Heng's use of Quadratic Irrationals.* The following passage from the commentary we are reading is important for the additional testimony it provides on the use of such numbers in ancient China, since Liu Hui reports thereafter on the use Zhang Heng 張衡 made of such quantities as "side of $N$" for dealing also, within a cosmological framework, with the relationship between the cube and the sphere.[34] Zhang Heng (78–142), whose name is mainly known in connection with astronomy, constitutes yet another example of a scholar in ancient China manipulating quantities like "side of $N$" between the time of the composition of *The Nine Chapters* and Liu Hui's time.

Zhang Heng, Liu Hui reports, stated that the circumscribed cube corresponds to the "side of 64" when the sphere inscribed to it corresponds to the "side of 25." These figures may seem surprising — we meet again with integers expressed as the "side of a square" — but they are highlighted, if we follow Liu Hui, by the fact that Zhang Heng also gives the "square" as corresponding to the "side of 8" whereas the "circle" corresponds to the "side of 5."

Let us first see how Liu Hui understands the latter statement, before we go back to Zhang Heng's treatment of the sphere and the cube, and to the relation between the two pairs of values. Liu Hui interprets the latter values as referring to some *lü* 率. Here we must momentarily interrupt our explanations and shed some light on the concept of *lü*. *The Nine Chapters* introduced this concept in the framework of the Rule of Three, or "procedure of suppose" as it is literally called in the Canon. There, *lü* qualifies the two explicit values that express how the two

34 The name of Zhang Heng is associated with an astronomical theory, *huntian* 渾天, "spherical sky": in opposition to those who held that the earth was a flat square, above which the sky was like a canopy (a theory named "sky as a canopy," *gaitian* 蓋天), the followers of *huntian* held that the earth was a sphere inside the sphere constituted by the sky. The former attributed the figure of the circle to the sky — the shape as well as the number 3 attached to the circumference of a circle of diameter 1 — whereas they attached to the earth the shape of the square, as well as the value of its perimeter (4, when dealing with the square circumscribed around the previous circle). On this, see *The Mathematical Classic of the Gnomon of the Zhou* (*Zhoubi suanjing* 周髀算經, ca. first century B.C.E or C.E.) [Qian 1963: 13]. One may therefore guess a relationship between the discussion we analyze here about spheres and cubes and the cosmological problem of determining shapes and figures attached to the earth and sky, according to the *huntian* theory.

types of things dealt with in a Rule of Three might be converted one into the other. The first example offered in *The Nine Chapters* concerns the conversion of grain: if unhusked grain corresponds to 50, when the coarsely husked grain corresponds to 30, both values suffice to compute the amount of husked grain obtained from any given amount of unhusked grain: these numbers are designated as *lü*. In other words, *lü* refers to a particular example of equivalence between the two kinds of things involved, and the Rule of Three enables us to extend this particular example to convert any quantity of one of the two things into the corresponding quantity of the other. As becomes clear later, the term *lü* connotes the property of these two particular values, possibly to be multiplied or divided by the same number, without the meaning of the pair, as expressing a relationship between two given entities, being changed. More generally, *lü* qualifies the elements of any set of numbers in as much as these numbers present the same ratios between them as the things they represent: the *lü* hence constitute what we shall call a "microcosm," reproducing at a given size the relations between a set of magnitudes in a given situation under scrutiny. They are thus defined up to a constant, and the Rule of Three appears to permit the connection of any two pairs of such values. *Lü*, as qualifying a type of number in a set of such numbers, and the Rule of Three, as a procedure ruling their changes as a set, are thus intimately linked.

We are now in a position to understand better Liu Hui's statement that "if one follows Zhang Heng's procedure, the *lü* of the perimeter of the square is the 'side of 8,' whereas the *lü* of the circumference of the circle is the 'side of 5'." In other words, this is simply a way of giving the ratio of the perimeter of the square to the circumference of the inscribed circle as the ratio between "side of 8" and "side of 5." Let us notice that, here, the numbers we are interested in, the quantities of the type "side of *N*," can therefore express the values of *lü* and, in correlation with this, enter Rules of Three: this confirms what we deduced above from the fact that they could be used as terms of ratios.

This concept of *lü*, which was until this point qualifying integers or rational numbers, is thus extended to qualify our quantities, the changes of which will follow their own patterns, as it appears from the following sentences in which Liu Hui deduces, from this, other ratios linked to the circle and the circumscribed square. He goes on:

> If we suppose that the perimeter of the square is the "side of 64 *chi*," then the circumference of the circle is the "side of 40 *chi*."

As we just recalled, the term *lü* connotes the property that the quantities it qualifies can be both multiplied or divided by the same number without the meaning of the pair being altered. This property is put into play here when the values of the *lü* are of the type "side of *N*." Accordingly, the following new algorithm is used for this first example of changing the values of such a pair of *lü*:

> If the *lü* of the perimeter of the square is the "side of 8" when the *lü* of the circumference of the circle is the "side of 5," then the *lü* of the perimeter of the square is the "side of 8 times 8" when the *lü* of the circumference of the circle is the "side of 5 times 8."

Or, symbolically:

If *lü* of $A$ is $\sqrt{a}$ when *lü* of $B$ is $\sqrt{b}$, then *lü* of $A$ is $\sqrt{a \cdot c}$ when *lü* of $B$ is $\sqrt{b \cdot c}$.

The transformation observed here is inverse to the one we described above, for simplifying ratios. Note that here $c$, by which the arguments are multiplied, is not a square. Note also that, as we saw for the terms of a ratio, when one of the *lü* is of the type "side of $N$," the corresponding *lü*'s are also all given in the same form. This accounts for the fact that "side of 64" is not changed into 8. This constraint is even more explicit when Liu Hui goes on:

> If, furthermore, one multiplies the diameter of 2 *chi* by itself, one gets, for the diameter, the side of 4 *chi*. This corresponds to the fact that the *lü* of the circumference of the circle is the "side of 10" when the *lü* of the diameter is the "side of 1."

Several facts are worth stressing here. Firstly, one notes how sets of *lü* are constituted: a pair of values was given for the *lü* of the perimeter of the square and the circumference of the circle. These values determine a given size for the situation under consideration (a circle inscribed within a square). In this example, these values are, respectively, "side of 64 *chi*" and "side of 40 *chi*." Now Liu Hui explores the microcosm constituted by all the geometrical elements of this situation at this given size. The values of these other geometrical elements in this situation constitute the values of their *lü* corresponding to the first two *lü*. He thus considers that the value of the diameter, in this very situation, can be taken as *lü* of the diameter in what now forms an extended set of *lü*'s. Liu Hui can then turn his attention to the pair of the *lü* of the circumference and diameter, and transform it by dividing both values by the same quantity. We see here the relationship between introducing the concept of *lü* and using particular examples in mathematical practice.

Secondly, to get from the pair of *lü* worth the "side of 4" and the "side of 40" for respectively the diameter and the circumference of the circle, to the pair "side of 1," "side of 10," Liu Hui applied the transformation inverse to the previous one, or symbolically:

If *lü* of $A$ is $\sqrt{a \cdot c}$ when *lü* of $B$ is $\sqrt{b \cdot c}$, then *lü* of $A$ is $\sqrt{a}$ when *lü* of $B$ is $\sqrt{b}$.

We have already met with this kind of reduction above, and this is why we admitted that what Liu Hui did amounted to introducing such quantities as "side of $N$" into the arithmetical procedure of the Rule of Three.

Thirdly, Zhang Heng's value for what we call $\pi$ is $\sqrt{10}$. As Liu Hui reports: "Zhang Heng also considered that the *lü* of 3 for the circumference and 1 for the diameter (i.e., those adopted by *The Nine Chapters* [Trans.]) were wrong. He thus rewrote this method. But he increased the circumference too much, this exceeds its actual value." We shall see below the possible importance of this remark.

Finally, we confirm that, whenever one of the *lü* is of the type "side of $N$," the *lü* given together with it must be of the same form. We have already seen four examples of this. Three of these occur in the passage relating Zhang Heng's values.

One of these cases, where the *lü* for the diameter is given as side of 1, when the *lü* of the circumference is given as side of 10, is most interesting for the way in which it expresses the unity. Above, the same phenomenon occurred when considering the ratio between the volumes of the inscribed and the circumscribed cubes as expressed by, respectively, $\sqrt{675\ chi}$ and $\sqrt{25\ chi}$.

This brings us back to the volumes, and to how Liu Hui restores the reasoning behind Zhang Heng's value for the ratio of the volume of the sphere to that of the circumscribed cube as that of $\sqrt{25}$ to $\sqrt{64}$. Indeed, Liu Hui points out that the structure of these two data is the same as the structure of the data underlying the algorithm provided by *The Nine Chapters*. This algorithm for extracting the "spherical root" amounted to stating that the ratio of the sphere to the circumscribed cube equaled 9/16. Given that in *The Nine Chapters* the ratio of the area of the circle to that of the circumscribed square is given as 3/4, and given also that the ratio of the volume of a cylinder to that of the circumscribed parallelepiped is given as equal to the ratio between the areas of the circle and the circumscribed square, Liu Hui reconstructs in the following way how the algorithm of *The Nine Chapters* was obtained. First, the cylinder inscribed in the cube was computed as having a volume equal to 3/4 of the volume of the cube. Secondly, if a cylinder is inscribed transversally within the same cube, the solid which is created as the intersection of both cylinders has, at each level, the same relation to the sphere as the area of the square has to the area of the circle. Hence the volume of the sphere is 3/4 the volume of this solid, whereas the first cylinder was itself 3/4 the volume of the original cube.

If the sphere is given as occupying 9/16 of the volume of the circumscribed cube, Liu Hui concludes, this means that, within the framework of the hypothesis for $\pi$, the ratio of the volume of the sphere to the first cylinder was taken as equivalent to its ratio to the solid introduced as the intersection of the two cylinders. After having recovered in this way the reasoning underlying the algorithm of *The Nine Chapters*, Liu Hui stresses that it is wrong, since it mistakenly identifies the first cylinder with the intersection of the two cylinders.

Zhang Heng's values for the ratio of the sphere to the circumscribed cube have the same structure, in that $\sqrt{25}$ and $\sqrt{64}$ are also both squares, and in fact the squares of the values given for expressing the ratio between the areas of the circle and the square.[35] Liu Hui comments, "Hence we know that he again takes the circular cylinder as represented by the *lü* of the square and the sphere by the *lü* of the circle. In that the error is important."

35 Liu Hui shows that the perimeters of the circle and the circumscribed square have the same ratio as their areas, since he shows that both areas are computed as the product of half the diameter by half the circumference. "Hence, he says, the perimeter of the square is taken as the *lü* of the area of the square, whereas the circumference of the circle is taken as the *lü* of the area of the circle." It is interesting to note here that the values of the *lü* are expressed by a given magnitude, and not by an actual numerical figure. This implies that the algorithms producing the two magnitudes of the areas were compared, and simplified, with respect to each other, not by a particular number, but by a given magnitude (one fourth of the diameter). This is how one could express the *lü* of these two magnitudes by two other magnitudes.

We can therefore surmise that Liu Hui's interpretation is that Zhang Heng applied to the quantities "side of $N$" the same reasoning in the following way:
The volume of a cylinder is to the circumscribed cube as "side of 5" to "side of 8."
The volume of the sphere is again taken to have the same ratio with respect to the volume of a cylinder as "side of 5" to "side of 8" (the *lü* of the circle to the *lü* of the square).
Consequently, the volume of the cube is to that of the sphere as

"(side of 8)$^2$" to "(side of 5)$^2$",

which is

"side of (8)$^2$" to "side of (5)$^2$".

This highlights how such quantities as "side of $N$" enter computations with ratios. It is interesting to note that the ratio is expressed in this way, as a ratio between quantities of the type "side of $N$," instead of being simplified and given as the ratio of 8 to 5. This recalls what we repeatedly stressed above concerning sets of *lü*: as such, all these quantities ("side of 5," "side of 8," "side of 25," "side of 64") belong to the same set of *lü* and must thus be given in the same form.

*2.5 Approximations.* There is a last aspect worth inquiring into in this passage: how Liu Hui evaluates the error affecting the results obtained by Zhang Heng's formula, which, he says, "is important." This matters to us here, since it seems to involve a reasoning for comparing the result of a computation involving quantities such as "side of $N$," with the result of an algorithm applied to fractional quantities.

To explain this hypothesis, we need to come back to how, earlier in his commentary, Liu Hui commented on the algorithm given in *The Nine Chapters* to compute the volume of the sphere. He wrote, "Since, when one takes, as *lü* of the circle, 3 for the circumference and 1 for the diameter, the area of the circle is wrong by default and since, if we suppose that the circular cylinder is represented by the *lü* 率 of the square, then the volume of the ball is wrong by excess, they compensate each other; this is why the *lü* of 9 and 16 are by chance pretty close to reality. However, the ball is still wrong by excess."

By comparison, the error in the values obtained by Zhang Heng's algorithm is said to be significant. Two facts stressed by Liu Hui may have constituted the backbone of his analysis of why this was so. First, as shown above, the reasoning underlying Zhang Heng's algorithm amounts to taking the ratio of the sphere to the cylinder as that of the circle to the square to establish the algorithm, which is in error by excess. But, in addition to this, as Liu Hui stressed, the value for the circumference which corresponds to taking $\pi = \sqrt{10}$ also involves an error by excess. Hence, in contrast to the case of the algorithm in *The Nine Chapters*, the two errors now do not compensate, but compound each other. In fact, Liu Hui describes a computation, the aim of which he does not make explicit. We suggest that this computation may aim at highlighting how important the error is. Let us describe this computation.

Liu Hui starts from the values of "side of 675 *chi*" for the volume of the circumscribed cube and "side of 25 *chi*" for the volume of the inscribed cube. At the beginning of his report of what Zhang Heng did, Liu Hui seems to attribute to him the following remark: extracting both square roots, it would suffice to add an additional unit to 675, and "side of 676 *chi*" would yield 26, whereas "side of 25 *chi*" would immediately yield 5. Hence 26 is introduced as an integral value *by excess* for the volume of the circumscribed cube; and the fact that Liu Hui did not simplify these values above, as we noticed, might be explained by his intention to, later, use the value of 26, or to account for Zhang Heng's use of 26. In fact now — and this is where the computation we are interested in starts — Liu Hui sets out to compute what the procedure of *The Nine Chapters* — a procedure which, we saw, he knows to produce a value for the sphere by excess — would give if it were applied to a value of 26 for the circumscribed cube. This cannot but give a value for the sphere that exceeds the actual value. This is what Liu Hui concludes, without adding any comment. However, if one compares this value to what Zhang Heng's procedure would yield, if applied to a volume of "side of 675," one can see that $(5\sqrt{675})/8$ is even greater.[36] With this in mind we suggest interpreting the computation carried out by Liu Hui, when reporting Zhang Heng's views, as highlighting the importance of the error affecting Zhang Heng's way of computing the sphere.

Several facts led us to the hypothesis that the computation aims at showing how coarse Zhang Heng's formula is. The main reason is that this is the only way in which one can account for the presence of this computation in this context. But, in fact, another element supports this hypothesis: if Liu Hui had simplified the pair of quantities expressing the values of the volumes of the inscribed and the circumscribed cubes, and given them as being worth, respectively, "side of 27" and "side of 1," the smallest integral approximation by excess for "side of 27" would have been 6. However, in this case, the value computed with the algorithm of *The Nine Chapters* would not have been smaller than the value obtained by applying Zhang Heng's algorithm to "side of 27." Hence we may interpret Liu Hui's adoption of the value "side of 675," as well as the development of the computation we described above, as two indications converging towards the same conclusion: Liu Hui is developing an argument to assess the magnitude of error, i.e., the inexactness of the computation of the volume of the sphere using Zhang Heng's algorithm. In any case, the whole of this excerpt shows clearly the interplay between exegesis and mathematical practice in Liu Hui's commentary.

If the interpretation above is accepted, we then have a comparison between the results of two algorithms, one involving quantities of the type "side of *N*," and the other producing a fractional quantity. We shall meet below with yet another example of such a comparison in Liu Hui's commentaries.

Such is what we can know from the commentaries on *The Nine Chapters*, as to the manipulations of such quadratic irrationals. In addition to this, Liu Hui's

36 This is how we write the result of Zhang Heng's algorithm applied to the quantity "side of 675." Zhang Heng would probably have obtained it as the square root of $675 \cdot 25/64$, or, if we apply what the algorithm for square root extraction prescribes, the square root of 16 875, divided by 8. One may suppose that the results of the two algorithms were compared through their squares.

commentary sheds some light on how our third-century commentator conceived of such quantities and the reasons why they had to be introduced, which we shall now examine.

*2.6 Conceptions and Uses of the Quantities "Side of N."* Let us first examine an assertion in which Liu Hui comments simultaneously on two kinds of irrationals. We read in the commentary on Problem 11 of Chapter 9:

> Suppose the base (*gou* 句) and the height (*gu* 股, of a right-angled triangle) are each 5, the square of the hypotenuse is 50. If one extracts the square root of this, one gets 7 *chi*, and there is 1 left, hence (the number) is not exhausted. Suppose the hypotenuse is 10, its square is 100, halving it would make the two squares of the base or the height, obtaining for each 50, for which one must also not be able to extract (*dang yi bu ke kai* 當亦不可開). This is why one says the circumference is 3 when the diameter is 1, the side of the square is 5 when the diagonal is 7: even though one cannot exactly (*zheng* 正) manage to exhaust their inner constitution (*li* 理), one can still express an approximation to it, but that is all [Qian 1963: 247].

This passage, which we shall call (1), is interesting on several accounts. We shall only stress two aspects of it.

First, it is interesting that Liu Hui compares two cases where we know there is irrationality, even though of a different nature: the diagonal with respect to the side of the square, and the circumference with respect to the diameter of the circle. The fact that he considers the two situations, simultaneously *and* from the point of view of how to express these quantities, seems to indicate that, beyond the differences between them, he views the two situations in this regard as having a similar nature.

Secondly, the nature of this similarity is of the utmost importance for us. It is striking that, here, from noting that the number 50 is not exhausted by the root extraction algorithm, that one is not able to extract its root in the sense that the algorithm cannot yield it, the commentator does not draw the conclusion that the result should be given in the form "side of *N*." The two passages can be interpreted in a coherent way only if we entail that expressing the result as "side of *N*" cannot lead to "exactly exhausting the inner constitution (*li*)" of the diagonal and the side of the square. There is reason to believe that "exhausting the inner constitution" of the diameter and the circumference on the one hand, and of the side and the diagonal of the square on the other hand, is meant to convey a process whereby these magnitudes are cut into equal or proportional parts, in such a way that integers (*lü* 率) can express their relationship.[37] This interpretation is supported by the whole sentence that ends the quotation above: instead of "exactly exhausting the inner constitution," one may be able to "express it approximately," and this "ex-

37 On the interpretation, in early Chinese mathematical texts, of this word *li* 理 from classical Chinese philosophical vocabulary, see the glossary by K. Chemla (to appear). One may be tempted to interpret here *zheng* 正 "exactly" as "with integers," but this would require evidence to support it that has yet to be found.

pression" refers to the pairs of integers that can be given as *lü* for the magnitudes. Expressing approximately the "inner constitution" of two magnitudes with respect to each other is equivalent to providing a pair of integers. In the case of the side and the diagonal of the square, instead of giving "side of 50" as the value of the diagonal when the side equals 5 — or "side of 25" —, or else as the value of the side when the diagonal is worth 10 — or "side of 100" —, the inner constitution is expressed approximately by attaching integers to *both* the side and the diagonal. If this interpretation holds true, then the commentator states here that in both cases, one *cannot* express both magnitudes by *lü* which would be integers and would have the same ratio as the magnitudes themselves, and that *the only thing one can do is to express their inner constitution with integers approximately.*

In the two commentaries where he deals with, respectively, the algorithm to compute the area of the circle and the algorithm for extracting square roots, Liu Hui makes assertions concerning the nature of the quantities involved here that nicely complement the passage (1).

Let us first recall the case of the area of the circle [Qian 1963: 102–104].[38] The algorithm given in *The Nine Chapters* to compute the area of the circle prescribes multiplying "half the diameter by half the circumference of the circle." After having accounted for the correctness of the algorithm as such, Liu Hui addresses the following question: the algorithm that is proved to be correct is the one involving the diameter and the circumference of the circle *as such*, and *not* the algorithm as applied to the problems in the context of which *The Nine Chapters* spells it out and which provide values for the circumference and the diameter of the circle having a ratio of 3 to 1. Here is how Liu Hui states it, in a passage we designate as (2):

> Here, by circumference and diameter we designate (*wei* 謂) the values (*shu* 數) that attain what is so (*zhi ran* 至然), what the *lü* of 3 for circumference and 1 for diameter are not.

Hence, making use of passages (1) and (2) together, we deduce that, according to Liu Hui, there *are* "values," "quantities" (*shu*) for both the diameter and the circumference that "attain what is so," and these "values" are precisely those put into play by the algorithm shown to be correct. However, one *cannot express* them simultaneously *but in an approximate way*, and this is why *The Nine Chapters* provides values that do not conform to the real objects. Moreover, after the passage (2), Liu Hui works out a pair of *lü* for the diameter and the circumference, i.e., a pair of integers that give a better approximation of the ratio, or of the "inner constitution" of the diameter and the circumference.

Secondly, when Liu Hui comments on the introduction of quadratic irrationals in *The Nine Chapters*, he also makes clear-cut assertions about their nature. In the first place, for cases when the number $N$ is not exhausted by the square root algorithm, he considers fractional approximations, one of which he finds to be always

38 For a more detailed discussion, see [Chemla 1996a]. It is interesting to note that it is in the framework of this commentary that Liu Hui considers the relationship between the circle and inscribed as well as circumscribed squares.

smaller $(a + (A - a^2) / 2a + 1)$, and the other always larger $(a + (A - a^2) / 2a)$, than the root. Thereafter, in what we refer to as passage (3), Liu Hui states:

> One *cannot determine its value* (*shu*, i.e., the value, the quantity of the root). Therefore, *it is only when* "one names it with 'side'" that one does not make any mistake (or, that there is no error) (emphasis ours) [Qian 1963: 150].[39]

Again, if we consider passages (1) and (3) together, we deduce that, in this case, the root being sought has a "value," a "quantity" (*shu*), even if it cannot be expressed in a way that "exhausts the inner constitution" of the magnitude considered with respect to unity. Note that our interpretation of the impossibility of "exactly" "exhausting the inner constitution" above fits with Liu Hui's discarding of fractional quantities as a possible result in such cases here. Moreover, the only solution for stating the value in an exact way is to introduce a way of naming it, as "side of *N*." However, one can also express the inner constitution of the magnitude with respect to unity approximately, by a pair of integers.

Hence the two cases compared by Liu Hui above, i.e., the circumference and the diameter of the circle, and the side and the diagonal of the square, are different. In the former case, one simply designates the two magnitudes by their names to refer to their value; in the latter, one introduces a new type of quantity to express their value. At the same time, the former ones are not taken as terms of arithmetical operations, whereas the latter ones are. It is all the more interesting then that Liu Hui stressed that both situations fall under the category of *shu*, i.e., "value," "quantity," and they have in common the fact that their "inner constitution" cannot be "exactly exhausted," or can only be "expressed approximately." This helps to make clearer the statement in Liu Hui's commentary that comes closest to an assertion of the irrational character of some ratios between magnitudes.

It is important to stress both the similarity and the difference between the two kinds of value in order to understand another comparison that Liu Hui develops between quadratic irrationals and fractions. If the side and the diagonal of the square were compared to the values attached to a circle as regards the impossibility of exhausting their inner constitution, quadratic irrationals will be compared to fractions in another respect. This relates to the double line of discussion of Liu Hui's commentary on the introduction of quadratic irrationals. Passage (3) quoted above reflects the part of the commentary devoted to analyzing the *nature* of the result. However, Liu Hui is also interested, and this is the second line of his commentary, in accounting for the introduction of this new type of quantity: why does *The Nine Chapters* prescribe giving the result in this form, when the algorithm

39 For more details, see [Chemla 1997/98]. After having commented on the last sentence of the algorithm given in *The Nine Chapters*, Liu Hui also suggests that one can go on computing digits beyond the units: using the same algorithm, the first digit computed should be associated with the denominator 10, the two first digits with the denominator 100, and so on, he states. The values of 5 and 7, given above for, respectively, the side and the diagonal of the square correspond to this latter computation: the square root of 2 is computed as 1 + 4/10, or else 7/5.

fails to produce the root? We can read passage (4) of Liu Hui's commentary as addressing this question:

> Every time that one extracts the root of a number to make the side of a square, the multiplication of this side by itself *must restore* (this number) [Qian 1963: 150].[40]

It is from *this* point of view, namely, that the inverse operation applied to the result must restore the number with which one started, that Liu Hui accounts for the necessity of discarding fractional results, when a square root algorithm fails to produce the root, and that he draws a comparison with fractions, as results of division: "This is analogous to, when one divides 10 by 3, taking the remainder equal to $1/3$, therefore one is again able to restore its value." One may be tempted to conclude that Liu Hui does not distinguish between the types of quantity $\sqrt{2}$ and $1/3$, but we have already seen that this would be wrong. Here, in order not to misunderstand the sentence, one must take care not to confuse the discussion concerning the *nature* of the result introduced — the only one relating to the irrational character of the magnitudes —and the *analysis of the reasons for introducing an exact result*, which raises another subject of concern. As shown in passage (1), concerning the possibility of exhausting the "inner constitution," Liu Hui makes a statement about the side and the diagonal, the diameter and the circumference, but does not include any fraction (numerator and denominator) among the examples. This is where he deals with the *nature* of the quantities. It is only when it relates to accounting for the introduction of such quantities as "side of *N*" that Liu Hui compares quadratic irrationals — and *not* the diameter and the circumference of the circle — with fractions.[41] His interest in explaining *why* one needs to introduce quantities as "side of *N*" must be treated separately from his analysis of *what* these quantities are, once they are introduced. To understand the meaning of this comparison between quadratic irrationals and fractions, one must reconstruct the reason why it is important, in Liu Hui's view, to obtain results for divisions and square root extractions that allow the restoration of the numbers from which one started. This can be done by following the use of the operation of "restoring" throughout the commentary. On the basis of such an analysis, which we cannot repeat here,[42] one can conclude that the property of the inverse operation to restore the number from which one started is essentially put into play in a certain kind of proof establishing the correctness of the algorithms. This kind of proof, an "algebraic proof within an algorithmic context," is carried out by transforming algorithms, i.e., rewriting them as lists of operations. The operations it makes use of in this context — deleting, grouping or inverting operations that are inverse to each

40 Our emphasis. The term used (*ji* 積) means that this number has been produced by a multiplication, which is the case for an area or a volume. Moreover, the term used below for "square" can also mean "cube." Hence the sentence may have a meaning that exceeds the context of the square root extraction within which it is stated.

41 As noted by a referee, there is again a "similarity here with Sanskrit mathematical typology of operations": division and root extraction both have their "restorative inverses," in contrast to the relationship between circumference and diameter of the circle.

42 For the details of the analysis, see [Chemla 1997/98].

other — are valid only because the results of these operations, given as fractions or quadratic irrationals, when necessary, are exact. Hence the importance Liu Hui grants to this property of exactness shared by fractions and quadratic irrationals.

A remark can shed some light on the relation between the introduction of quadratic irrationals and proving the correctness of algorithms. Even though the algorithm for cube root extraction offered in the same Chapter 4 of *The Nine Chapters* is, in every respect, described in parallel with the algorithm for square root extraction, it fails to reproduce one of the latter's key aspects: it does not include, at the end of the algorithm, in case the extraction algorithm fails to produce the root, a statement similar to the one introducing quadratic irrationals.[43] This may relate to the fact that, except for the "extraction of spherical root," no other algorithm in *The Nine Chapters* involves a cube root extraction, and, as a consequence, this operation does not appear in establishing the correctness of the algorithms contained in *The Nine Chapters*.

In conclusion, quadratic irrationals introduced in *The Nine Chapters* are at the center of two different subjects of interest. On the one hand, they enter in a comparison with the pair of values constituted by the diameter and the circumference of the circle, which confronts both situations from the point of view of the shared impossibility of "exhausting their inner constitution." This is the form used to assert irrationality in ancient China. On the other hand, quadratic irrationals are also considered with respect to their property of guaranteeing that the squaring of the result of a square root extraction restores the original number. And this operational property of theirs is not only the foundation for how one carries out arithmetical operations with them, but is also the perspective from which Liu Hui compares them to the other type of quantity introduced by *The Nine Chapters*, namely, the fractions. The first comparison addresses the nature of the quantity, and the impossibility of expressing ratios between magnitudes as ratios of integers. The second comparison addresses the way of carrying out operations with such values as well as the way of operating on algorithms as such, thanks to them.

## 3. Comparisons and Conclusions

Let us now summarize what we have established thus far, in order to confront the Chinese sources that we examined with what we have learned of Bhāskara's conception and manipulation of quadratic irrationals. Several striking similarities appear.

In both *The Nine Chapters* and Bhāskara's commentary, unless quadratic irrationals are produced as results of arithmetical operations applied to quantities of the same type, they are introduced as results of square root extraction that could not be carried out. In the latter case, they are thus given in similar ways in both ancient China and India: a way of naming them is devised that designates them indirectly, through their squares (*BAB* 2.7.cd). However, it is interesting — and

43 Since there is no part of Liu Hui's commentary discussing this possibility of introducing cubic irrationals, it seems that this does not relate to a philological problem.

this seems to hold true for all ancient traditions — that no similar phenomenon has as yet been found associated with cube root extractions.

Moreover, though their formulations differ, both Liu Hui and Bhāskara stress the impossibility of "stating" (*BAB* 2.10) or "determining" (Liu Hui in the commentary on square root extraction) quantities of the kind of quadratic irrationals.

Furthermore, in both contexts, the quadratic irrationals introduced are conceived of as being "quantities" (*shu* 數 in China, *rāśi* in India), like integers or fractions, and they can be given as values for lines (both straight or curved), areas, or even volumes. They are thus not attached to any particular geometrical meaning.

As quantities, they enter into arithmetical operations, and adequate new procedures are given to carry out these operations under these new circumstances. We find in Liu Hui's and Bhāskara's commentaries the same operations extended to quadratic irrationals: multiplication of one by its square, multiplication of two such quantities, Rules of Three.

These new procedures all carry out the extension by putting into practice the procedures that would have been applied to the squares. They are also determined by the fact that, in both contexts, the result must be given in the same and unique form: as the square root of a quantity, and never, as we often do, as the product of an integer or a fraction by a quadratic irrational.[44] Thus there is a community of practice. Another similarity in the practices with such quantities in both contexts consists in having all numbers expressed as square root numbers when one of them is of this type. This can be seen in the Rules of Three in both corpora where all the terms entering the rule are given as "side of $N$" in China, or *karaṇīs* in India: this implies that integers be transformed into the square roots of their squares, which is the case in both contexts.

In addition to this, the mathematical subjects for the treatment of which such quantities are introduced are also similar. In both ancient China and India, quadratic irrationals adhere to geometry. The main object which leads to introducing such quantities is the right-angled triangle, because of the application of algorithms linked to what is today usually called the "Pythagorean theorem." However, in both China and India, such quantities present themselves when dealing with solids and with the circle as well.

Last but not least, as far as we can tell from the surviving sources, the texts in ancient China and India that introduce, and compute with, quantities such as quadratic irrationals as they appear above *are the very ones* that also discuss the values of $\sqrt{10}$ and 1 for the ratio of the circumference of the circle to its diameter.[45] We

44 This seems to be what Brahmagupta refers to as an "abridgment"; see [Hayashi 1977: 52].

45 This correlation is not as obvious as it may seem. One may find expressions of such a ratio for the circumference and the diameter that do not make use of quadratic irrationals. Qin Jiushao 秦九韶 in the 13th century, as well as al-Khwārizmī in the ninth century (see below), both express this very ratio as a relation between the squares of the circumference and the diameter. Moreover, this correlation may extend to texts from India older than those examined in this paper. As stressed by a referee, we know that the *Śulbasūtras*, where we find the first references to *karaṇīs*, and the early-first millennium Jain texts, also use "square root of 10" in relation to the circle.

saw that Zhang Heng was reported to have introduced this pair of values, which Liu Hui criticized in his discussion of the sphere. On the other hand, it is within the context of his criticism of these values that Bhāskara mentions them (*BAB* 2.10). What is even more striking is that, in both commentaries, these values are criticized, whereas the *same alternative ratio* is supported: the commentary on the measure of the circle attributed to Liu Hui establishes the values of 3927 and 1250 as *lü*, respectively, for the circumference and the diameter [Qian 1963: 106].[46] These correspond exactly, up to a multiplicative factor, to the ratio offered by the *Āryabhaṭīya* and supported by Bhāskara, which gives the circumference as being 62 832/20 000 of the diameter [Shukla 1976: 72 ff.].[47]

In conclusion, the earliest Chinese and Indian texts that have come down to us and that introduce, and deal with, quadratic irrationals share a cluster of similar features that is relatively substantial. Many of these similarities are all the more interesting and striking because they are in opposition to what can be found in Greek sources. Let us review some of these in as much as this relates to this opposition we want to sketch.

In a most famous passage of Plato's dialogue *Theaetetus* (147C–148B),[48] Theaetetus is presented to report on Theodorus' contribution about the "three-foot-power" (*tripous dynamis*) and the "five-foot-power." These expressions refer to lines that are sides of squares having areas of, respectively, "three feet" and "five feet."[49] And, according to Theaetetus's report there, Theodorus proved that "these are not commensurable in length with the one-foot-power." What is interesting for us here is that magnitudes are introduced that are recognizable as quadratic irrationals. However, and the following sentences of the text confirm the fact, these magnitudes are *lines*, not quantities that can be abstracted from the geometrical form in which they are introduced, and it is *as lines* that the question of their commensurability to other lines is raised.

Moreover, in this passage of the *Theaetetus*, commensurability is considered to be possibly either in length (*mékei*) or "with respect to the planes which they (the lines) have the power to form." This last expression, which refers to the commensurability of lines from the point of view of the squares, resonates with the key concepts of Book X of Euclid's *Elements*, the most elaborate treatment to be found in ancient Greek mathematical literature of magnitudes that could be anachronistically read as quadratic irrationals. Indeed, here Euclid defines the

46 However, on this point, one should recall that some scholars have raised doubts as to the attribution of this commentary to Liu Hui.

47 Bhāskara's commentary, which works hard to explain why in this case Āryabhaṭa has not respected the principle of simplicity and has not given the simplest values possible, is translated and discussed in [Keller 2000, II: 136–152].

48 Among the numerous publications that discuss this passage, we refer the reader to [Knorr 1975: 62 ff.; Høyrup 1990; Caveing 1998: 164 ff.], where both a discussion and the references to the most important contributions to the topic can be found.

49 Note here the same phenomenon as mentioned above concerning the treatment of units in ancient Chinese texts: the same name of unit is given for one-dimensional and two-dimensional magnitudes.

commensurability between magnitudes as well as "commensurability in power (*dynamei*)." The latter relation applies only to lines, which are said to be "commensurable in power" if "the squares on them are measured by the same area" [Heath 1956: 10; Vitrac 1998: 27 ff.]. We recognize here the relation discussed in the *Theaetetus*. The introduction of this concept allows as "expressible" (Definition 3) lines that would be only commensurable in power, like the side and the diagonal of a square. However, it is only, again, *as lines* that such magnitudes can be considered. As Vitrac [1998: 39–40] rightly stresses, this entails that the notion of irrationality presents itself in different ways, according to whether one considers lines or squares: a line the length of which would be $\sqrt{2} \cdot E$, where $E$ is a length taken as reference, is "expressible," whereas a square the area of which would be equal to $\sqrt{2} \cdot T(E)$ ($T(E)$ being the square drawn on $E$) would be "irrational." This highlights the fact that, in this context, irrationality has to be understood with respect to the kind of magnitudes one is dealing with, and cannot be considered for quantities *in abstracto*. This opposes the ancient Chinese and Indian sources we considered, where quadratic irrationals, as we saw, are introduced as quantities, and ancient Greek elaborations, where these considerations are not detached from geometry. The sharp demarcation that ancient Greek mathematical authors made between number and magnitude and that relates to this treatment of irrationality is not to be met with in ancient Chinese and Indian texts.[50]

In view of this contrast, the similarity between Indian and Chinese sources in this respect thus appears to be less natural and calls for an explanation. One cannot rule out *a priori* the possibility of a transmission between China and India, even though so far, as we stressed right at the beginning, there is no direct evidence for it. We are yet far from being able to understand the historical processes that account for these similarities between the Indian and Chinese sources we examined. In particular, the state of the extant sources does not allow any hypothesis as to which direction this possible transmission may have followed. However, given the numerous similarities that are to be found between these Chinese and Indian mathematical sources, this seems to constitute a fruitful avenue for future research. We may hope that the detailed description of as many similar situations as possible may in the end shed light on this question more globally.

In the case in which we are more specifically interested here, in order to prepare the ground in a better way for this more global apprehension of the situation, we also need to point out some differences between our Chinese and Indian sources: they seem to indicate that, even though the similarities noticed may have been caused by a transmission, the sources we analyzed, i.e., *The Nine Chapters* and its commentaries as well as Bhāskara's commentary on the *Āryabhaṭīya*, do not appear to be direct witnesses for it.

50 We saw earlier another hint of the same phenomenon in ancient China when the ratio between two magnitudes was expressed as a fraction, a point which we stressed above. This opposition between Indian and Greek mathematical texts was grasped by [Juschkewitsch 1964: 128], even though he could not be aware at the time of the similarity between Chinese and Indian early sources in this respect.

As far as the quantities involved are concerned, we saw above that *BAB* was regularly introducing irrationals of the type of square roots of quantities such as "integers increased/decreased by fractions," whereas the Chinese sources kept to magnitudes of the kind "side of *N*," where *N* is an integer. This refers to a more general contrast: Indian sources seem to make use of quantities of the type integer ± fraction directly, whereas Chinese sources attest to as wide a use as possible of integers [Chemla *et al.* 1992]. This latter specificity relates to the central character, in early China's mathematics, of such concepts as *lü*.[51] This implies that the quadratic irrationals which we encountered in both contexts fit with more general specificities of mathematical practice.

From the point of view of the operations carried out, there is one which is not to be met with in the extant Chinese sources, whereas, on the contrary, it appears to have been a topic of much research for centuries in India, namely, addition. As we saw above, *BAB* gives an algorithm for adding, in specific cases, quadratic irrationals in such a way as to obtain a result which would be a unique quadratic irrational. In contemporary texts such as the *BSS*, one finds a more general algorithm to the same end, attesting to a continuous interest in the problem. Furthermore, developing algorithms to perform such additions in wider and wider sets of cases seems to have been a concern of mathematicians as late as the 16th century in India [Hayashi 1977; Juschkewitsch 1964: 127–128].

This relates to another difference between our two corpora of text. We could describe the introduction of, and computations with, quadratic irrationals in relation to three Chinese sources: *The Nine Chapters*, completed around the beginning of the common era; Zhang Heng's writings on mathematics, as attested to by Liu Hui; and the commentary ascribed to Liu Hui and hence completed in 263. After the third century, it seems that no other mention is to be found of the use of such quantities in China: the tradition on this topic seems to have been interrupted. This may be part of a much more general phenomenon: several mathematical concepts and algorithms contained in *The Nine Chapters* or in the commentaries seem not to have been understood in depth by mathematicians of the subsequent centuries, and were, hence, transmitted in a coarser form.[52] By contrast, in India, seventh-century mathematicians developed procedures to compute with such quadratic irrationals. Since they do not appear to depend on a common contemporary source, it seems that the topic was one that had already been studied in the previous centuries. Moreover, as we just mentioned, this topic appears to have remained an active area of research interest relatively continuously until quite late. This may also explain another difference between our Chinese and our Indian corpus: in contrast to the computations we described in the Chinese sources, the Indian texts regularly express procedures for dealing with quadratic irrationals in general terms, before applying them to a specific situation. This may relate to the existence of a more developed tradition of such procedures.

51 In relation to this, the Chinese sources discussed above presented the simultaneous reduction of pairs of *lü* 率 of the type "*side of N*."

52 Some of them were recovered during the Song-Yuan period, others not. See, for examples, [Chemla 1996b and 1997b].

Such is thus the first picture of similarities and contrasts we get if we concentrate on Chinese and Indian sources. However, there are reasons that should compel us not to stop here. Due to the limitations inherent in the format of a paper, we shall simply list them, and we shall outline questions that should be pursued in order to deal with this topic in a more exhaustive way. We plan to push this research program further and deal with these questions in a later sequel to the present paper.

First, the present paper has emphasized the serious problems of interpretation raised by the Sanskrit word *karaṇī*. As has already been mentioned, several authors have tackled this issue without managing to provide a unified presentation of all the meanings one is tempted to attach to this term on the basis of its various occurrences. However, the difficulties encountered when dealing with the word *karaṇī* strikingly recall the problems historians faced when trying to provide a unified picture of the term *dynamis*, which was used in early Greek texts in connection with what we call irrational lines. The term *dynamis* appears to have also taken on the two meanings of "square"[53] as well as "square root" — since it could designate a line by the square it has the power to form.[54] This makes it difficult for the exegetes to account for what appears as a contradiction and leads to debates interestingly similar to those of the Indologists. Furthermore, in the same way as the Sanskrit verb formed on the radical of *karaṇīs* was used in singular formulations of the procedure corresponding to the so-called "Pythagorean theorem," the Greek verb corresponding to *dynamis*, *dynasthai*, was used in some Greek enunciations of the theorem or similar ones [Høyrup 1990: 203–205; Vitrac 1998: 32, 126–138, n. 145]. As a consequence, if, on the one hand, computations with quadratic irrationals designated with the word *karaṇī* in ancient India present striking similarities with what can be found in ancient Chinese sources and not ancient Greek sources, on the other hand, the uncommon cluster of meanings taken on by the term used to refer to these numbers, *karaṇī*, does not evoke at all the Chinese term *mian* 面, but the cluster of meanings which developed in ancient Greek mathematical literature around the word *dynamis*.

The picture gets even more complex if one recalls that, as Jens Høyrup [1990] noticed, the interpretation of the Akkadian *mithartum* presented the same difficulties, the very ones we are interested in, as that of the Greek *dynamis*. To deal with these facts, Høyrup developed his analysis along two lines. First, he suggested putting into play this conceptual parallel in such a way as to use what wc know of the Babylonian concepts to shed light on the Greek cases. Secondly, he wondered whether this coincidence could be accounted for in terms of transmission from

53 This is the meaning one can attribute to the term as used in Diophantos' *Arithmetics* [Rashed 1984a: 112 ff.], where it refers to the square of a number, as well as in late commentaries, where it can designate geometrical squares [Høyrup 1990: 206; Vitrac 1998: 28 and n. 20]. One can attach the Euclidean qualification of a relation between ratios as *dynamei* to this range of meaning.

54 We met above with an example in Plato's *Theaetetus*. [Høyrup 1990: 206] signals a use of *dynamei* in Hero's *Metrica* that falls under this rubric.

Old Babylonian to Greek mathematics, and then discussed the possible scenarios for this transmission. Now, if we add the term *karaṇī* to the debate, if we ask why the interpretation of the *three* terms *mithartum*, *dynamis* and *karaṇī* raise similar difficulties, the issue gets much more intricate. In any case, this collection of facts seems to indicate that the picture in terms of transmission may be more complex than we first thought and calls for a more global approach.

This is the first extension which the topic requires. It is clear that discussing all the problems raised by these additional similarities would exceed the scope of this paper. But, at first sight, the two lines of inquiry followed by J. Høyrup reasonably define the program of research that would also be worth pursuing here. First, from a conceptual point of view, which light could each of these situations shed on the other? Secondly, are we to account for these similarities in terms of transmission, and if so, how?

There is a second series of reasons that indicate that the problems of transmission raised by our topic are worth exploring further. These concern the Arabic world in the Middle Ages. It is generally admitted that Arabic mathematicians distinguished themselves from Greek mathematicians such as Euclid in that they carried out an "arithmetization" of the theory of quadratic irrationals: they introduced algebraic irrationals as numerical quantities, developed algorithms for carrying out ordinary arithmetical computations with them, thereby widening the range of "numbers," and, in parallel, gave an algebraic interpretation for Book X of Euclid's *Elements*, thus extending its meaning to bear upon magnitudes in general, numerical as well as geometrical.[55] Both Chakrabarti[56] [1934: 36–38] and Juschkewitsch [1964: 128, 250] noticed that this conception was in continuity with the way in which Indian mathematicians of whom we know conceived of such objects. We can now add that it was also in continuity with the way in which such quadratic irrationals were introduced in early imperial China. This naturally leads to a question: was this an independent development that was made in the Arabic-speaking world, based only on Greek mathematical texts and the inner logic of Arabic mathematics? Or did Arabic mathematicians also inherit, in one way or another, these earlier Asian ways of conceiving of, and computing with, irrationals, as they inherited concepts and procedures for integers and fractions? In the latter case, we would have, for the irrationals, a history in which different objects were later synthesized into a single one, rather than a linear account of a step-wise development.

In order to raise this question more specifically, we need to take a closer look, from this perspective, at al-Khwārizmī's *Concise Book on the Operations of al-jabr* ("algebra") *and al-muqābala*, completed between 813 and 833. A first examination seems to indicate that a more detailed analysis of how it is correlated,

55 On these commentaries, the reader is referred to [Matvievskaya 1987; Ben Miled 1999]. See also [Juschkewitsch 1964: 248 ff.; Ahmad & Rashed 1972: 37 ff.].

56 This was noted by [Hayashi 1977: 58, n. 28]. In general, [Chakrabarti 1934] does not provide much evidence for his statements.

in this respect, with the sources we examined would be worth carrying out.[57] Let us concentrate in the following lines on the hints of correlation that we found.[58] As is well known, after having introduced the elements with which to compose any quadratic equation, and after having presented the algorithms to solve the six normal forms they can take and proved them, al-Khwārizmī turns to examining computations with binomials (composed, strikingly, of numbers and "roots") and with square roots.[59] In the latter section, which touches the main topic of this paper, al-Khwārizmī considers quantities that are square roots of integers or of fractionary quantities and square roots of perfect squares as well as of other numbers.

A sentence, which apparently was interpreted in diverging ways by different translators, seems to indicate how al-Khwārizmī conceives of these quantities. If we understand the sentence as Rosen [1831: 27] does, it would read: "If you require to double the root of any known or unknown square ...." However, if we follow Ruska [1917: 63–64], the sentence, when speaking of such an object, should read: "If you require to double the rational or irrational root of a number ....."[60] This does not seem to be how Robert of Chester understood the original text.[61] However, Ruska's interpretation is in agreement with Gerard of Cremona's twelfth-century translation [Hughes 1986: 243],[62] when the latter mentions *quamlibet census radicem notam sive surdam*, i.e., any root, known (understood by Hughes [1986: 263] as rational) or irrational, of a wealth. We thus see that Gerard relates this sentence of al-Khwārizmī's to the question of irrational quantities.

57 Since neither author can read Arabic, the sketch of analysis we develop below depends on translations of al-Khwārizmī's book [Rosen 1831], as well as the critical editions of its Latin translations [Karpinski 1915; Hughes 1986, 1989]. However, we tried to check our descriptions against the secondary literature, and on some key points that we shall indicate, this led us not to follow the translations mentioned. In any case, our reflections are to be taken with caution.

58 Note that we want to deal here only with this aspect of al-Khwārizmī's book, leaving aside its most important part, i.e., the theoretical breakthrough which al-Khwārizmī's algebra represents. For this other aspect, see [Rashed 1983].

59 [Rosen 1831: 27–31], described in [Anbouba 1978: 71].

60 Ruska criticizes this point in Rosen's translation [1831: 27], noticing that, in a note, [Rosen 1831: 192] improves, without indicating it, his translation. There Rosen comments on the two qualifications attached to the word "root" by reformulating them as "audible" versus "inaudible," or "surd."

61 See [Hughes 1989: 49], whose critical edition diverges on this point from [Karpinski 1915: 96–97]. Interestingly enough, this text correlates here with Rosen's translation. This may indicate a noteworthy relation between, on the one hand, "known," "given," and rational, and, on the other hand, "unknown" and "irrational," in line with the designation of the unknown as root.

62 Exactly the same opposition recurs later ([Rosen 1831: 29; Hughes 1986: 244]). More generally, Ruska [1917: 65] finds himself in agreement more with medieval Latin translators than he does with Rosen. Juschkewitsch [1964: 209] also chooses to understand here that reference is made to "numerical quadratic irrationals." He then immediately concludes that this most probably relates to a translation of the Greek concept of *alogos*. Given the greater affinity of al-Khwārizmī's treatment of such quantities with earlier Asian sources, we would refrain from jumping to this conclusion. In general, all the possible links of al-Khwārizmī's algebra with Euclid's *Elements* seem to us to require a cautious analysis.

In this section of his text, al-Khwārizmī describes algorithms to perform some ordinary computations, not on integers or on fractions, but on square roots. The algorithms described are general: they apply indistinctly to roots of square as well as non-square numbers. These algorithms (multiplying a root by an integer, by a fraction; dividing or multiplying roots by one another — the very operations which are common to our Chinese and Indian sources[63]) always prescribe carrying out the computations on the squares, and end the procedure by taking the square root of the result. This implies that all numbers entering the computations must first be transformed into a root form. For instance, the integers are transformed into the roots of their squares. Moreover, as a consequence, all results are given as roots of integers, or roots of fractional quantities. Here two cases can occur. Either applying the square root algorithm to the integer or the fractional quantity produces an integer or a fraction, and this is the form in which the result is given in the end. Or, such is not the case, and the result is given as "the root of" the integer or the fraction.[64] One recognizes the form of result common to both our Chinese and Indian sources. In the "various questions" that follow, in which the reduction of a problem to an equation is exemplified, the result on several occasions is given in this form.[65] What is especially interesting is that the formulation of the problem is now no longer specifically geometrical. However, such quantities also occur in

63 We do not consider the actual examples where quantities of the form "integers ± roots of integers" are added or subtracted, as algorithms for adding or subtracting roots [Rosen 1831: 27]. Indeed, in the computations, only identical roots are considered, and hence simply counted; moreover, numbers and roots are kept apart, added or subtracted between likes. The only transformation affecting a root amounts to changing $n\sqrt{N}$ into $\sqrt{n^2N}$. In a sense, in these examples, except for the last transformation, $\sqrt{200}$ is handled like $x$ would be. This recalls Ruska's remark [1917: 61–65] that al-Khwārizmī's terminology might be better interpreted if, instead of taking, with Rosen, *māl* as the "square," and *jidhr* as the "root," understood as the first power of the unknown, we would interpret *māl* as "wealth," the unknown number, as opposed to the "absolute number" (the constant term), and *jidhr*, its "root." Al-Khwārizmī's equations would thus establish a relation between a number, an unknown number and its root. According to Ruska, this makes it possible to avoid many of the numerous exchanges between "number" or "square root," and square, which Rosen needed to perform in order to make sense of the text.

64 The result is thus, for example, given as the "root of one-sixth," see [Rosen 1831: 30]. See also [Hughes 1986: 244; Karpinski 1915: 101–103; Hughes 1989: 52].

65 [Rosen 1831: 53–54] shows two problems for which the results are, respectively, "root of five" and "root of thirty." [Rosen 1831: 62] shows a problem the solution of which is obtained by taking the root of "seven and a half." The Latin translations of al-Khwārizmī's book do not all contain the same problems. However, they contain problems for which the solutions are given in this form: [Karpinski 1915: 118–119; Hughes 1989: 61–62]. [Juschkewitsch 1964: 209] notes that once, given a problem with the solution $15-5\sqrt{5}$, al-Khwārizmī does not mention it [Rosen 1831: 51]; the problem is also found in Gerard's version [Hughes 1986: 257]. One may see here a confirmation of the fact that, as in early China or India, the only admissible results had to have the form "root of $N$." Since, in this case, the root could not be expressed in this form, this may have been a reason to discard it. Moreover, [Juschkewitsch 1964: 209] rightly emphasizes that such quantities occur as results, never as data in the problem: this is also true for both the Chinese and Indian sources we examined. Abū Kāmil does away with this restriction, as we shall see below.

the geometrical part of al-Khwārizmī's book, which was not translated into Latin. The application of the algorithm corresponding to the "Pythagorean theorem" hence gives him the opportunity to give the "root of 75" as the height of a given triangle [Rosen 1831: 79].[66] Concerning quadratic irrationals, we therefore find again here the same cluster of similarities that have been observed in the Chinese and Indian sources discussed earlier. More generally, several other hints indicate a close relationship between al-Khwārizmī's algebra and Indian sources.[67] To mention only one, it is interesting for us here to recall that al-Khwārizmī gives two approximate values for the ratio of the circumference of a circle to its diameter,[68] i.e., $\sqrt{10}$, and 62 832/20 000! The first attested occurrence of quadratic irrationals in Arabic sources may thus well be linked to earlier Asian treatments elaborated either in China or in India.

This introduction of quadratic irrationals in al-Khwārizmī's algebra opened the way to a steady development of their use and treatment in later algebraic treatises. This is the case in Abū Kāmil's *Book on al-jabr and al-muqābala*, completed at the end of the ninth or beginning of the tenth century. In a chapter clearly based on al-Khwārizmī's section on binomials and square roots alluded to above, Abū Kāmil systematizes and develops the computations with quadratic irrationals considered by his predecessor [Levey 1966: 52–82]. In contrast to his mention of Book II of Euclid's *Elements* in the preceding chapter of his *Book on al-jabr and al-muqābala*, this one contains no reference to the Greek text. Moreover, Abū Kāmil now considers equations with irrational coefficients, and extends the range of algebraic irrationals considered [Anbouba 1978: 83–84, 96–97; Juschkewitsch 1964: 224–228]. It is again striking that the algorithms he provides to add two quadratic irrationals and transform them into a unique square root recalls the algorithms expounded in later Sanskrit sources.[69]

In parallel to this development, al-Khwārizmī's book opens the way for a new tradition of reading Book X of Euclid's *Elements*: al-Māhānī, one of his contemporaries, in the oldest extant Arabic commentary on Book X, interprets it as dealing not only with geometrical magnitudes, but with integers, fractional quantities and their roots. Moreover, he makes use of algebra to translate geometrical prob-

66 Later on, al-Khwārizmī also considers roots of quantities composed of "things and numbers" [Rosen 1831: 81].

67 [Ruska 1917] contains many other interesting hints. Some 19th- and early 20th-century authors, including Rosen [1831] and Ruska [1917], insisted on al-Khwārizmī's dependence on Indian sources. These facts are less discussed in today's historiography, and one may wonder about the reasons for this "forgetting." Could it be that the discovery and interpretation, in the 1930s, of Babylonian clay-tablets dealing, roughly speaking, with quadratic equations somehow diverted attention from this fact? It may be worth inquiring further into this issue. In any case, the question of dealing and computing with quadratic irrationals may constitute an additional hint for this continuity.

68 [Anbouba 1978: 72] confirms that al-Khwārizmī explicitly attributes these values to the Indians.

69 [Hayashi 1977: 53–54] describes the same algorithm in Bhāskara II's *Bījagaṇita*. This was also noted by [Juschkewitsch 1964: 249].

lems into equations.[70] Matvievskaya [1987] follows the development of this line of research in the subsequent extant Arabic commentaries on Book X.

It therefore seems, as R. Rashed has pointed out, that we can see the development in the Arabic world of two interacting traditions of dealing with what we now read as irrationals, one relating to a reading of Book X, the other one independent from it [Rashed 1997: 37]. Al-Karajī's contribution may be considered as the merging of the two traditions and their culmination [Ahmad & Rashed 1972: 37–52]. On the basis of this rough sketch, the question which seems to us worth raising can be formulated in a slightly more specific way: is it possible that the development of the computations for algebraic irrationals, as they can be traced in the earliest Arabic algebraic treatises onwards, may have been inspired at least in part by earlier Asian sources? This would mean that, in order to capture the historical processes of the unfolding of mathematical research on irrational entities, Greek sources could no longer be considered as the unique point of departure.

Clearly many questions are still open, ones we shall not try to solve here. However, are not echoes such as these, which reverberate through Old-Babylonian, Greek and Indian, Chinese and Arabic texts, worth pondering, in that they are likely to help us break down reductionist and all too linear representations of the history of ancient mathematics? In the case of irrational magnitudes, it may well be that the history will reveal a more complex pattern of developments, with transmissions in more directions than usually believed.

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# Rule of Three and its Variations in India

by SREERAMULA RAJESWARA SARMA

*trairāśikenaiva yad etad uktaṃ vyāptaṃ svabhedair hariṇeva viśvam.*
Just as the universe is pervaded by Hari with His manifestations, even so all that has been taught [in arithmetic] is pervaded by the Rule of Three with its variations.

Bhāskarācārya, *Līlāvatī* (ca. 1150)

Uns haben die Meister der freien Kunst von der Zahl eine Regel gefunden, die heisst Goldene Regel, davon, dass sie so kostbar und nützlich ist gegenüber allen anderen Regeln von gleicher Art, wie Gold alle anderen Metalle übertrifft. Sie wird auch genannt Regeldetri nach welscher Zunge, weil sie aussagt von dreierlei und beschliesst drei Zahlen in sich.

Ulrich Wagner, *Das Bamberger Rechenbuch von 1483*

In the history of transmission of mathematical ideas, the Rule of Three forms an interesting case. It was known in China as early as the first century A.D. Indian texts dwell on it from the fifth century onwards. It was introduced into the Islamic world in about the eighth century. Renaissance Europe hailed it as the Golden Rule. In this paper, I propose to discuss the history of this rule and its variations in India as discussed in the texts in Sanskrit and other languages. I shall dwell on the theoretical deliberations, the types of problems and the mechanical processes by which these are solved by the rule. I conclude the paper with brief notes on the transmission of the rule to the Islamic world and thence to Europe.

In der Geschichte der Übermittlung von mathematischen Ideen ist die Dreisatz-Regel ein interessanter Fall. In China war sie schon im ersten Jahrhundert n. Chr. bekannt. Indische Texte behandeln sie seit dem fünften Jahrhundert. In die islamische Welt wurde sie um das achte Jahrhundert eingeführt. Zur Zeit der Renaissance wurde sie in Europa als „Goldene Regel" bekannt. Die Geschichte dieser Regel und ihrer Varianten werden im folgenden Aufsatz behandelt. Der Aufsatz erklärt die theoretischen Überlegungen, die Arten von Aufgaben, die mit dieser Regel gelöst werden können, die mechanischen Vorgänge, die bei der Lösung der Aufgaben zu benutzen sind, und erörtert dann kurz die Verbreitung der Regel in der islamischen Welt und in Europa.

## Introduction

In the history of transmission of mathematical ideas, the Rule of Three forms an interesting case. It was known in China as early as the first century A.D. Indian texts dwell on it from the fifth century onwards but a rudimentary form of the rule was available much earlier. It was introduced into the Islamic world in about the eighth century. Europe hailed it as the Golden Rule. The importance of the rule lies not so much in the subtlety of its theory as in the simple process of solving problems. This process consists of writing down the three given terms in a linear sequence (A → B → C) and then, proceeding in the reverse direction, multiplying the last term with the middle term and dividing their product by the first term (C × B ÷ A). With this rule one can easily solve several types of problems even without a knowledge of the general theory of proportion.

The writers in Sanskrit, however, were well aware of the theory. Commenting on the rule given by Āryabhaṭa, Bhāskara I notes that this rule encompasses Rules of Five, Seven and others because these are special cases of the Rule of Three itself. Bhāskara II even declares that the Rule of Three pervades the whole field of arithmetic with its many variations just as Viṣṇu pervades the entire universe through his countless manifestations. Leaving aside the poetic hyperbole, there is no doubt that the mechanical methods provided by the Rule of Three and its variations offer quick solutions to nearly all problems concerning commercial transactions.

This mechanical rule held sway over large parts of the world for nearly two thousand years. If the price of A things is B coins, the issue of determining the price of C things must have been one of the earliest exercises in the realm of computation. Problems of this nature being fundamental to every commercial transaction, their solution must have been available in all early civilisations. As Tropfke observes, "the logic on which the Rule of Three is based must belong to the earliest realisations of the counting man" [Tropfke 1930: 187]. However, the earliest records concerning the theoretical deliberations about the problems and their solution emanate from China and India, although their mutual relationship awaits further investigation.

## Early History of the Rule of Three

In India the Rule of Three was first mentioned by Āryabhaṭa I in his *Āryabhaṭīya* (A.D. 499):

> Now having multiplied the quantity known as fruit (*phala-rāśi*) pertaining to the Rule of Three (*trairāśika*) by the quantity known as requisition (*icchā-rāśi*), the obtained result (*labdha*) should be divided by the argument (*pramāṇa*). [What is obtained] from this [operation] is the fruit corresponding to the requisition (*icchā-phala*).[1]

Here Āryabhaṭa not only gives the name *Trairāśika* (that which consists of three numerical quantities or terms) for the Rule of Three, but mentions as well the technical terms for the four numerical quantities involved (*pramāṇa*, *phala-rāśi*,

1 [*Āryabhaṭīya, Gaṇitapāda* 26]:
*trairāśikaphalarāśiṃ tam athecchārāśinā hataṃ kṛtvā /*
*labdhaṃ pramāṇabhajitaṃ tasmād icchāphalam idam syāt //*

*icchā-rāśi*, *icchā-phala*) and gives the formula for solving the problem. Subsequent writers, notably Brahmagupta in his *Brāhmasphuṭasiddhānta* (A.D. 628) and Bhāskara I in his commentary (A.D. 629) on the *Āryabhaṭīya* elaborate upon this brief statement by Āryabhaṭa, but employ the same terminology, albeit with slight modifications. It is on the basis of the writings of these mathematicians that histories of mathematics generally trace the origin of the Rule of Three to India.[2] The brief manner in which Āryabhaṭa presents the rule in his work implies that he is referring to an already well known rule which he is restating here in order to employ it in astronomical computations. Therefore, it is tempting to look for the antecedents for Āryabhaṭa's rule.[3]

Kuppanna Sastry sees the first mention of the Rule of Three in the following verse of the *Vedāṅgajyotiṣa*, "The known result is to be multiplied by the quantity for which the result is wanted, and divided by the quantity for which the known result is given."[4] Sastry goes on to say:

> The instruction is concise and looks like an aphorism. There are four items in a proportion, three known and one unknown, which is obtained from the three known. Hence the rule to get this is called the "Rule of Three." The four items are: (a) If for so much quantity, (b) so much result is got, (c) for this much quantity given now, (d) how much is the result that will be got? The first two are called *jñāta-rāśis* and the next two are called *jñeya-rāśis*. The application of the rule is: Take the known result, i.e. (b), multiply it with the quantity (c) for which the result is to be known, and divide by the quantity (a) for which the result is given; thus the result to be known, i.e. (d), is got.

It is obvious that we have here a rudimentary form of the Rule of Three and that the rule was needed for the computations envisaged in the text. Although Indians developed special terminology for the Rule of Three in later times, the general terms used here, *jñāna⟨ta⟩rāśi* (the quantity that is known or given) and *jñeya-rāśi* (the quantity that is to be known), are also frequently employed in later times.[5] Indeed, it is conceivable that the term *jñāna* gave rise to the later term *pramāṇa*. However, the date of this text, available in two recensions, is uncertain. Kuppanna Sastry himself would like to place the composition of the text in the period between 1370–1150 B.C.; others assign it to 500 B.C. In either case, Āryabhaṭa's rule appears to have a long prehistory in India.

2 Thus, D. E. Smith, "The mercantile Rule of Three seems to have originated among Hindus. It was called by this name by Brahmagupta (ca. 628) and Bhāskara (ca. 1150), and the name is also found among the Arab and medieval Latin writers" [Smith 1925: 483].

3 The word *rāśi* occurs in [*Chāndogya Upaniṣad* 7.1.2] as the name of a *vidyā* (subject of study) and in [*Ṭhāṇaṃga Sutta*: 747] as the title of one of the topics of mathematics. There is no reason to suppose that either of these occurrences denotes the Rule of Three. The former might mean mathematics in general and the latter "heap," "group," or "set."

4 Ṛk-recension 24; Yajus-recension 42: *jñeyarāśi-gatābhystā⟨taṃ⟩ vibhajet jñāna⟨ta⟩rāśinā* [*Vedāṅga Jyotiṣa*: 40–41].

5 Cf. the anonymous commentary on the *Pāṭīgaṇita*, [*Bakhshālī*].

However, Joseph Needham observes that the "Rule of Three, though generally attributed to India, is found in the Han *Chiu Chang*, earlier than in any Sanskrit text. Noteworthy is the fact that the technical term for the numerator is the same in both languages — *shih* and *phala*, both meaning "fruit." So also for the denominator, *fa* and *pramāṇa*, both representing standard unit measures of length" [Needham 1959: 146]. Needham goes on to add that "Even the third known term in the relationship can be identified in the two languages. For *icchā*, 'wish, or requisition' reflects *so chhiu lü*, i.e. ratio, number sought for" [Needham 1959: 146, note i].

In a recent article, N. L. Maiti draws attention to the passage of the *Vedāṅga Jyotiṣa*, in order to counter Needham's claim of Chinese priority. Maiti also disputes Needham's linguistic equation *fa = pramāṇa; shih = phala; so chhiu lü = icchā*. Finally, he tries to clinch the issue by citing [Maiti 1996: 7] the view of a Chinese scholar from Singapore, Lam Lay-Yong: "The rule of three which originated among the Hindus is a device used by oriental merchants to secure results to certain numerical problems" [Lam 1977: 329].

I am not competent to judge in favour or against Needham's linguistic equation, except to say that the word *pramāṇa* does not represent "standard unit measures of length" as words like *aṅgula* or *hasta* do; it merely means, among other things, "size" or "measure." I am sure there are others who can determine the precise meaning of the Chinese *fa* and decide whether the two sets of terminology have the same connotation. But there is no denying the fact that the Rule of Three had an important place in Chinese mathematics as well. Even if the verse from the *Vedāṅga Jyotiṣa* alludes in a rudimentary form to the Rule of Three and thus testifies to the existence of the rule in the centuries before the Christian era, there is nothing to prevent the knowledge of the Chinese *Jiu Zhang* (i.e., Needham's *Chiu Chang*, or *Nine Chapters*) to travel to India in the early centuries of the Christian era and to give impetus to the development of the Rule in India.[6]

## Development of the Rule in India

We have seen that the Rule of Three occurs in a rudimentary form in the *Vedāṅga Jyotiṣa* towards 500 B.C. and about a thousand years later it appears in a fully developed form with all the technical terminology in the *Āryabhaṭīya* of Āryabhaṭa. In his commentary on the *Āryabhaṭīya*, Bhāskara I dwells at length on the full implication of the rule given by Āryabhaṭa [*Āryabhaṭīya-Bhāskara*: 116–122]. According to him, Āryabhaṭa's rule encompasses Rules of Five, Seven, etc. This point will be discussed below in detail. Bhāskara also cites a stanza, which states:

6 If India received impetus from China in this process of development and then transmitted an elaborate system to the Middle East and Europe, then this would testify to both the receptivity and creativity in mathematical thought in India. It would, however, be nice if the transmission could be mapped in detail.

> In solving problems connected with the Rule of Three, when the numbers are written down (*sthāpanā*), the wise should know that the two like quantities should be set down in the first and in the last places, and the unlike quantity in the middle [*Āryabhaṭīya-Bhāskara*: 117].[7]

This stanza is preceded by the expression *uktam ca*, "It has also been said." This suggests that the verse was composed by somebody else before Bhāskara's time. This is an important citation, not for the method of solving, but for the method of writing down the numbers. In his rule, Āryabhaṭa did not explain how to set down the three given numerical quantities. This anonymous verse provides therefore the first extant statement that the three quantities should be set down in a certain sequence, viz. $A \rightarrow B \rightarrow C$, and then be worked out in a certain other sequence: $C \times B \div A$. Logically, B should first be divided to get the price of one item (i.e., the rate) and then, the quotient should be multiplied by the requisition ($B \div A \times C$) to obtain the price of the required number of items. But division may produce fractions and operating with them is more difficult than working with integers. Therefore this anonymous verse prescribes multiplication first and division next, and this procedure is followed by all subsequent writers. Besides increasing the chances of computing with integers, this procedure has the added advantage of mechanical neatness in execution: a forward motion from left to right while setting down the quantities and a contrary motion from the right to the left while working out the problem:

| | |
|---|---|
| forward motion in setting down, viz. | $A \rightarrow B \rightarrow C$ |
| backward motion in computation, viz. | $A \leftarrow B \leftarrow C$. |

This anonymous statement is authenticated by Brahmagupta, who formally restates the sequence in these words:

> In the Rule of Three, argument, fruit and requisition [are the names of terms]: the first and last terms must be similar. Requisition, multiplied by the fruit, and divided by the argument is the result.[8]

Brahmagupta is also the first to state that in the Inverse Rule of Three, the direction of the operation will be the reverse (of what it was in the direct Rule of Three). Thus here, the direction of the setting down and computation will be the same as shown below.

| | | |
|---|---|---|
| Inverse Rule of Three: | setting | $A \rightarrow B \rightarrow C$ |
| | computation | $A \rightarrow B \rightarrow C \quad (= A \times B \div C)$. |

7 [*Āryabhaṭīya-Bhāskara*: 117]:

*ādyantayos tu sadṛśau vijñeyau sthāpanāsu rāśīnām /*
*asadṛśarāśir madhye trairāśikasādhanāya budhaih //*

In later times, the terms were also called *ādi* / *ādya* (first), *madhya* (middle) and *antya* (last).

8 [*Brāhmasphuṭasiddhānta* 12.10]:

*trairāśike pramāṇa-phalam-icchādyantayoḥ sadṛśarāśih /*
*icchā phalena guṇitā pramāṇabhaktā phalaṃ bhavati // 12.10 //*

Brahmagupta's formulation of the Rule of Three became the model for the subsequent writers who all underscore that the three given terms should be set down in such a way that the first and last be of like denomination and the middle one be of a different kind. Thus in his *Pāṭīgaṇita*, Śrīdhara (ca. A.D. 750) reiterates these points:

> In [solving problems on] the Rule of Three, the argument (*pramāṇa*) and the requisition (*icchā*), which are of the same denomination, should be set down in the first and last places; the fruit (*phala*), which is of a different denomination, should be set down in the middle. [This having been done], that [middle quantity] multiplied by the last quantity should be divided by the first quantity [*Pāṭīgaṇita*, translation: 23].[9]

An anonymous commentary on the *Pāṭīgaṇita*, composed in Kashmir in the ninth or tenth century, has a valuable discussion:

> All this is *trairāśika*. In this method, in the first and last positions, set down the quantities which are of the same class or denomination (*jāti*). Furthermore, in the first place, set down the quantity which is the argument, then [in the last position] the quantity which is the requisition and between these two the fruit which belongs to a different denomination. Having done this, the fruit is multiplied by the last term, namely the quantity called requisition and then divided by the first term, namely the quantity called argument. Thus the desired (*jijñāsita*) result is obtained. Here *jāti* refers to the denomination connected with the commercial transactions. Being of the same denomination or of different denominations is to be understood in this sense and not in the sense of caste as applicable to Brahmins etc. ... If the argument is a commodity (*paṇanīya*) and the requisition is also a commodity, they are of the same class. Then the middle term will be the price (*mūlya*), because of the mutual dependence (*parasparāpekṣitatva*) of the commodity (*paṇya*) and price. Likewise, if the first and last terms are the prices, then the commodity will be the middle term. If the two [first and last terms] refer to artisans of the same denomination, then their wages (*bhṛti*) is the middle term. Or the amount of work done by the artisans is of a different denomination. Therefore [a statement related to their amount of work] is the middle term. While the first and the last terms are of like denomination, the middle term is of the same denomination as the quantity which is desired to be known [*Pāṭīgaṇita*: 37].

Other mathematicians, for example, Mahāvīra (ca. A.D. 850) and the second Āryabhaṭa (ca. A.D. 950), do not add much to the theory of the Rule of Three.[10] The celebrated Bhāskara II also follows suit in his *Līlāvatī*, but with a certain economy of expression:

> The argument and requisition are of like denomination; they are to be set down in the first and the last places. The fruit, which is of a different denomination, is set down in the middle. That [middle term], being multiplied by the requisition

9 [*Pāṭīgaṇita*: 37 (Rule 43)]:

*ādyantayos trirāśāv abhinnajātī pramāṇam icchā ca /*
*phalam anyajāti madhye tad antyaguṇam ādinā vibhajet //*

The translation is by K. S. Shukla [*Pāṭīgaṇita*: 23].

> and divided by the first term, gives the fruit of the requisition. The operation is reversed in the inverse (*viloma*) method.[11]

Bhāskara II deserves special notice because, towards the close of his *Līlāvatī*, he declares that nearly the entire arithmetic is based on the Rule of Three and that most of the topics are but variations of this Rule of Three:

> Just as the universe is pervaded by Hari with His manifestations, even so all that has been taught [in arithmetic] is pervaded by the Rule of Three with its variations.[12]

He goes on to elaborate this further in the following words:

> As Lord Śrī Nārāyaṇa, who relieves the sufferings of birth and death, who is the sole primary cause of the creation of the universe, pervades this universe through His own manifestations as worlds, paradises, mountains, rivers, gods, men, demons, etc., so does the Rule of Three pervade the whole of the science of calculation. ... Whatever is computed whether in algebra or in this [arithmetic] by means of multiplication and division may be comprehended by the sagacious learned as the Rule of Three. What has been composed by the sages through the multifarious methods and operations such as miscellaneous rules, etc., teaching its easy variations, is simply with the object of increasing the comprehension of duller intellects like ourselves [*Līlāvatī*, auto-commentary on 240, and verse 241; trans. Datta & Singh 1962, I: 209].

Again, in the *Siddhāntaśiromaṇi*, Bhāskara II reiterates that arithmetic is basically the Rule of Three only and goes on to say that:

> Leaving squaring, square-root, cubing and cube-root, whatever is calculated is certainly a variation of the Rule of Three, nothing else. For increasing the comprehension of duller intellects like ours, what has been written in various ways by the learned sages ..., has become arithmetic [*Siddhāntaśiromaṇi*, Golādhyāya, Praśnādhyāya, 3–4; trans. Datta & Singh 1962, I: 210].

## Rule of Three and Proportion

Were these writers aware that the Rule of Three is based on proportion? D.E. Smith observes that "Proportion was thus concealed in the form of an arbitrary rule, and the fundamental connection between the two did not attract much notice until, in the Renaissance period, mathematicians began to give some attention to commercial arithmetic" [Smith 1925: 484]. This statement is not based on valid grounds.

10 The second Āryabhaṭa, however, uses two different terms: *māna* for the first term and *vinimaya* for the middle term; cf. [*Mahāsiddhānta* 15.24–15.25].

11 [*Līlāvatī*: 73]:
*pramāṇam icchā ca samānajātī ādyantayoḥ staḥ phalam anyajātiḥ /*
*madhye tadicchāhatam ādihṛt syād icchāphalaṃ vystavidhir vilome //*

12 [*Līlāvatī*: 239]:
*trairāśikenaiva yad etad uktaṃ vyāptaṃ svabhedair hariṇeva viśvam //*

That the Rule of Three (*trairāśika*) was a case of proportion (*anupāta*) is well known, even if the formula does not show it in the manner in which the West is accustomed to see it ($a : b :: c : d$).[13] This will be evident from Bhāskara's comments on Āryabhaṭa's rule: "In this rule, Āryabhaṭa described only the fundamentals of proportion (*anupāta*). All others such as the Rule of Five etc. follow from that fundamental rule of proportion" [*Āryabhaṭīya-Bhāskara*: 116].

Commenting on the same passage, Sūryadeva Yajvan states:

> Here we have the logical proposition (*vācoyukti*) — if by so many coins so many things are obtained, by so many coins how many things will be obtained? Here the first quantity is called *pramāṇa*; the second quantity is *phala* and the third is *icchā*. With these three quantities, the fourth is determined.[14]

Indeed, while solving the problems posed in the texts, the commentaries often state such propositions in order to show the logic behind the various steps of computation.[15] Bhaṭṭotpala clearly states that the proportions are called the mathematics of the Rule of Three.[16]

Furthermore, the Rule of Three is often used as a means of verification in solving other problems. Thus for instance, Bhāskara I, in his commentary on *Āryabhaṭīya* 2.25 which contains a problem on interest, employs the Rule of Five for verification.[17] The *Bakhshālī Manuscript* frequently employs the Rule of Three for verification with the words *pratyaya(s) trairāśikena* [*Bakhshālī*: 157, 159 *et passim*]. So does Gaṇeśa in his commentary *Buddhivilāsinī* on the *Līlāvatī* [*Buddhivilāsinī*: 83 *et passim*]. Already in the ninth century, Govindasvāmin attempts to relate the rule to the science of logic. He sees it as a case of inference. Just as there is an invariable concomitance between smoke and fire, so it is between the argument and the fruit. Just as this invariable concomitance allows one to infer that there is fire on the mountain because there is smoke, so does the relation between the argument and fruit allow us to compute the fruit of requisition from the requisition.[18]

13 But is this not so with every formula, that it conceals the logic behind it? See [Datta and Singh, I: 217]. See also [Juschkewitsch 1964: 119–120]: "Die Erläuterung der Regeln war in den indischen Werken formaler Art, doch hatten die indischen Mathematiker unzweifelhaft Verständnis für deren gemeinsame Grundlage und deren Querverbindungen."

14 [*Āryabhaṭīya-Sūryadeva*: 65]:

> *atra hīyam vācoyuktiḥ etāvadbhir etāvanti labhyante etāvadbhiḥ kiyantīti / tatra prathamaḥ pramāṇarāśiḥ dvitīyaḥ phalarāśiḥ tṛtīya icchārāśiḥ / taiś caturtho rāśiḥ sādhyate /*

15 Cf. [Nīlakaṇṭha, III: 49]: *trairāśika-vācoyuktiś caivam*. Throughout this commentary, there are such statements of proportion.

16 In his commentary on Varāhamihira's *Laghujātaka* 6.2 he states: *anupātās trairāśika-gaṇitam abhidhīyate*, as cited in the *Petersburger Wörterbuch*, s.v. *trairāśika*.

17 [*Āryabhaṭīya-Bhāskara*: 114–115]: *pratyayakaraṇam pañcarāśikena*.

18 His views are cited in [*Kriyākramakarī*: 179–183]. For a lucid exposition of these views, see [Hayashi 2000: 210–226]. Here, Hayashi wishes to render the term *trairāśika* as "three-quantity operation."

## Applications of the Rule of Three

After thus discussing the views of various mathematicians, we may now look at the areas where this rule is applied, in other words, the problems concerning the Rule of Three. It is obvious that Āryabhaṭa's main purpose in enunciating the Rule of Three is to employ it in astronomical computations. One of these is the computation of the mean position of a planet from the number of its revolutions in a *Kalpa* of 4,320,000,000 years.[19] Many of the problems of spherical astronomy are also solved by the application of the Rule of Three to the similar triangles called *akṣakṣetra*, "latitude-triangles" [*Āryabhaṭīya-Trans.*: 130–132]. Likewise, the Rule of Three forms the basis for trigonometrical ratios [Gupta 1997: 73–87]. Therefore, Nīlakaṇṭha Somasutvan (b. A.D. 1444) declares, in his commentary on the *Āryabhaṭīya*, that the entire mathematical astronomy (*graha-gaṇita*) is pervaded by two fundamental laws: by the law of [the relation between] the base, perpendicular and hypotenuse [in a right-angled triangle] and by the Rule of Three.[20]

However, the Rule of Three became well known outside India for its application in the so-called commercial problems in arithmetic.[21] It is Bhāskara I who, in the course of his commentary on Āryabhaṭa's rule, gives various examples of such commercial problems for the first time [*Āryabhaṭīya-Bhāskara*: 116–122]. Since Brahmagupta did not give any examples of his own, these given by his contemporary Bhāskara are the earliest. Therefore, it is necessary to look more closely at these earliest examples provided by Bhāskara and his methods of solving them.

Bhāskara gives altogether seven problems or examples (*uddeśaka*): (i) price and quantity of sandalwood; (ii) price and weight of ginger (the problem has fractions and illustrates Āryabhaṭa's rule for fractions); (iii) price and quantity of musk also with fractions; (iv) time taken by a snake in entering a hole; (v) mixed quantities (*miśrakarāśis*); (vi) partnership (*prakṣepakaraṇa*); (vii) partnership expressed as fractions (*bhinna*). Of these, (i) is a case of simple proportion; (iii) is similar to (ii). We shall examine how Bhāskara solves the other five problems. We shall also see how some of these became new topics in later times.

*Price and weight of ginger.* Bhāskara's second problem involves fractions:

> Ginger, of 1 *bhāra* weight, was sold at 10 and 1/5 coins. What is the price of ginger of 100 and 1/2 *pala*s? [*Āryabhaṭīya-Bhāskara*: 117]

Since 1 *bhāra* equals 2000 *pala*s, the three terms are set down as follows. This process of setting down the given terms in the prescribed order is called *nyāsa*:

19 Cf. [Nīlakaṇṭha, III: 1]: *ahargaṇāt trairāśikena madhyamam ānīya sphuṭīkriyate.*

20 [Nīlakaṇṭha, I: 100]: *bhujākoṭikarṇanyāyena trairāśikanyāyena cobhābhyāṃ sakalaṃ graha-gaṇitaṃ vyāptam.* See also [Nīlakaṇṭha, I: 19].

21 Even non-mathematical texts in India stress the importance of the Rule of Three for solving commercial problems. Thus King Someśvara lays down in his *Mānasollāsa* (A.D. 1129) that the accountant in the royal treasury should be thoroughly versed in multiplication and division [with fractions] and the method of [applying] the Rule of Three (*trairāśika-vidhāna*); cf. [*Mānasollāsa* 2.2.124 (p. 40)]; see also [*id.* 2.2.113–2.2.116 (p. 39)] where the rules of three, five, seven and nine terms are discussed.

| | | |
|---|---|---|
| 2000 | 10 | 100 |
| | 1 | 1 |
| | 5 | 2. |

It may be noted that this is how mixed fractions are written in India: integer, numerator and denominator one below the other without any lines separating them. Bhāskara explains the next step thus:

> After assimilating the integers with the related fractions by reducing them to a common denominator (*savarṇita*), the three quantities are set down once more in the following manner:
>
> | | | |
> |---|---|---|
> | 2000 | 51 | 201 |
> | | 5 | 2. |
>
> Then, following Āryabhaṭa's rule, the denominators of the two multipliers are transposed to the divisor. Thus by the two denominators 5 and 2, the divisor (2000) is multiplied, it becomes 20000. The product of 201 and 51 is 10251. This is divided by 20000. The result is 10251/20000 coins ...

*Snake entering a hole*. Bhāskara's fourth problem is the following:

> A snake, twenty cubits long, enters a hole at the rate of half *aṅgula* per *muhūrta* and it comes out by one-fifth *aṅgula*. How many days does it take to enter the hole? Setting down: the snake, 480 *aṅgula*s long, goes in by 1/2 *aṅgula*, comes out by 1/5 *aṅgula* in 1 *muhūrta*. Thus its rate of entering per *muhūrta* is half *aṅgula* diminished by one-fifth *aṅgula*. So subtract one-fifth from one-half, and set the terms down:
>
> rate of entry 3/10 *aṅgula* in *muhūrta* 1, snake's length in *aṅgula*s 480
>
> [*Āryabhaṭīya-Bhāskara*: 118].

The calculation implied is: $480 \times 1 \times 10/3 = 4800/3 = 1600$ *uhūrtas* = 1600/30 days = 53 1/3 days.

Śrīdhara treats this as an independent topic called *gati-nivṛtti* (forward and backward motion) and provides a separate rule for solving it, "On subtracting the backward motion per day from the forward motion per day, the remainder is the (true) motion per day" [*Pāṭīgaṇita*: Rule 44ab (p. 41)]. Then this net daily motion is treated as the argument.

Why do we need this new rule? Can we not solve this as Bhāskara did above? The commentary on the *Pāṭīgaṇita* says that it is for the convenience of pupils (*śiṣyahitārthatvāt*):

> Subtract the backward motion per day from the daily forward motion. The remainder will be the net daily motion. Take that as argument, 1 day as the fruit, and demand the time taken for the distance to be traversed. Thus there are three Rules of Three: the first to derive the daily forward motion, the second to get the daily backward motion, and the third to compute the time taken for the distance to be traversed. These can of course be worked out according to the previously given general rule. Even so, it is convenient for the pupils to have a new rule

which provides for the argument and fruit to be used for deriving the time taken for the distance to be traversed. Being ignorant, the pupils might try to compute the time for the distance to be traversed, without first subtracting the rate of backward motion, and then compute the amount of backward motion for the above period of time, and then take their difference; but then that would not be correct [*Pāṭīgaṇita*: 41].

Śrīdhara gives two examples, the second of which reads as follows:

> A man earns 1/2 less 8 silver pieces in 1 and 1/3 days; he spends 1/2 silver per day on his food. In how many days will he become the lord of one hundred pieces of silver (*śateśvara*)?[22]

Mahāvīra's *Gaṇitasārasaṃgraha* treats this topic rather elaborately with one rule and several lengthy examples. His rule is as follows:

> Write down the net daily movement, as derived from the difference of [the given rates of] forward and backward movements, each [of these rates] being [first] divided by its own [specified] time; and then in relation to this (net daily movement), carry out the operation of the rule-of-three [*Gaṇitasārasaṃgraha* 5.23].

Of the various examples given by Mahāvīra, the following may be cited:

> A well completely filled with water is 10 *daṇḍa*s (= 960 *aṅgula*s) in depth. A lotus sprouting therein grows from the bottom at the rate of 2 1/2 *aṅgula*s in a day and half; the water thereof flows out through a pump at the rate of 2 1/20 *aṅgulas* [of the depth of the well] in 1 1/2 days; while 1 1/5 *aṅgula*s water are lost in a day by evaporation; a tortoise below pulls down 5 1/4 *aṅgula*s of the stalk of the lotus plant in 3 1/2 days. By what time will the lotus be at the same level with the water in the well? [*Gaṇitasārasaṃgraha* 5.28–30]

These recall the lion in the pit problem posed by Leonardo of Pisa in the *Liber Abbaci:*

> The pit is 50 feet deep. The lion climbs up 1/7 of a foot each day and then falls back 1/9 of a foot each night. How long will it take him to climb out of the pit?

However, Leonardo does not employ the Indian method for solving this problem, instead he "uses a version of false position. He assumes the answer to be 63 days, since 63 is divisible by both 7 and 9. Thus in 63 days the lion will climb up 9 feet and fall down 7, for a net gain of 2 feet. By proportionality, then, to climb 50 feet, the lion will take 1575 days" [Katz 1993: 283–284].[23]

*Mixed quantities.* Going back to Bhāskara, his fifth problem concerns "mixed quantities" (*miśrarāśi*s). He adds that here also the same principle of proportion (*anupātābhāym*) applies. The problem is as follows:

22 Note the expression *śateśvara*, "lord of one hundred pieces of money," apparently the eighth-century version of "millionaire."

23 Katz adds: "By the way, Leonardo's answer is incorrect. At the end of 1571 days the lion will be only 8/63 of a foot from the top. On the next day he will reach the top."

> Eight bulls are tame and three are untame: thus it has been said about [a group of] bulls. Then in 1001, how many are tame and how many otherwise?
> Statement: tame 8, to be tamed 3, total tame and untame 1001.
> Now we set down the terms for the Rule of Three thus:
> tame and untame 11, tame 8, group of all 1001.
> Here the proposition (*vācoyukti*) is: if out of eleven bulls which are tame and untame, eight are tame, then out of 1001 how many are tame?
> The result is (1001 × 8 / 11 =) 728. Subtract this number from the total to get (1001 − 728 =) 273 untame bulls [*Āryabhaṭīya-Bhāskara*: 118–119].

This class of problems is styled *miśraka* "mixed" because one of the three terms is the sum of two different quantities; in the present case, the numbers of tame and untame bulls; and the problem involves finding these two numbers separately. However, the only mixed quantity which has any practical relevance would be the sum of the principal and the interest accrued at the end of a given period.

*Partnership.* The sixth problem, related to *prakṣepa-karaṇa* "partnership", is as follows:

> Five merchants in partnership with capitals amounting to 1 increased by 1 each time, earned a profit of 1000. Tell the share of each person.
> Statement: capitals 1, 2, 3, 4, 5 / profit 1000 /
> Computation (*karaṇa*) — one should formulate a series of statements of proportion, such as: From the sum of the capitals invested of 15, there accrues a profit of 1000. Then from the capital of 1 how much profit accrues; from the capital of 2 how much profit accrues, and so on.
> For the first proposition, the answer is 66 2/3; second 133 1/3; third 200; fourth 266 2/3; fifth 333 1/3 [*Āryabhaṭīya-Bhāskara*: 119].

The last example also deals with partnership, but here the shares of investment are expressed in fractions (*bhinna*):

> Merchants who invested 1/2, 1/3 and 1/8 earned a profit of 69. What are the individual shares?
>
> | Statement: | 1 | 1 | 1 | profit |
> |---|---|---|---|---|
> | | 2 | 3 | 8 | |
>
> By taking a common denominator for all these fractions, we get 12/24, 8/24 and 3/24. The denominators are of no account and therefore we take only the numerators 12, 8, 3. As in the previous case of partnership, take their sum 23. For this 23 capital invested (*prakṣepa*) the profit is 69, then how much for each individual; by the Rule of Three we get 36, 24, 9 [*Āryabhaṭīya-Bhāskara*: 119].

Bhāskara's examples thus cover the whole range where the direct Rule of Three can be applied. Later mathematicians like Śrīdhara and Mahāvīra created independent topics out of these variations, such as *gati-nivṛtti* "foreward and backward motion," *prakṣepa-karaṇa* "partnership" and several types of *miśraka* "mixed quantities," and formulated separate rules for their solution.

## Rules of Five, Seven or More Terms

Another set of important variations of the Rule of Three are the rules for five, seven and more terms, called respectively *Pañcarāśika*, *Saptarāśika*, *Navarāśika*, etc. In his commentary on the *Āryabhaṭīya*, Bhāskara argues that these are just special cases of the Rule of Three:

> Here Ācārya Āryabhaṭa had described the Rule of Three only. How are the well-known Rules of Five, etc., to be obtained? I say this: The Ācārya [Master] has described only the fundamentals of *anupāta* (proportion). All others such as the Rule of Five, etc., follow from that fundamental rule of proportion. How? The Rule of Five, etc., consists of combination of the Rule of Three ... In the Rule of Five there are two Rules of Three, in the Rule of Seven, three Rules of Three, in the Rule of Nine, four Rules of Three, and so on [*Āryabhaṭīya-Bhāskara*: 116; trans. Datta and Singh 1962, I: 211].

Bhāskara gives the following example for the Rule of Five:

> On 100 the interest in 1 month is 5. Then what is the interest on 20 for 6 months? Tell if you understood [Ārya]bhaṭa's mathematics [*Āryabhaṭīya-Bhāskara*: 119–120].

and explains the solution in the following words:

> Computation (*karaṇa*). First Rule of Three: 100, 5, 20; result 1 silver coin. Second Rule of Three: if the interest on 20 is 1, how much in 6 months? result 6 silver coins.

Bhāskara goes on to add:

> If the same computation is performed in one go, it becomes the Rule of Five. There also, we have two numerical quantities as argument (*pramāṇa-rāśis*), namely (100 and 1); 5 is the fruit; [we ask] how much on 20 in 6 months; thus 20 and 6 come under requisition (*icchārāśi*). As before, quantities of requisition (*icchārāśis*) are multiplied by the quantity of fruit (*phalarāśi*) and divided by the two quantities of argument (*pramāṇa-rāśis*). We get the same result as before: $(20 \times 6)(5) \div (100 \times 1) = 6$. This is just the Rule of Three set down in two different ways. As before, the denominators of fractions also are mutually transposed from division and multiplication.[24]

Bhāskara's second example reads thus:

> If on 20 and 1/2 in 1 and 1/5 month the interest is 1 and 1/3 silver, on 1/4 less 7 what is the interest in 6 and 1/10 months? [*Āryabhaṭīya-Bhāskara*: 121]

Bhāskara does not state all the steps, but he would probably proceed as follows:

24 Strangely enough, *Bakhshālī* does not treat the Rule of Five as a separate entity but employs two successive Rules of Three for solving problems with five terms; cf. [*Bakhshālī*: example 38 (X-21), p. 427]; it does the same, as will be shown below, also with the Inverse Rule of Three.

After converting all the quanties into regular fractions, set down the terms in a row:

41/2, 6/5, 4/3, 27/4, 61/10 I

Multiply the fruit with the two quantities of requisition and divide by the two quantities of argument:

$(4/3 \times 27/4 \times 61/10) \div (41/2 \times 6/5)$ II

Now the denominators of one group become the multipliers of the numerators of the other. Transpose these accordingly:

$(4 \times 27 \times 61 \times 2 \times 5) \div (41 \times 6 \times 3 \times 4 \times 10)$ III

The result then is 2 171/1226.

Brahmagupta condenses this process and tells us how to reach step III. This is achieved by: (i) arranging the given data in two columns, the first column containing the data concerning the argument, the second containing that concerning the requisition; (ii) transposing the two fruits; (iii) transposing the denominators. Then the product of the column with more numerous terms is divided by that of the column with the lesser number of terms. His rule reads as follows:

> In the case of uneven terms, from three up to eleven, transpose the fruit on both sides. The product of the more numerous terms on one side, divided by that of the fewer terms on the other, provides the answer. In all the fractions, transpose the denominators, in a like manner, on both sides.[25]

Thus, by reading Bhāskara and Brahmagupta together, we can see the rationale of this method. The texts do not expressly state that the quantities should be set down in two vertical columns. Gaṇeśa, however, says in his commentary on Bhāskara's *Līlāvatī* that the quantities on the argument side should be written one below the other; so also the quantities on the requisition side. This is tantamount to two vertical columns [*Buddhivilāsinī*: 76].

The manuscripts consistently set down the quantities in this manner in two columns; sometimes the quantities are enclosed in boxes with single or double borders. Though the extant manuscripts are not very old, they may be following a

25 [*Brāhmasphuṭasiddhānta* 12.11–12.12]:

*trairāśikādiṣu phalam viṣameṣv ekādaśānteṣu* //11//
*phalasaṅkramaṇam ubhayato bahurāśivadho 'lpavadhahṛto jñeyam* /
*sakaleṣv evam bhinneṣūbhayataś chedasaṅkramaṇam* //12//

Usually, there is only one fruit, which pertains to the argument side. Then why does Brahmagupta ask us to transpose the fruits? Because, there can also be problems where the fruits or the interests of the two sides are given and one is asked to compute the principal or the time on the requisition side. Again, when the fruit is transposed from the first column to the second, the second will have more numerous terms. Then why does he not say: divide the product of the second column by that of the first column? This is usually the case, but there can be instances of more terms in the first column. It must also be kept in mind that terms mean only the numerators and not the denominators.

continuous tradition, just as they do while setting down the terms in the Rule of Three in a horizontal sequence.[26]

Thus two methods seem to be prevalent for setting down the quantities: a horizontal one for the Rule of Three and a vertical double column for Rules of Five and others. However, it appears that the three terms of the *trairāśika* were occasionally written in a vertical column as well. In *Bakhshālī*, there are some 44 problems where the Rule of Three is applied [*Bakhshālī*: 421–429]. The terms are set down in horizontal rows in all the cases, except in one where they are set in a vertical column as shown below [*Bakhshālī*: 270; Maiti 1996: 3–4].

| |
|---|
| 1<br>*dramma* |
| 100<br>*trapusa* |
| 1<br>2 |

The problem here is, "One hundred pieces of tin are obtained for one *dramma*. How many are obtained for a half [*dramma*]." Note that here the denominations are also set down along with the numerical quantities.

Another case occurs in the recently published mathematical text *Caturacintāmaṇi* of Giridharabhaṭṭa (*fl.* 1587). In an example dealing with Rules of Three, Five and Seven, the terms in the three problems are arranged vertically: the terms of the Rule of Three in a single column, those of Five and Seven in two columns [*Caturacintāmaṇi*, verse 34, p. 142 (text), p. 164 (translation)].

> If six is [obtained] by means of five, what is [obtained] by means of eight? Or else, if [that is obtained] in one month, then what is [obtained] in ten [months]? If the result [is obtained] from three people, then what is [obtained] from five? say separately.

| Rule of Three | Rule of Five | | | Rule of Seven | | |
|---|---|---|---|---|---|---|
| 5 | month | 1 | 10 | people | 3 | 5 |
| 6 | | 5 | 8 | month | 1 | 10 |
| 8 | | 6 | | | 5 | 8 |
| | | | | | 6 | |

On the other hand, in his *Rāshikāt al-Hind*, al-Bīrūnī consistently uses vertical double columns for setting down the terms whether they are three or seventeen. For example, he sets down the three terms of the Rule of Three in the following manner:

$$\begin{array}{r|l} 15 & 5 \\ \hline & 3 \end{array},$$

26 Again, in some manuscripts, in the cell for the unknown quantity (*jñeya*) a zero is written, just as we write an *x* today. This zero is a mere symbol. It should not be confused with the quantity zero and used in the multiplication along with the other numbers in the column.

where the argument is 5, fruit is 3 and the requisition 15. He says expressly that the terms are arranged with two mutually intersecting lines. Surely, this arrangement also must have had an Indian prototype. It may be recalled that Brahmagupta prescribes vertical double columns for all rules with odd terms from three up to eleven. Some Indian manuscripts must have followed this custom, which al-Bīrūnī emulated.

Thus, whatever may have been the arrangement employed for the Rule of Three, the arrangement finally adopted for the Rule of Five and others was evolving in the seventh century. Writing in 629, Bhāskara was not fully aware of this; but writing in 628 Brahmagupta prescribes it. Therefore, it is quite certain that Brahmagupta himself must have invented this method of writing down in two columns. Later writers, such as the anonymous commentator on the *Pāṭīgaṇita*, designate these two columns clearly as *pramāṇa-rāśi-pakṣa* (the side containing quantities belonging to the argument) and *icchā-rāśi-pakṣa* (the side containing quantities belonging to the requisition).

But is it necessary to have a new rule and a new arrangement? Why cannot these problems be solved by a series of Rules of Three? To this, the commentary on the *Pāṭīgaṇita* again provides the answer:

> The Rule of Five, etc., can be solved by postulating several successive Rules of Three. But identifying in each case correctly the argument, fruit and requisition, requires certain logical ability, which the pupil might not have. Therefore this new rule.[27]

Besides the economy in time and space, there is another advantage which the commentary repeatedly emphasises while working out the problems. After arranging the terms in two columns, and transposing the fruit and the denominators, common factors in the two columns can be cancelled out more easily. Then computing the products in the two columns becomes that much simpler.

Although these problems may contain any odd number of terms, the writers usually go up to eleven terms only. Al-Bīrūnī states that he encountered in India problems containing more than eleven terms and gives problems containing up to seventeen terms in his treatise [Juschkewitsch 1964: 214]. However, the only example that I have yet found with more than eleven terms occurs in the *Gaṇitalatā*, composed by Vallabha in 1841:

> If 1 house, 5 cubits in width, 16 cubits long, with 2 storeys and 2 inner courts, is available at the rate of 3 *niṣkas* per month, how much money is needed for 4 houses, each 6 cubits 'vide and 18 cubits long, with 4 storeys and 3 inner courts for 5 months? [*Gaṇitalatā*: *Trairāśikakusuma*, verse 13 (f. 26v)]

The thirteen terms can be arranged in two vertical columns as shown below:

27 [*Pāṭīgaṇita*: 45]: *pañcarāśyādiphalam anekatrairāśika-karma-sādhyam / pramāṇaphalecchayā vyavasthayā cānayanasaraṇiḥ śiṣyasya durjñeya iti sūtrāntarārambhaḥ.*

| | |
|---|---|
| 1 | 4 |
| 5 | 6 |
| 16 | 18 |
| 2 | 4 |
| 2 | 3 |
| 1 | 5 |
| 3 | |

After transposing the fruit and cancelling out the common factors, what remain are 3, 9, 3, 3 in the right column. Their product 243 is the result.

We do not know much about the state of mathematics in the regional languages of India. While looking for Telugu manuscripts, I found an elaborate classification of the variations of the Rule of Three in one manuscript. Because of the disjointed nature of the manuscript, all the variations are not clear to me. Therefore, I shall not enumerate them now. But what is clear is a simpler method of solving the problems of Rule of Five, etc. Set down all the terms horizontally in a sequence. If there are $n$ terms, take the product of the last $(n + 1)/2$ and divide this by the product of the first $(n - 1)/2$ terms. Thus in the case of the Rule of Five, the product of the last three terms is divided by the product of the first two terms; or in the case of the Rule of Seven, the product of the fourth, fifth, sixth and seventh terms is divided by the product of the first, second, and third terms.[28] This may be illustrated by working out the previous example. First we set down all the thirteen terms in the proper sequence:

/1 / 5 / 16 / 2 / 2 / 1/ 3 / 4 / 6/ 18 / 4 / 3 / 5/

We divide the product of the last seven terms by the product of the first six terms:

$$(3 \times 4 \times 6 \times 18 \times 4 \times 3 \times 5) \div (1 \times 5 \times 16 \times 2 \times 2 \times 1)$$

This, in effect, is what Bhāskara I seemed to suggest, before Brahmagupta proposed the arrangement of two vertical columns. The Telugu solution then, is the ultimate stage of mechanical solution, especially if there are no fractions. Unfortunately, neither the date nor the author of this text fragment is known.

## Barter and other Types of Problems

Barter (*bhāṇḍapratibhāṇḍaka*) is treated as an extension (*atideśa*) of the Rule of Five,[29] so also several other types of problems. Mathematicians enjoyed formulating separate rules for solving these. Brahmagupta provides a special rule for barter:

28 In a manuscript owned by Mantri Gopalakrishna Murthy the rule reads thus in Telugu:

*kaḍarāsul avi nālgu kramamuto guṇiyiṃci*
*dravyamaṃdu beṃci dānikriṃda* /
*modalu rāsulu mūḍu mudam oppa guṇiyiṃci*
*pālubuccavaccu saptarāśī* //

29 [*Buddivilāsinī*: 83]: *atropapattis trairāśikadvayena* /

> In the barter of commodities, transposition of prices being first terms takes place; and the rest of the process is the same as above directed [*Brāhmasphuṭa-siddhānta* 12.13].

This can be demonstrated with the following example:

> If a hundred of mangoes be purchased for ten *paṇa*s, and of pomegranates for eight, how many pomegranates [should be exchanged] for twenty mangoes?
> Statement:

| | |
|---|---|
| 10 | 8 |
| 100 | 100 |
| 20 | |

After transposition of prices and also of the fruit,

| | |
|---|---|
| 8 | 10 |
| 100 | 100 |
| | 20 |

Result: 25 pomegranates.

## Inverse Rule of Three

While the elder Bhāskara merely states that the Inverse Rule of Three (*vyasta-trairāśika* or *viloma-trairāśika*) is the reverse of the direct rule, Brahmagupta is the first to spell out the rule in full, "In the inverse rule of three terms, the product of argument and fruit, being divided by the requisition, is the answer."[30]

The younger Bhāskara lays down where this inverse rule is to be employed:

> When there is diminution of fruit if there be increase of requisition, and increase of fruit if there be diminution of requisition, then the inverse rule of three is [employed]. For instance, when the value of living beings is regulated by their age; and in the case of gold, where the weight and touch are compared; or when heaps are subdivided; let the inverted rule of three terms be [used] [*Līlāvatī*: 77–78; trans. Colebrooke 1973: 34].

Of all these varieties of problems dealt with under the Inverse Rule of Three, those concerning the sale of women became notorious. Al-Bīrūnī mentions the following in his [*India* I: 313], "If the price of a harlot of 15 years be, e.g., 10 denars, how much will it be when she is 40 years old?"

Al-Bīrūnī's source must be similar to the one given by Pṛthūdaka in his commentary on the *Brāhmasphuṭasiddhānta*:

> If a sixteen year old wench, her voice sweet like that of a cuckoo and Saurus crane, dancing and chirping like a peacock, receives 600 coins, what would one of 25 years get? [*Brāhmasphuṭasiddhānta*: 769]

30 [*Brāhmasphuṭasiddhānta* 12.11]: *vyastatrairāśikaphalam icchābhaktaḥ pramānaphala-ghātaḥ*. Śrīdhara also gives a similar definition, substituting, however, argument, etc., by first, last and middle terms [*Pāṭīgaṇita*: Rule 44 cd].

Nārāyaṇa, the author of the *Gaṇitakaumudī*, has a more modest aim; he does not compute the price for the purchase, but only wants to know the fee for one night:

> If a woman of sixteen, with agreeable gestures, wit and coquetry, gets a fee (*bhāṭi*) of ten *niṣkas*, then tell me quickly, how much should the customer give to one of twenty years? [*Gaṇitakaumudī*, I: 49]

As Bhāskara noted, the price of living beings, be they slaves or draught animals, is inversely proportional to their age. But the price does not increase indefinitely as the age decreases. There is the optimum age that receives the maximum price. In the case of female slaves to be employed for manual labour or for sexual enjoyment, sixteen appears to be the optimum age. Gaṇeśa Daivajña notes that, "A woman of sixteen reaches the optimum as regards her body and qualities. Therefore she receives the maximum price" [*Buddhivilāsinī*, Part I: 74].

One of the problems given by Śrīdhara under Inverse Rule of Three has five terms:

> Some quantity of yarn was used in weaving blankets of breadth 3 and length 9, the blankets thus woven are 200. With the same yarn how many blankets can be woven of breadth 2 and length 6 units [*Pāṭīgaṇita*: example 37 (p. 44)].

However, both Śrīdhara and the commentator treat this as a case of the Inverse Rule of Three; they first calculate the area of each type of carpet and put down these areas under argument and requisition. Thus states the commentary:

> Here by the multiplication of the breadth and length, the area is known. Thus the product of 3 and 9 is 27. The product of 2 and 6 is 12. The area measure of 27 is the given quantity (*jñātameyasaṃkhyā*). Therefore it is the argument. The quantity 12 is the requisition by inversion. The known number is the middle term. Statement: 27 / 200 / 12. By proceeding according to the given rule, the result obtained is 450.[31]

Mahāvīra envisages, besides the Inverse Rule of Three (*vyasta-trairāśika*), also the Inverse Rule of Five (*vyasta-pañca-rāśika*), the Inverse Rule of Seven (*vyasta-sapta-rāśika*), and the Inverse Rule of Nine (*vyasta-nava-rāśika*), but does not explain how these have to be worked out. Probably he would solve these in the same way as Inverse Rule of Three. Since the additional terms are attributes to either the argument or to the requisition and are directly proportional to the same, the various terms under the argument are multiplied and their product is treated as the argument in the Inverse Rule of Three; the same is done with the terms under the requisition; so that finally, there are only three terms, which are then dealt with according to the Inverse Rule of Three. We consider his problem on the Inverse Rule of Seven:

31 [*Pāṭīgaṇita*, Commentary: 44]:
*atra viṣkambhāyāmayor vadhān mānaparicchedāḥ / tena trikanavavadhaḥ saptaviṃśatiḥ dvikaṣaṭkavadho dvādaśa / saptaviṃśatimānasya jñātameyasaṃkhyatvād pramāṇatvaṃ dvādaśakasya viparyāsād icchātvaṃ jñātā saṃkhyā madhyamo rāśiḥ /*

> Out of a gigantic ruby, measuring 4, 9, 8 cubits respectively in length, breadth and height, how many icons can be carved of Tīrthaṅkaras, each measuring 2, 6, 1 cubits respectively in length, breadth and height? [*Gaṇitasārasaṃgraha* 5.21]

Probably, Mahāvīra would set down the numbers thus:

| | | |
|---|---|---|
| 4, 9, 8 | 1 | 2, 6, 1 |

Take the product of each group and proceed as in the Inverse Rule of Three:

$$(4 \times 9 \times 8) \times 1 \div (2 \times 6 \times 1) = 288 \div 12 = 24.$$

We have seen that Rules of Five, Seven, etc., do not appear in the *Bakhshālī Manuscript*. This text does not seem to be aware of the Inverse Rule of Three either. The two problems that are available in this (somewhat mutilated) work are solved by employing the Rule of Three successively two times [*Bakhshālī*: 426–427, examples 35–38]. Let us take the following problem, "If one person can live on eight *drammas* for forty-two days, then how long can seventy persons live on the same amount of money?"

In the method of the Inverse Rule of Three, the three terms are 1 / 42 / 70 and the result is $1 \times 42 \div 70 = 3/5$ days. The *Bakhshālī Manuscript* solves it in two steps by the direct Rule of Three thus:

(i) If 1 person lives on 8 *drammas*, 70 persons live on how much?
The three terms are 1 / 8 / 70 and the result $70 \times 8 \div 1 = 560$ *drammas*.
(ii) If (70 persons can live) on 560 for 42 days, 8 *drammas* will last how many days?
The terms are 560 / 42 / 8 and the result $8 \times 42 \div 560 = 3/5$ days.

This is interesting because this is precisely the procedure adopted in the commentaries of other texts for verifying the results of extended problems with five terms such as barter, mixtures and the like.[32]

## Rule of Three in the Islamic World

About the transmission of the Rule of Three to the Islamic world and thence to Europe, I have nothing to add, but I will summarise the little I know and raise one or two questions. We have seen that by the time of Brahmagupta in the early seventh century, the Rule of Three and its variations reached full development. In the next century, various elements of Indian mathematics and astronomy were disseminated to the Islamic world. The Rule of Three seems to be one of these elements thus transmitted. From the ninth century onwards, Arab mathematicians began to discuss the Rule of Three and other variants [Ansari & Hussain 1994: 223].

Thus al-Khwārizmī (ca. 850) discusses the Rule of Three in his book on Algebra. This treatise contains a small chapter on commerical problems including the

32 See, for example, [*Buddhivilāsinī*: 83 (barter), 91 (cistern problem), etc.].

simple Rule of Three according to the Indian model [Juschkewitsch 1964: 204]. We have noted al-Bīrūnī's (973–1048) reference to the Inverse Rule of Three. He also composed an exclusive tract on the Rule of Three entitled *Fī Rāshikāt al-Hind* [*al-Bīrūnī*]. Here he discusses direct and inverse Rule of Three as well as the rules for five, seven and more terms up to seventeen [Juschkewitsch 1964: 214].

But I do not quite know whether the Arab texts discuss all the Indian variations. Regarding the method of setting down the terms, al-Bīrūnī arranged them in two vertical columns, even for the Rule of Three. But this vertical arrangement did not reach Europe, where emphasis is laid on the horizontal arrangement of the three terms as in Indian *Trairāśika*. So there must be at least onc Arabic sourcc which retained the Indian system of arranging the three terms in a horizontal row and transmitted it to the West. Again, European sources prescribe a method of verification for the Rule of Three that is not found in Sanskrit texts. Do the Arabic texts have such a system?

## Rule of Three in Europe

The story of the dissemination of Indian numerals as well as commercial problems to Europe, especially through the mediation of Leonardo of Pisa, is well documented as is the praise of the Rule of Three as the Golden Rule [Tropfke 1930: 190–191; Juschkewitsch 1964: 120; Smith 1925: 486 ff.; Wagner 1988: 181]. The Inverse Rule of Three was also known in Europe [Smith 1925: 490–491], but other variations such as the Rules of Five, Seven, etc., do not seem to have enjoyed as much popularity as the direct Rule of Three. Even when these variations did occur, they were not solved in the way they were solved in India. Recall the lion in the pit problem and its solution by Leonardo.

The only original source that I had access to, physically and linguistically, is the *Bamberger Rechenbuch* (1483) attributed to Ulrich Wagner. Three points struck me in this book as remarkable: (i) Stress is laid on the linear arrangement of the three terms;[33] (ii) If one of the terms is unity, it is expressely stated not to multiply with it or divide by it as the case may be;[34] (iii) Verification of the Rule of Three: Interchange the positions of the argument and requisition, set down the result in the middle, and apply the Rule of Three. The new result must be the same as the old fruit.[35] That is to say, set down C, D, A; A × D ÷ C = B.

33 [*Bamberger Rechenbuch*: 181–182]: "Und sind es drei Dinge, die du setzt. Darunter muss das erste und letzte allemal gleich sein. Und zur letzten sollst du setzen was du wissen willst. Dasselbe und das mittlere sollst du miteinander multiplizieren und durch das erste teilen."

34 [*Bamberger Rechenbuch*: 183]: "...wenn 1 am letzten steht, so multipliziere nicht, ... Wenn aber 1 an der ersten Stelle steht, so teils nicht in das erste, ..."

35 [*Bamberger Rechenbuch*: 187]: "Willst du probieren, was du mit der Goldnen Regel gemacht hast, so kehre es um. Also, was du an der ersten Stelle gehabt hast, setze an die dritte. Und was an der dritten Stelle gestanden ist, setze an die erste. Und was kostet, an die Mitte. Und dann mache es nach der oben mitgeteilten Regel. Und es muss gerade soviel kommen, wie vorher in der Mitte gestanden ist."

## Conclusion

We have seen that in the Rule of Three and other rules, it is the mechanical arrangement of the given quantities that is important, as is also the equally mechanical process of solution. Thus in India, the terms in the Rule of Three are generally written in a horizontal row, while those in the Rules of Five, etc., are set down in two vertical columns. It is possible to conceive of an intermediate stage when the terms of the Rule of Three were also written in two vertical columns as in the Rule of Five, etc.

It was probably Brahmagupta who introduced the custom of writing the terms in two vertical columns. One can argue that Brahmagupta employs both styles of writing. In [*Brāhmasphuṭasiddhānta* 12.10] the terms of the Rule of Three are to be set down in a horizontal sequence, so also the terms of the Inverse Rule of Three in [*Brāhmasphuṭasiddhānta* 12.11ab]. According to [*Brāhmasphuṭasiddhānta* 12.11cd–12], however, the terms are to be set down in two columns for all rules with odd terms, from three to eleven.

Thus, for the following sum there are many ways of setting down terms: If 5 mangoes cost 10 rupees, how much do 8 mangoes cost?

| | | |
|---|---|---|
| 5 | 10 | 8 |

| |
|---|
| 5 |
| 10 |
| 8 |

| | |
|---|---|
| 5 | 8 |
| 10 | |

| | |
|---|---|
| 5 | 8 |
| 10 | 0 |

In spite of the high encomiums it received in India and Europe, the Rule of Three, or more specifically the method of solution proposed in India for the Rule of Three, does not seem to be in use anywhere. In Europe, it was dropped from the school books in the last century. Possibly it was dropped in India also in the last century when modern mathematics was introduced here.

In the north Indian schools, I am told, the problems related to the Rule of Three are solved today by what is known as "One-One-Rule" (*ek-ek-niyam*). What it is will be clear from the following example:

| | |
|---|---|
| 5 mangoes cost | 10 rupees |
| 1 mango | 10/5 rupees |
| 8 mangoes | $10/5 \times 8 = 16$ rupees. |

It is, of course, the more logical method. But when I went to school in south India, more than half a century ago, we skipped the middle line. We first wrote the proposition thus:

| | | |
|---|---|---|
| 5 mangoes cost | $\leftarrow$ | 10 rupees |
| 8 mangoes | $\nearrow$ | ? |

The answer, we were taught, is obtained by multiplying the 8 by the second term in the first line and then dividing the product by the first term in the first line. One drew an arrow diagonally from 8 to 10 and another arrow horizontally from 10 to 5. It was thus a compromise between the one line system of old and the three line system of modern times. I still practice this method. I must, however, confess that until now I never asked where it came from.

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# A Homecoming Stranger: Transmission of the Method of Double False Position and the Story of Hiero's Crown

by LIU DUN

The two major subjects of this article have no direct relation to one another, but they are both interesting examples of the transmission of mathematics. The earliest extant reference of the *Yingbuzu* (Method of Double False Position) occurs in Chapter 7 of the *Jiuzhang suanshu* (Nine Chapters on the Mathematical Art, ca. A.D. 50). Subsequently, this method was transmitted to Europe by Muslim mathematicians in medieval times. When Jesuits introduced Western knowledge of mathematics to China in the early seventeenth century, they incorrectly claimed that the method was actually discovered by Western mathematicians, and was not to be found in the "old text" of the *Nine Chapters*. On the other hand, a famous story about Hiero's crown was transmitted from the West to China, along with problems illustrating the so-called "Golden Method," a Renaissance European nickname for the Method of Double False Position.

## 1

The *Yingbuzu* (Method of Double False Position) first appeared in China in the ancient mathematical classic text, the *Jiuzhang suanshu* (Nine Chapters on the Mathematical Art, ca. A.D. 50). The method then spread from China into central Asia and then to Europe between the 9[th] and 13[th] centuries. Among recent studies concerning the nature of this method and the context of its transmission, the following three contributions to the subject are worthy of special attention:

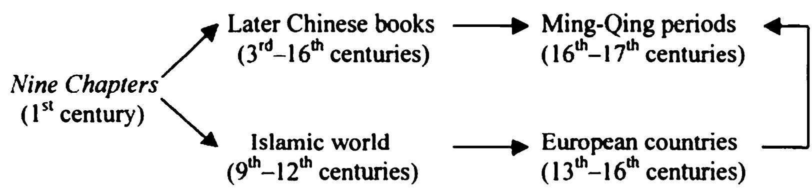
1.) *Jiuzhang suanshu yingbuzushu liuchuan ouzhou kao* 九章算术盈不足术流传欧洲考 (Research on the Transmission of the Double False Position Method from China to Europe) [Qian Baocong 1927].
This was the first comprehensive study on the *Yingbuzu* method, its origin, development, and transmission from East to West. Qian divided the 20 problems in the chapter entitled *Yingbuzu* of the *Nine Chapters* into two types: the first eight are directly indicated as belonging to type one and deal with "excess" and "deficit" (*ying* meaning excess, or "too much" and *buzu* meaning deficit, or "not enough"); the others belong to type two, in which the problems do not obviously deal with "excess" and "deficit." Meanwhile, Qian points out that the *Yingbuzu* method is an equivalent of linear interpolation.

2.) *Shilun songyuan shiqi zhongguo he yisilan guojia jian de shuxue jiaoliu* 试论宋元时期中国和伊斯兰国家间的数学交流 (On the Exchange of Mathematics between China and Islamic Countries During the Song-Yuan Periods) [Du Shiran 1966].

Based upon Qian's work and the studies of some Russian historians of mathematics (A. P. Yushkevich and M. Y. Vygodskii), Du's contribution provides evidence of the transmission of the *Yingbuzu* method from China to the Islamic world.

3.) "Reflections on the World-Wide History of the Rule of False Double Position, or: How a Loop Was Closed" [Chemla 1997].
Chemla divides the 20 problems of the *Yingbuzu* chapter into two kinds, according to the procedures they employ. The first group consists of problems in which procedures I and II are applied, while the second group of problems apply procedures I and III. Chemla argues that distinguishing between the two kinds of problems not only makes it possible to understand the subtlety of the *Nine Chapters*, but it also offers a means of describing their transmission. Moreover, she also investigates the so-called "closed loop" course by introducing the relevant references.

As a result of these three studies, we now have a good overview of the transmission of the Method of Double False Position, although at present the links of transmission are still rather vague. One plausible route of transmission is the following:




The purpose of this presentation is neither to reinforce evidence for transmission by adding new data, nor to repeat the works of mathematicians such as Qusṭā ibn Lūqā (9th century), Abū 'l-Faraj al-Nadīm (10th century), al-Ḥaṣṣār (12th century), and Leonardo Fibonacci (1170?–1250), which have already been discussed at length in previous studies; instead, the special focus here will be upon the significance of distinction drawn by Qian Baocong between the two types of problem, and how the general Method of Double False Position was transformed and transmitted in the later periods.

## II

The first type of problem categorized by Qian Baocong is typified by Problem 1 in the *Yingbuzu* chapter of the *Nine Chapters*:

> A number of people are going to purchase goods. If everyone pays 8 [$A_1$], there is an excess of 3 [$B_1$]; if everyone pays 7 [$A_2$], there is a deficit of 4 [$B_2$]. How many people are involved and how much do the goods cost? [Qian Baocong 1963, 1: 205]

The solution procedure as given in the *Nine Chapters* is as follows:

(1) Put the sums [supposed to be paid] on the counting board, i.e.,

$$A_1 \qquad A_2$$

(2) Put the corresponding excess and deficit below the sums respectively, i.e.,

$$\begin{array}{cc} A_1 & A_2 \\ B_1 & B_2 \end{array}$$

(3) Cross-multiply them and then add the results together to get the dividend,

$$\begin{array}{cc} A_1 & A_2 \\ B_1 & B_2 \end{array}$$

i.e., $(A_1B_2 + B_1A_2)$

(4) Add excess and deficit to get the divisor, i.e.,

$$B_2 + B_1$$

(5) Perform the division (to get the amount of money each person ought to pay), i.e.,

$$(A_1B_2 + B_1A_2) / (B_2 + B_1)$$

(6) Subtract the lesser sum (of money) from the larger to get the divisor, i.e.,

$$A_1 - A_2$$

(7) Divide the previous "dividend" and "divisor" to get the price of goods (Y), and the number of people involved (X) respectively, i.e.,

$$X = (A_1B_2 + B_1A_2) / (A_1 - A_2)$$
$$Y = (B_2 + B_1) / (A_1 - A_2)$$

In Problem 1 in the *Yingbuzu* chapter, the number of people involved is thus found as $(3 + 4)/(8 - 7) = 7$; and the price of the goods as: $(8 \times 4) + (7 \times 3)/(8 - 7) = 53$.

The most sophisticated application of this method in the *Yingbuzu* chapter of the *Nine Chapters* involves the solution of problems which do not contain the data of excess and deficit, i.e., problems of the second type according to Qian Baocong's categorization. Take Problem 16 as an example:

> Assume that 1 cubic *cun* of jade weighs 7 *liang*, and 1 cubic *cun* of stone weighs 6 *liang*; there are $3^3$ cubic *cun* of mineral composed of jade and stone which weigh 11 *jin* [i.e., 176 *liang*], what is the proportion of jade and stone? [Qian Baocong 1963: 214]

The secret to solving a problem of this kind is to "construct" the data of excess and deficit by making two assumptions. For instance, in the concrete case of Problem 16, we first suppose this $3^3$ cubic *cun* to be composed of pure jade, in which case its weight would be: $3^3 \times 7 = 189$ *liang*; comparing this number with the given one, we get an "excess": $189 - 176 = 13$. Next, we suppose the $3^3$ cubic *cun* to be composed of pure stone, in which case its weight would be: $3^3 \times 6 = 162$, with a corresponding "deficit" of $176 - 162 = 14$. Consequently, the original problem has been transformed as follows:

> A certain number of people is going to purchase a certain amount of goods. If everyone pays 7 [$A_1$], there is an excess of 13 [$B_1$]; if everyone pays 6 [$A_2$], there is a deficit of 14 [$B_2$]. How much should each person pay?

It is easy to find appropriate answers by applying the procedures discussed above as type one. Here the volume of jade (or stone) is treated as in the problem dealing with what "each person ought to pay," according to formula (5), as given above, i.e.,

volume of jade: $(27 \times 14 + 0 \times 13) / (13 + 14) = 14$ cubic *cun*,

or,

volume of stone: $(0 \times 14 + 27 \times 13) / (13 + 14) = 13$ cubic *cun*.

However, this kind of variation makes it possible to apply the *Yingbuzu* method efficiently to a great extent. As a matter of fact, the general method can be used to solve all linear problems, although it is only an approximation method when used in a non-linear situation.[1]

## III

In addition to those in the *Nine Chapters*, problems involving the *Yingbuzu* method also appear in later Chinese mathematical books, which include the *Sunzi suanjing* (Master Sun's Mathematical Classic, $4^{th}$–$5^{th}$ centuries), the *Zhang Qiujian suanjing* (Zhang Qiujian's Mathematical Classic, $5^{th}$ century), and in various writings in the Song-Yuan periods (ca. $13^{th}$–$14^{th}$ centuries).

Unfortunately, no description of this method as systematic and comprehensive as that in the *Nine Chapters* appears in later writings. In particular, problems of the second type seem to have been forgotten entirely by mathematicians of the Ming dynasty (1368–1644). For example, although Cheng Dawei (1533–1606) devoted a specific chapter to this topic, the ingenious applications of type two are absent from his *Suanfa tongzong* (Systematic Treatise on Algorithms, 1592). In fact, scholars of

1 Martzloff introduces this method as "Linear Interpolation" in a chapter titled "Approximation Formulae"; see [Martzloff 1997: 336–338].

the Ming dynasty had only a superficial knowledge of earlier mathematics — most important mathematical works, along with their outstanding achievements, were virtually unknown, and some had even been lost [Liu Dun 1994].

However, by the end of the $16^{th}$ century, Western mathematical knowledge was being introduced into China by the Jesuits. In collaboration with Li Zhizao (1565–1630), Matteo Ricci (1552–1610) translated the *Epitome Arithmeticae Practicae* into Chinese, which was written and published by his former teacher Christopher Clavius (1537–1612) in 1583. Three years after Ricci's death Li Zhizao published the Chinese version under the title *Tongwen suanzhi* (A Guide to Arithmetic in Common Language, 1613), which is considered to be the first book introducing Western arithmetic into China.

In volume 4 of the *Tongwen suanzhi tongbian*, there is a section called *Diejie huzheng*, which means "Doubly Borrow and Mutually Compare," probably a literary translation of the "Method of Double False Position."[2] However, Li Zhizao not only presents the treatment of this method due to Clavius which he translated with Matteo Ricci, but he also includes those *Yingbuzu* problems which he called *jiufa*, namely the "old method" found in the *Suanfa tongzong*.

Obviously, Li Zhizao had no direct access to the *Nine Chapters* and he believed that the *jiufa* only explicitly gave data of excess and deficit, whereas the Western method was more efficient. As he concludes:

> In the past there was a chapter called *Yingnü*, in which most (problems) were of this kind. Nevertheless this (new method in our book) borrows (the false) numbers in advance, then deduces the excess and deficit, after which we get the unknowns. Therefore it is called "Borrow and Compare." In regard to those problems directly providing excess and/or deficit, it can be solved by only using the old method. [Li Zhizao, 4: 165]

Of course Li Zhizao was wrong. For example, consider a problem of "Western" character in his *Tongwen suanzhi*:

> A golden stove was ordered to be made from 100 *jin* of gold. After it was done, (someone) suspected that the goldsmith had stolen some of the gold and replaced it with (the same quantity of) silver. Without damaging the stove, how is it possible to know how much silver had been used? [Li Zhizao, 4: 177]

In fact, this problem is the same as Problem 16 in the *Yingbuzu* chapter of the *Nine Chapters*, the only difference being that "gold" and "silver" replace "jade" and "stone," respectively.

The solution relies on the following prerequisite: as the objects were entirely immersed in water, 100 *jin* of gold, of silver, and of the stove made by the goldsmith, water would be displaced weighing respectively 60, 90, and 65 *jin*.

2 Li explains the phrase *Diejie huzheng* as "borrow the false to obtain the true, the method is marvellous ...... borrow the false twice and then compare with the true, there are differences." In Chinese: 借虚征实，其术精矣…两借虚数以征之，征之于实尚远也.

Suppose 40 *jin* of gold was replaced by silver, then 60 *jin* of gold remained. According to the above prerequisite, the displaced water would be:

$$60 \times (60/100) + 40 \times (90/100) = 72\,jin.$$

Comparing the number with 65 results in an excess of 7.

Now suppose 30 *jin* of gold was replaced by silver, then 70 *jin* of gold remained. According to the same prerequisite: the displaced water would be:

$$70 \times (60/100) + 30 \times (90/100) = 69\,jin.$$

Comparing this number with 65 results in an excess of 4.

Therefore the problem has been transformed into "a number of people are going to purchase goods":

> A certain number of people are going to purchase a certain amount of goods. If everyone pays 40 [$A_1$], there is an excess of 7 [$B_1$]; if everyone pays 30 [$A_2$], there is an excess of 4 [$B_2$]. How much should each person pay?

According to formula (5) as given above, "the amount of money each person ought to pay" is:

$$[(40 \times 4) + (30 \times -7)] / [-7 + 4] = 50/3.$$

In fact, this is the quantity of gold (unit *jin*) that has been replaced by silver in the original problem.

## IV

This problem and its solution in the *Tongwen suanzhi* depends upon knowledge of floating bodies, and specifically the well known Principle of Archimedes. This in turn raises the closely related question of the famous story of Hiero's crown.

The story is first known from Marcus Vitruvius, the Roman architect and engineer of the first century before the Christian era. In the Preface to Book IX of his *De Architectura Libri X* (ca. 20 B.C.), Vitruvius explains how Archimedes discovered the principle of floating bodies when he happened to sit in his bath and subsequently considered the displacement of water, which led him to the principle whereby he could solve the problem given him by the King of Syracuse, Hiero, namely to determine whether the king's crown of gold was pure or whether he had been cheated by the goldsmith [Vitruvius 1955, 2: 202–205].

The story is even more vividly described in a Chinese book with the title *Ouluoba xijing lu* (The Western Mirror in Europe):

> The king ordered a goldsmith to cast a tripod using 100 [*jin* of] gold. The goldsmith withheld a certain amount of gold and mixed the rest with silver. When the tripod was completed, it was presented to the king. On finding the color quite dull, the king instructed a person named *Ya-Er-Ri-Bai-La*, who was skilled in mathematics and astronomy, to figure out how much gold had been stolen. ...

> *Ya-Er-Ri-Bai-La* was ordered to determine the truth. At first he could not think of a way to do so, although he thought carefully about this from many different points of view. One day when he went for a bath, as he watched the water rise and eventually overflow from the pool, how to solve the problem suddenly came to him in a flash, and he was so overjoyed by this that without realizing it he ran naked all the way home. [*Ouluoba xijing lu*, 4: 302]

Up to this point, the basic core of the Vitruvian account of Archimedes and Hiero's crown has been faithfully introduced to China — except for two minor details: the one that Archimedes becomes *Ya-Er-Ri-Bai-La*, and the second, that the crown seems inexplicably to have been changed into a gold tripod. But the tripod, rather than a gold crown, is the appropriate symbol of imperial authority in ancient China, and thus the translation while not literal is in fact comparable, exactly in keeping with the Chinese context.

Similarly, while at first there may seem to be no connection between Archimedes and *Ya-Er-Ri-Bai-La*, the latter may in fact be some kind of transliteration of the word *algebra*. This could have happened possibly in two ways. It may have been that the story was dictated by a Westerner and written down by a Chinese copyist. This was a standard method of translating foreign works into Chinese in the $16^{th}$–$17^{th}$ centuries, and in this way "Archimedes" may have been replaced by "*Ya-Er-Ri-Bai-La*." It is also possible that the name *Ya-Er-Ri-Bai-La* was deliberately created with an intentional subtle or hidden meaning, just as the ancient Chinese practice used the words *Shao Gao* (which literally means "Discuss the Height") for the name of a great geometer in the age of *Zhougong*.[3]

Altogether there are nine problems in the *Ouluoba xijing lu*, including the problem of the gold tripod, all of which are categorized as belonging to the *Shuangfa* (Double Method). Here *Shuangfa* is clearly a direct translation of "Method of Double False Position."

The fact that the data given and the solutions of the nine problems in the *Ouluoba xijing lu* are exactly the same as those to be found in the *Tongwen suanzhi*, makes it natural to suppose that the two books are directly related. Unfortunately, even now, it is not known who the author(s) of the *Ouluoba xijing lu* may have been.

## V

Although its authorship remains unknown, the *Ouluoba xijing lu* enjoyed a wide circulation in China during the Ming-Qing periods. As a mathematical work, its title refers directly to Western learning, introduced in the late $16^{th}$ century when the Jesuits first arrived in China. As the Preface says:

> The translation of Matteo Ricci involves division and multiplication, extraction of square and cubic roots, surveying, the *Gougu* (or Pythagorean) theorem, and the

3 See *Zhoubi suanjing* (Mathematical Classic of the Zhou Gnomon, ca. 50 B.C.) in [Qian Baocong 1963, 1: 13–22].

> golden method, etc. All are extremely concise and easy to learn. Moreover, they only rely on pen and paper instead of using counting rods. No matter whether it is the height of mountains and towers, the depth and width of wells, valleys, rivers and marshes, the distance of places, length of cotton and silk, amount of rice and millet, or weights and prices, all can be deduced. What [Ricci introduces] can even compare favorably with the various methods in our Chinese *Nine Chapters.* It is really a south-pointing carriage and a mirror of the Western learning for mathematicians. [*Ouluoba xijing lu*, 4: 281]

Here the reference to "our Chinese" suggests the writer of the Preface must be Chinese. On the other hand, from the transformed story of Hiero's crown and the transliteration *Ya-Er-Ri-Bai-La*, there can be no doubt that there must been have a foreigner involved in the composition of the book. Thus it is all the more likely a collaborative effort, just like the earliest Chinese version of Euclid's *Elements* (1607), which was translated by Matteo Ricci with the help of Xu Guangqi (1562–1633). Needless to say, the *Ouluoba xijing lu* is a valuable source for examining the exchange of mathematical knowledge between China and the West during the late Ming and early Qing dynasties.

This book has been largely ignored since the middle of the Qing dynasty. In 1946, the late historian of mathematics Yan Dunjie (1917–1988) wrote a paper entitled *Xijing lu mingqiu* (Searching for the *Xijing lu* in the Dark), in which an outline of the book is sketched on the basis of records from various Qing scholars, although Yan at the time had no idea whether there was any copy of the book still in existence or not [Yan Dunjie 1946: 25]. Four years later in 1950, he in fact found a manuscript version of the *Ouluoba xijing lu* in the Beijing University Library. This transcript, to which was attached a note written by Qian Daxin (1728–1804), had been copied by Jiao Xun (1763–1820). Both Jiao and Qian were great masters of the *Qian-Jia* School, which sought to recover from ancient Chinese classic mathematical texts results and methods comparable to those in Western resources. From the notes of Jiao Xun and Qian Daxin it is clear that another two celebrated Chinese mathematicians, Mei Wending (1633–1721) and Li Rui (1769–1817) also studied the *Ouluoba xijing lu* [Liu Dun 1993]. In fact, Mei Wending wrote an essay on *Xijing lu dingbu* (A Commentary on the *Xijing lu*). Subsequently, Li Rui purchased a copy of the *Ouluoba xijing lu* which seems to have been the one that Mei owned. Jiao Xun later borrowed this copy from Li Rui in order to make another copy, and it is this copy that is the only one at this point known to have survived, namely the unique copy preserved in the Beijing University Library.

Devised at some time prior to when the *Nine Chapters* was compiled, the *Yingbuzu* method had survived a long journey. The loop was eventually closed in the Ming-Qing period, as Karine Chemla pointed out. In the opinion of some late-Ming scholars such as Li Zhizao, the general method looks like a newcomer, a stranger. But this is not true. As He Zhizhang (659–744), a famous Tang poet, wrote:

I left home when I was a lad;
    Now old, and home's again in sight.
I have my native tongue as I had,
    But my temple hair is sparse and white,
Confronted by the children small,
    I am a stranger to the place.
"And where do you come from at all?"
    One asks with a brightly smiling face [He Zhizhang 1997: 32].

But this time the homecoming stranger also brought a mathematical companion, who could tell the story of Hiero's crown.

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## Glossary of Chinese Text

### *Persons*

Cheng Dawei 程大位
Du Shiran 杜石然
Guo Shuchun 郭书春
He Zhizhang 贺知章
Jiao Xun 焦循
Li Rui 李锐
Li Zhizao 李之藻
Liu Dun 刘钝
Mei Wending 梅文鼎
Qian Baocong 钱宝琮
Qian Daxin 钱大昕
Shao Gao 商高
Xu Guangqi 徐光启
Ya-Er-Ri-Bai-La 亚尔日白腊
Yan Dunjie 严敦杰
Zhougong (Duke of Zhou) 周公

### *Titles of Works, Chapters etc.*

*Diejie huzheng* ("Doubly Borrow and Mutually compare") 迭借互征
*Jiuzhang suanshu* (Nine Chapters on the Mathematical Art) 九章算术
*[Ouluoba] xijing lu* (The Western Mirror [in Europe]) 欧罗巴西镜录 or 西镜录
*Suanfa tongzong* (Systematic Treatise on Algorithms) 算法统宗
*Sunzi suanjing* (Master Sun's Mathematical Classic) 孙子算经
*Tongwen suanzhi* (A Guide to Arithmetic in Common Language) 同文算指
*Tongwen suanzhi tongbian* (A Guide to Arithmetic in Common Language, General Section) 同文算指通编
*Xijing lu mingqiu* (Searching for the *Xijing lu* in the Dark) 西镜录冥求
*Xijing lu dingbu* (A Commentary on the *Xijing lu*) 西镜录订补
*Zhangqiujian suanjing* (Zhang Qiujian's Mathematical Classic) 张丘建算经
*Zhoubi suanjing* (Mathematical Classic of the Zhou Gnomon) 周髀算经

### *Terminology*

*cun* (measure of length) 寸
*Gougu* (Pythagorean theorem) 勾股定理
*liang* (measure of weight) 两
*Jiufa* ("Old method") 旧法
*Jin* (measure of weight) 斤
*Qian-Jia* School 钱嘉学派
*Shuangfa* ("Double Method") 双法
*Xixue* (Western Learning) 西学
*Yingbuzu* ("Method of Double False Position") 盈不足
*Yingnü* ("Excess and deficit") 盈朒

# Use and Transmission of Iterative Approximations in India and the Islamic World

by KIM PLOFKER

Iterative approximation methods are one of the computational staples of the medieval Indian mathematical tradition, particularly in mathematical astronomy. Yet up to this point, there have been few attempts to survey this class of techniques as a whole within the Indian texts, or to trace their connections to similar mathematical tools in other traditions. This paper describes the kinds of iterative algorithms that Indian mathematicians employed and discusses specific examples of each, as well as the overall development of their use. In addition, it mentions some highlights of the history of iterations in other ancient and medieval cultures, and their possible relationships to Indian techniques.

## Introduction

While mathematics in general is usually described as logically precise, rigorously demonstrable, and universally applicable, mathematicians have frequently availed themselves of approximation techniques whose behavior they cannot really understand or prove, and which they sometimes arbitrarily restrict to certain types of problems. These peculiarities make approximation methods a very useful tool for studying the history of mathematical transmission: whereas demonstrably exact rules from one mathematical tradition can be readily absorbed into another so that they soon look like indigenous developments, special tricks for approximating solutions often preserve more traces of their alien origin. Unfortunately, the major historical study of this subject at present [Goldstine 1977] does not pursue it further into the past than the sixteenth century, or beyond the boundaries of Europe; and studies that do look at earlier developments tend not to delve into cross-cultural transmission.[1] This paper will attempt to sketch part of the history of the subset of approximation methods called iterative approximations, as they were known in the mathematical science of medieval India and the Islamic world.

## Mathematical Overview of Iterative Approximations

An iterative technique approximates an exact solution to the desired degree of precision by first applying a given function to an initial estimate of the answer, and then reapplying the same function to the result of the first application, and so on indefinitely. Not all repetitive algorithms — for example, continued fractions or the

1 [Rashed 1994] devotes a chapter to Islamic numerical analysis, as does [al-Daffa & Stroyls 1984].

method of exhaustion — necessarily fall into this category; it is limited here to the sort of techniques specifically identified in Sanskrit texts as "asakṛt" ("not just once," iterative). The following discussions of these techniques in terms of modern mathematics divide them for convenience into "fixed-point" and "two-point" techniques, a distinction not recognized by medieval mathematicians.

*Fixed-Point Approximations.* First in chronology as well as importance are fixed-point iterations, in which the desired root of some given function $f$ is found by means of an auxiliary function $g$ which has a fixed or stationary point where $f$ has the root: in other words, $g(x) = x$ when $f(x) = 0$. Once the single initial estimate $x_0$ is chosen, each subsequent approximate value $x_k$ is given by

$$x_k = g(x_{k-1}). \tag{1}$$

The behavior of such functions does not seem to have been studied as a formal mathematical subject until well into the twentieth century. At present, of course, they form a prominent feature of the theory of dynamical systems, which defines certain aspects of their behavior as follows (see [Devaney 1992] for a typical discussion of the subject in more detail):

- The *orbit* of a seed $x_0$ under the iteration of $g$ is the set of all successive values $x_k$ resulting from that iteration.
- A fixed point $X$ of $g$ is an *attracting* fixed point if the slope of the function (assuming $g$ differentiable) at that point is greater than $-1$ and less than $+1$; i.e., $|g'(X)| < 1$. Then there will be an interval containing $X$ such that an iteration of $g$ beginning with any seed $x_0$ in that interval will converge to the fixed point. A fixed point is called *repelling* if $|g'(X)| > 1$, and it occurs in an interval within which iteration from any seed $x_0$ will diverge away from the fixed point. (A fixed point where $|g'(X)| = 1$ is called *neutral*.) The speed of an iteration's convergence to an attracting fixed point $X$ is inversely dependent upon $|g'(X)|$; the measure of that speed is called the *order of convergence*.

*Two-Point Approximations: Regula Falsi.* Another class of iterative techniques is the so-called "two-point" approximations, which require two initial estimates of the answer instead of one, and use linear interpolation between successive pairs of estimates to produce more accurate estimates. The most important method in this category is generally known as "Regula Falsi," which must not be confused with ancient non-iterative techniques with similar names, forerunners of algebraic procedures for solving equations in one unknown. In Regula Falsi, given two initial estimates $x_0, x_1$ of a root $x$ of $f$, the $(k+1)$th approximation to the root is given by

$$x_{k+1} = x_0 - \frac{f(x_0) \cdot (x_k - x_0)}{f(x_k) - f(x_0)}. \tag{2}$$

For any continuous $f$, as long as the initial estimates bracket the root $x$ (i.e., $f(x_0) \cdot f(x_1) < 0$), and the designations $x_0$ and $x_1$ are assigned to these points so that $x_0$

and $x_2$ also bracket the root, then all $x_k$ will approach progressively closer to the root, and the method will converge. The order of convergence for Regula Falsi is 1.[2]

*Two-Point Approximations: the Secant Method.* This iteration is almost identical to Regula Falsi, except that when $x_0$, $x_1$ are initially given, the $(k+1)$th approximation to $x$ is found from

$$x_{k+1} = x_k - \frac{f(x_k) \cdot (x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}. \tag{3}$$

In other words, each estimated value is derived by interpolation (or extrapolation) from the two previous estimates $x_k$ and $x_{k-1}$, not $x_k$ and one of the initial estimates $x_0$ or $x_1$. It can be shown that the method will always converge, if $x_0$ and $x_1$ are chosen sufficiently close to the root, with order $(1+\sqrt{5})/2$.

## Historical Overview of Ancient Iterations

Naturally, the above presentation bears very little relation to the pre-modern understanding of iteration, which is innocent of the notions of roots, derivatives, or graphical representation of functions. In this context, iteration is much more properly thought of simply as a repeated application of an algorithm to changing values of a wrong answer until gradually a fixed right answer emerges.

The oldest of the iterative techniques discussed here is undoubtedly fixed-point approximation. Possibly the earliest such method extant is an algorithm for finding square roots; some evidence for its use appears in Babylonian parameters[3] and a version of it is explicitly given by Heron [Heath 1921, II: 323], whose formula for successive approximations $x_k$ to the square root of a constant $C$ is

$$x_{k+1} = \frac{x_k + C/x_k}{2}. \tag{4}$$

Fitting this formula to our definitions of fixed-point iteration is easily done by means of a little algebraic manipulation confirming that $g(x_k) = x_{k+1}$ has a fixed point $x_{k+1} = x$ where $f(x) = x^2 - C$ has a root.

Other such algorithms are comparatively infrequent in the legacy of Greek mathematics. One example is the iterative method explicitly used by Ptolemy in *Almagest* X–XI to compute the eccentricities and apsidal lines of the superior planets.[4] In both

2 The non-iterative "Rule of (Single) False Position," on the other hand, merely uses direct proportion between values of a variable and some multiple of that variable. That is, where some (nonzero) value $f(x)$ of a function $f = mx$ is known, only an initial guess $x_0$ is required to find $x = (f(x) \cdot x_0)/f(x_0)$. Perhaps ultimately derived from this is the "Rule of Double False Position" for a root $x$ of a linear function $f$ by means of two guesses $x_0$ and $x_1$, whose result is identical to the right-hand side of (2) if $k = 1$; for higher-order functions, the method yields an approximate result.

3 See [Neugebauer & Sachs 1945: 43], but note the caveat in [Robson 1999: 44]; see also [Berggren 2002]. It has also been suggested [Heath 1921, II: 51] that Archimedes' value for the square root of 3 was obtained in this way.

4 [Toomer 1984]; the technique is discussed in [Neugebauer 1975, I: 174 ff.].

this and the square-root formula, the iterative functions are well behaved and converge quite rapidly; we have no evidence that this behavior was ever considered by those who exploited it to be mathematically interesting in itself. And it may be that Greek knowledge of or interest in iteration methods subsequently declined; at least, it appears that in the fourth century, the iterative nature of a fixed-point technique for constructing two mean proportionals (imperfectly applied by a pupil of Pandrosion) was not even recognized by Pappus (nor, presumably, by Pandrosion herself).[5] Certainly, such techniques could not have been very satisfying to the Greek inclination for geometrical rigor, which might have made the mathematical climate somewhat inhospitable for them.

## Fixed-Point Iteration in Indian Astronomy

The exploration of fixed-point iteration apparently begins in India at about the same time it seems to have ended in Greece. Among strictly mathematical methods, an Indian variant of Heron's square-root formula probably dates back at least to the middle of the first millennium CE [Hayashi 1995: 100–108], and in mathematical astronomy iterative rules are extremely common, both for lack of and in addition to equivalent analytical solutions. We will attempt to give an idea of both their astronomical context and their mathematical form.

*Planetary Longitudes.* The earliest such rules involve calculations of planetary positions, and with Ptolemy's applications of iterative techniques in mind, it is somewhat tempting to think of them as possible legacies from the pre-Ptolemaic astronomy that gave Indian celestial models so many of their characteristic features [Pingree 1976, 1974]; but it has been suggested [Yano 1990, 1997] that they are probably Indian developments of static Greek models. Ingenious formulas for iteratively computing latitudes and true longitudes, as well as other planetary parameters, appear in astronomical texts from the fifth and sixth centuries;[6] to illustrate the approach, we will focus on a somewhat simpler rule from a seventh-century text, the *Brāhmasphuṭasiddhānta* of Brahmagupta, which seeks to calculate the mean longitude of the sun if its true longitude is known.

Ordinarily, the practice for computing planetary positions when using geocentric geometric models such as the Greek and the Indian is to find the planet's mean longitude — which is easily computed from its uniform rate of mean motion if the mean longitude at a known prior time is given — and then correct it to the true position, in accordance with the geometry of the given model. A simple version of the Indian solar model is shown in Figure 1. The sun's mean longitude $\bar{\lambda}$ along the ecliptic centered on the earth at $O$ changes uniformly with time, but its true longitude $\lambda$ is displaced from $\bar{\lambda}$ by an amount $\mu$ determined by the location of a

5 [Heath 1921, I: 268–270; Jones 1986, I: 4–5].

6 E.g., *Pañcasiddhāntikā* X, 2–4 [Neugebauer & Pingree 1970-71: I, 100; II, 78], *Paitāmahasiddhānta* IV, 11–12 [Pingree 1967/68: 493–495]; see [Plofker 1995] for a discussion of these methods.

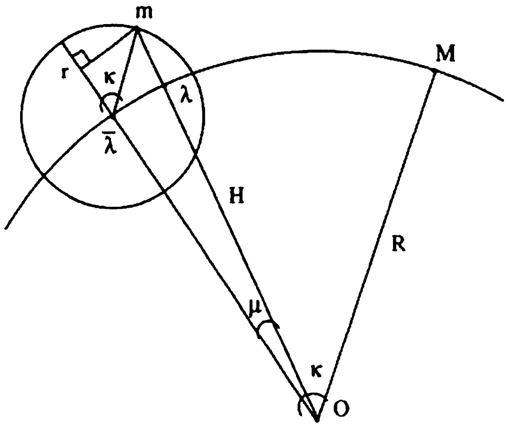


**Figure 1.** Mean and true solar positions

point $m$ on its epicycle. This point lies on the epicycle radius parallel to the ecliptic radius $OM$ upon which the orbital "apex" $M$, where $\mu = 0$, lies at longitude $\lambda_M$.[7] Finding the amount of this displacement $\mu$ (known as the "equation") and thus the true longitude $\lambda$ is trigonometrically quite simple. Considering for computational purposes that the ecliptic radius $OM$ is equal to the standard trigonometric radius $R$, and writing the correspondingly scaled functions with initial capitals (as "Sin" for "$R$ sin"), we have

$$\begin{gathered} \kappa = \bar{\lambda} - \lambda_M, \\ \operatorname{Sin}\mu = \operatorname{Sin}\kappa \cdot \frac{r}{R} \cdot \frac{R}{H} = \operatorname{Sin}\kappa \cdot \frac{r}{H}, \\ \lambda = \bar{\lambda} - \mu, \end{gathered} \tag{5}$$

where $H$ is the hypotenuse $Om$ of the large right triangle and $r$ the epicycle radius. Writing this as a single expression for $\lambda$ in terms of $\bar{\lambda}$ (which requires the hypotenuse $H$ to be represented in terms of the other two sides), we get

$$\lambda = \bar{\lambda} - \operatorname{Sin}^{-1}\left(\frac{\operatorname{Sin}(\bar{\lambda} - \lambda_M) \cdot r}{\sqrt{\left(\operatorname{Sin}(\bar{\lambda} - \lambda_M) \cdot \frac{r}{R}\right)^2 + \left(R + \operatorname{Cos}(\bar{\lambda} - \lambda_M) \cdot \frac{r}{R}\right)^2}}\right), \tag{6}$$

and thus $\lambda$ is known.

7 The physical cosmology of the Indian system is more flexible than that usually associated with the Ptolemaic: the orbital circle may be considered an eccentric, in which case $M$ is its apogee; or it may be viewed as a concentric bearing an epicycle as shown, with the sun actually moving upon the epicycle; or the sun may be taken (as in the *Sūryasiddhānta*, 2, 1–5 [Dvivedī 1987: 31–32]) to be on the concentric at the center of the epicycle but displaced to $\lambda$ by the pull on a cord of wind by a semidivine being at the point $m$. All these different physical models are computationally equivalent.

However, as noted above, Brahmagupta wants to invert this procedure, that is, to solve for the mean longitude $\bar{\lambda}$ given the true longitude $\lambda$. Not surprisingly, solving equation 6 directly for $\bar{\lambda}$ is well beyond the scope of Brahmagupta's available mathematical tools. The solution he prescribes for getting around this difficulty is as follows:

> The [Sine of the sun's] declination is multiplied by the Radius and divided by the Sine of 24 degrees [i.e., the obliquity of the ecliptic, $\varepsilon$]. The arc [of that ...] is the accurate [longitude]. [The equation] is repeatedly added [to that] when negative and subtracted when positive; from this, increased or decreased by the longitudinal difference [i.e., a correction depending upon the distance of the observer's terrestrial longitude from the prime meridian], the [longitude of the] sun is mean, as originally.[8]

In other words, after giving the standard Indian formula for computing the sun's true longitude from its declination determined from observation, $\operatorname{Sin}\lambda = R \cdot \operatorname{Sin}\delta / \operatorname{Sin}\varepsilon$, Brahmagupta directs the user to treat that value as though it were the mean longitude and then compute the correction $\mu$ for it, to apply that with reversed sign to $\lambda$, and iterate the process to get successively better values of $\bar{\lambda}$. That is, setting the initial estimate $\bar{\lambda}_0 = \lambda$, we find the succeeding estimates to be

$$\bar{\lambda}_{k+1} = \lambda + \mu(\bar{\lambda}_k) =$$

$$= \lambda + \operatorname{Sin}^{-1}\left(\frac{\operatorname{Sin}(\bar{\lambda}_k - \lambda_M)\cdot r}{\sqrt{\left(\operatorname{Sin}(\bar{\lambda}_k - \lambda_M)\cdot \frac{r}{R}\right)^2 + \left(R + \operatorname{Cos}(\bar{\lambda}_k - \lambda_M)\cdot \frac{r}{R}\right)^2}}\right). \quad (7)$$

Obviously, this auxiliary function $\bar{\lambda}_{k+1} = \lambda + \mu(\bar{\lambda}_k)$ has its fixed point $\bar{\lambda}$ where $\lambda = \bar{\lambda} - \mu(\bar{\lambda})$, that is, at the desired mean longitude at the given time with respect to the observer's location (which is subsequently adjusted to give the value at that time with respect to the prime meridian, this being the standard definition of the uncorrected mean longitude with which typical astronomical computations commence). Thus Brahmagupta has avoided dealing with an impossible closed-form solution by simply iterating to reach the fixed point of his approximate solution.

*"Three Questions" Iterations and the "Iteration Explosion."* The early Sanskrit astronomical texts mentioned above limit their use of fixed-point iteration to genuinely insoluble problems involving planetary positions, for which exact solutions are unobtainable via the mathematics available. From about the eighth century onwards, however, Indian astronomers drastically broaden the applications of these techniques to include cases that might be described as "artificially insoluble," that is, where simple closed-form solutions are possible but a necessary parameter (often one that could readily be found from a single observation) is considered to be unknown. Problems presented in this way are very often drawn from the field of Indian mathematical astronomy called "Three Questions": computing from knowledge of

8 *Brāhmasphuṭasiddhānta* 3, 61–62 [Dvivedi 1902: 69].

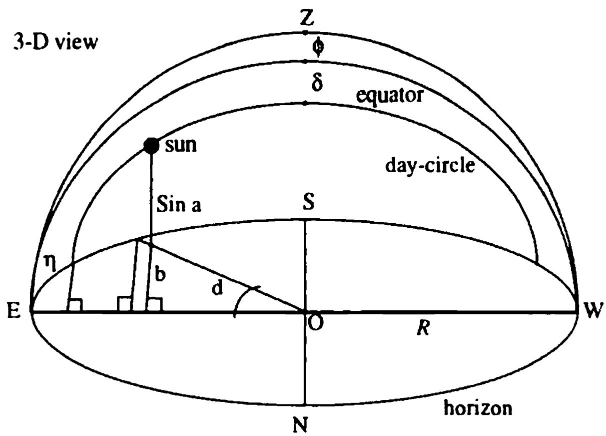


**Figure 2.** The celestial hemisphere above the horizon

the sun's ecliptic position a) one's orientation with respect to the directions, b) terrestrial latitude, and c) current time. The example we will use as an illustration is one of several such rules provided in the middle of the ninth century by Govindasvāmin in his commentary on the seventh-century *Mahābhāskarīya* of Bhāskara.

The rule in question determines the local terrestrial latitude $\phi$ by means of an iterative calculation of the angle $\eta$ (the so-called "rising amplitude") between the east point on the local horizon and the point at which the sun rises, when the solar declination $\delta$, altitude $a$, and modified azimuth $d$ (namely, "east azimuth," or azimuth minus 90 degrees) are given. The sun's daily path is here taken to be a small celestial circle, the "day-circle," parallel to the celestial equator and separated from it by the amount of its declination, as shown in Figure 2; the sun's altitude at any point in the day is its angular distance from the plane of the horizon, and the corresponding $d$ its angular distance from the prime vertical or great circle ($EZW$ in the figure) through the zenith and the east and west points. It is clear from the similar right triangles in the corresponding analemma projection in Figure 3 that the latitude $\phi$ or angle between the celestial equator and the local zenith is straightforwardly related to $\eta$ and $\delta$ as follows:

$$\frac{\operatorname{Sin}\eta}{\operatorname{Sin}\delta} = \frac{R}{\operatorname{Cos}\phi}. \tag{8}$$

The declination $\delta$ can be computed from the given solar longitude by means of the well-known rule in the above quotation from Brahmagupta. And the rising amplitude $\eta$, of course, could be obtained by a simple observation of the sun's position with respect to the east point on the horizon at sunrise. Govindasvāmin, however, considers the case where the user does not know $\eta$ but does know $a$ and $d$ at some arbitrary time during the day, and can find the distance $b$ between the foot of the imaginary perpendicular dropped from the sun onto the plane of the horizon

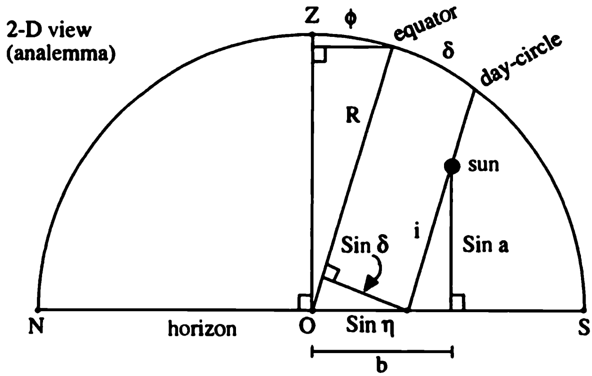


**Figure 3.** Analemma projection of the celestial hemisphere

and the east-west line $EW$ from another pair of similar right triangles:

$$\frac{b}{\mathrm{Cos}\, a} = \frac{\mathrm{Sin}\, d}{R}. \tag{9}$$

He says:

> The Sine of declination is [arbitrarily] increased by some [amount]; that should be [the estimated Sine of] the sun's rising amplitude. And the leg [$b$] diminished by that is [the Sine of] the "amplitude of the upright [i.e., of the current altitude]" when [the sun is] in the south [i.e., when the declination is southern]. [For a northern declination,] when [the sun is] in the north with respect to the prime vertical, the amplitude of the upright is the Sine of the Day-lord's rising amplitude diminished by the leg, or the sum of those two if the sun has gone to the south. If the sun stands on the prime vertical, then the unchanged rising amplitude is the amplitude of the upright of the sun. The square-root of the sum of the squares of that [quantity] and the upright is the Sine produced from the circle of the day-radius [i.e., from the day-circle] which is above the horizon.
>
> When one has divided the Sine of declination [corresponding to] the desired Sine [and] multiplied by that [square-root] by the Sine of altitude, the quotient is the Sine of the sun's rising amplitude. With that, one should operate again just as before, until the Sine of the sun's rising amplitude is the same as the one [previously] produced from the rising amplitude. When one has subtracted the square of [the Sine of] the declination from the square of [the Sine of] the sun's rising amplitude, the square-root [of the result] is the "earth-sine." He [i.e., the author on whom Govindasvāmin is commenting] will state that method for the latitude by means of the sun's rising amplitude and the earth-sine.[9]

The first step is thus to choose some arbitrary initial Sin $\eta_0$ such that $|\eta_0| > |\delta|$: a wise decision, since unless the celestial equator is perpendicular to the horizon (namely, at $\phi = 0°$), the angular distance $\eta$ between the celestial equator and the day-circle measured along the horizon will be greater than their angular separation $\delta$

9 *Mahābhāskarīya* 3, 41 [Kuppanna Sastri 1957: 156–157].

measured perpendicular to both. One must then compute an initial value of the north-south distance $b + \operatorname{Sin}\eta_0$ between the sun's current position and its rising point. (For the sake of generality and to avoid Govindasvāmin's rather complicated enumeration of the sign conditions, we will consider the sign of $\eta$ to correspond to that of $\delta$, which is always negative when south of the equator, since only positive, i.e. northern, terrestrial latitudes are considered; and we will take $b$ to be negative when north of the east-west line. In Figure 2, therefore, $b$ is positive and $\eta$ negative.) Then one computes the first approximation to the "Sine produced from the day-circle," that is, the hypotenuse $i$ in Figure 3 of the two legs $\operatorname{Sin} a$ and $|b + \operatorname{Sin}\eta|$, by the Pythagorean theorem:

$$i_0 = \sqrt{(b + \operatorname{Sin}\eta_0)^2 + \operatorname{Sin}^2 a}. \tag{10}$$

A new value for $\eta$ can then be computed from the similarity of the right triangles with hypotenuses $i$ and $|\operatorname{Sin}\eta|$:

$$\operatorname{Sin}\eta_1 = \frac{i_0}{\operatorname{Sin} a}\cdot\operatorname{Sin}\delta. \tag{11}$$

"With that, one should operate just as before": i.e., these two steps are repeated until $\operatorname{Sin}\eta$ is fixed, as we may represent by the following iterative function $g$:

$$\operatorname{Sin}\eta_{k+1} = g(\operatorname{Sin}\eta_k) = \frac{\sqrt{(b + \operatorname{Sin}\eta_k)^2 + \operatorname{Sin}^2 a}}{\operatorname{Sin} a}\cdot\operatorname{Sin}\delta. \tag{12}$$

Then the desired quantity $\operatorname{Sin}\phi$ falls easily out of a similar-triangle proportion complementary to the one given in equation 8:

$$\frac{\operatorname{Sin}\phi}{R} = \frac{\sqrt{\operatorname{Sin}^2\eta - \operatorname{Sin}^2\delta}}{\operatorname{Sin}\eta}. \tag{13}$$

Since $i/\operatorname{Sin} a = R/\operatorname{Cos}\phi$, the final form of equation 11 will be equivalent to equation 8. That is, its fixed point $\eta$ as obtained from the iteration of equation 12 will produce the required $\phi$.

But will the iteration in all cases obtain that fixed point — that is, is the fixed point always attracting? So far, we have paid little attention to the issue of convergence of these iterative techniques: since medieval mathematicians were primarily concerned with obtaining usable results rather than with investigating the complexities of dynamical systems, the iterative techniques that they preserved are typically reliable. The criteria defined in our mathematical overview of fixed-point approximations, however, enable us to see where such methods sometimes fail. We noted there that the value of the first derivative of the iterative function determines the nature of the fixed point; so, differentiating the right side of equation 12 with respect to $\operatorname{Sin}\eta$, we find

$$g'(\operatorname{Sin}\eta) = \frac{\operatorname{Sin}\delta}{\operatorname{Sin} a}\cdot\frac{b + \operatorname{Sin}\eta}{\sqrt{(b + \operatorname{Sin}\eta)^2 + \operatorname{Sin}^2 a}}. \tag{14}$$

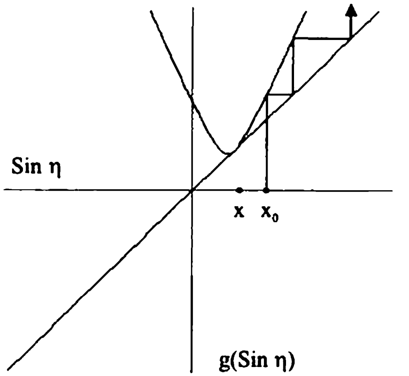


**Figure 4.** Orbit of a non-convergent iteration

Under what circumstances would the absolute value of this expression be greater than 1 at the fixed point $\eta$, causing the iteration to diverge? Obviously, the value of the right-triangle leg $|b + \mathrm{Sin}\,\eta|$ will always be less than that of its hypotenuse $i$ appearing in the denominator, so their ratio will be less than 1. But if $\mathrm{Sin}\,a$ is quite small in relation to $\mathrm{Sin}\,\delta$, the large coefficient $\mathrm{Sin}\,\delta/\,\mathrm{Sin}\,a$ may make the entire expression greater than 1. In other words, when the sun's declination is large (near a solstice) but its altitude sufficiently small (near sunrise or sunset), this iterative approximation cannot work; see the orbit diagram in Figure 4 for an illustration of the repelling fixed point in such a case. There is no clear evidence that Govindasvāmin or anyone else who worked with this iteration commented on, or even noticed, this problem. In similar cases, however, a good deal of attention appears to have been paid to non-convergent iterations and ways to diagnose and / or fix them.[10]

*Transmission to the Islamic World: an Iterative Rule for Eclipse Computations.* Given this evidence for the sizable role of fixed-point iteration in early Sanskrit works, it is natural to wonder what influence it may have had on mathematical science in the Islamic world. Several fixed-point computations used by Islamic mathematicians (such as al-Kāshī's algorithm for the sine of 1°) have been discussed elsewhere (e.g., in [Yushkevich & Rozenfeld 1973; Rashed 1994: 88–114]), but none of them is conclusively connected to an Indian prototype. To find such a connection, we must turn back to astronomy.[11]

At the beginning of the tenth century, Vaṭeśvara wrote the huge compendium of mathematical astronomy known as the *Vaṭeśvarasiddhānta*, containing most of

10 Some examples are examined by this author in [Plofker 1996] and in a forthcoming article, "The Problem of the Sun's Corner Altitude and Convergence of Fixed-Point Approximations in Indian Astronomy."

11 The strong dependence of early Islamic astronomy in general upon Indian sources is discussed in, for example, [Pingree 1996]. In particular, some Islamic authors computed mean from true solar longitudes iteratively after the fashion of Brahmagupta in equation 7, as described in [Kennedy 1969; Kennedy, Pingree & Haddad 1981: 301–302].

the standard computational techniques known to Indian astronomers at that period. Among the listed rules are several formulas for the parallax of the sun and moon: i.e., the angular shift between their positions as computed with respect to the center of the earth and as perceived from somewhere on the surface of the earth. Because the sun and moon are so close to the earth, this displacement is significant for computing the time of a perceived solar eclipse, which would coincide with the true ecliptic conjunction of the bodies only if they were at the observer's zenith — that is, if the observer were on the line between the center of the earth and the conjoined moon and sun. The parallax thus measures the difference between the luminaries' positions at true conjunction and their positions at apparent conjunction, that is, where the observer would be able to see them in eclipse, at that same time. An estimate of the time of apparent conjunction can therefore be obtained by computing how long it will take the sun and moon to cover that parallactic distance by their motion in their orbits, and adding or subtracting the resulting interval with respect to the time of true conjunction.

But this computation is not in fact a one-step solution: for by the time the sun and moon have covered the distance in parallax computed for their true conjunction, their motion and that of the heavens will also have shifted them with respect to the observer and thus changed the angle of parallax. To compute the time difference exactly, the observer would have to work backwards: namely, to find the parallax angle for the time of apparent conjunction, calculate the time required for the sun and moon to traverse it, and thence obtain the time of true conjunction (which, unlike apparent conjunction, will not be affected by the bodies' shift with respect to the observer in the meantime). Unfortunately, since planetary positions are calculated relative to the center of the earth and not to the observer's surface location, the time of apparent conjunction is not independently accessible via mathematical prediction.

We are therefore in a position very similar to that of Brahmagupta when he sought to compute mean from true longitudes, as discussed above. That is, we have two positions which are related by a computable correction term, but the position we know is the one that the correction should produce, not the one to which the correction should be applied. The means Vaṭeśvara uses for getting around this problem in one of his parallax rules are precisely the same as Brahmagupta's: we take the known value as a first approximation to the desired one, compute and apply the correction term with reversed sign, and repeat the process until the desired value becomes fixed. The following explanation of the details of the method refers to Figure 5; for simplicity's sake, we will restrict it to consideration of the moon's parallax in longitude only.

The point $Z$ is the observer's local zenith, and $M$ is the position of the moon on the ecliptic, which appears to be displaced by parallax to the point $M'$. The arc $ZM$ is then the moon's zenith-distance $\zeta$, whose component in longitude $\zeta_\lambda$ is the ecliptic arc $VM$, and the arc $MM'$ is the lunar parallax $p$, whose longitudinal component $p_\lambda$ is $MF$. It is clear that if the moon were at $V$, that is, on the vertical circle passing through both the zenith and the ecliptic pole $P'$, there would be no longitudinal component of zenith-distance and no longitudinal parallax either (i.e., the observer would see the moon depressed below the ecliptic, but not east or west of its actual

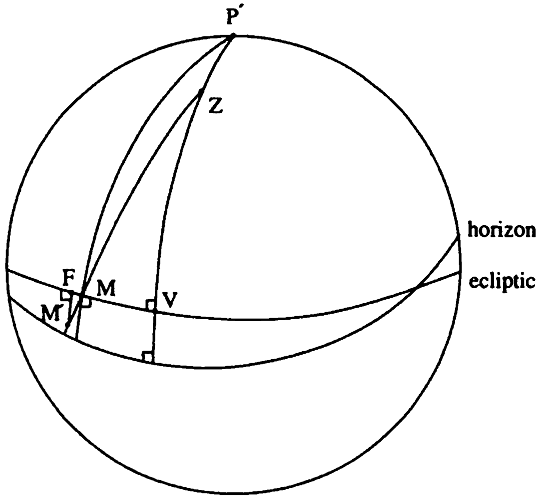


**Figure 5.** Parallax of the moon

longitude). Conversely, if the moon were 90° away from $V$ along the ecliptic, the longitudinal parallax would be at its maximum, which is taken to be an amount equal to its motion in 4 *ghaṭikās* (a time-unit equal to 1/60 of a day or 24 minutes). We therefore employ the classic Indian technique of sinusoidal interpolation, assuming that $p_\lambda$ just varies as the Sine of $\zeta_\lambda$; then we apply that $p_\lambda$, converted to time-units, to the known time of true conjunction. Or as Vaṭeśvara puts it:

> The Sine of the longitudinal component of the zenith-distance multiplied by four and divided by the Radius [is] the longitudinal parallax in terms of *ghaṭikās*. [When that is] subtracted from the time of conjunction or added to it in the manner stated before, repeatedly, [it is] here stated [to be] the true longitudinal parallax.[12]

Rephrasing this in mathematical notation, we first define the time in *ghaṭikās* corresponding to the longitudinal parallax, $p_\lambda{}^{(g)}$, by the proportion

$$\frac{p_\lambda{}^{(g)}}{4} = \frac{\operatorname{Sin}\zeta_\lambda}{R} \quad \text{or} \quad p_\lambda{}^{(g)} = 4\sin\zeta\lambda. \tag{15}$$

Then if the time of ecliptic conjunction is $T$ and that of apparent conjunction $T'$, at which the parallactic interval is $p_\lambda{}^{(g)}(T')$,

$$T = T' \pm p_\lambda{}^{(g)}(T'), \tag{16}$$

but as noted above, we know $T$ but not $T'$. So instead, taking $T'_0 = T$ as a first approximation to the desired value $T'$, we iterate

$$T'_{k+1} = T \mp p_\lambda{}^{(g)}(T'_k) \tag{17}$$

12 *Vaṭeśvarasiddhānta* V, 16 [Shukla 1986: I, 244].

until the value of $T'$ becomes fixed. Then we will have the correct value of the parallax in time $p_\lambda{}^{(g)}$ between true and apparent conjunction.

A very similar approach to dealing with the same problem in Islamic astronomy is taken in the ninth-century *zīj* of Ḥabash al-Hāsib, a set of astronomical tables heavily influenced by Indian methods [Kennedy 1956; Kennedy & Transue 1956]. Here too the user is required to find the correct longitudinal parallax in terms of time (though in this case the time-units are minutes rather than *ghaṭikā*s), but the iterative technique is applied instead to the calculation of $\zeta_\lambda$. A time interval $t$ corresponding to the desired $\zeta_\lambda$ is known, but it is related to $\zeta_\lambda$ by

$$t = \zeta_\lambda - 24 \sin \zeta_\lambda, \tag{18}$$

which can not be directly solved for $\zeta_\lambda$. Once again, the solution is to assume a first approximation $\zeta_{\lambda 0} = t$ and iterate

$$\zeta_{\lambda k+1} = t + 24 \sin \zeta_{\lambda k} \tag{19}$$

until the value of $\zeta_\lambda$ is fixed.[13] Then the longitudinal parallax in minutes is defined by

$$p_\lambda{}^{(\mathrm{min})} = 96 \sin \zeta_\lambda, \tag{20}$$

which, since $96 = 24 \times 4$, is precisely equivalent to the Indian rule for the longitudinal parallax in (24-minute) *ghaṭikā*s from equation 15.[14]

To sum up, we have seen that an Arabic *zīj* in the middle of the ninth century solves the problem of computing lunar parallax in a way very similar to that followed by a Sanskrit *siddhānta* at the beginning of the tenth century (most of whose material is much older): their formulas for the parallax are essentially identical, and they both exploit an iterative trick already well-known in Indian astronomy for several centuries.[15] Although Islamic mathematicians certainly encountered some iterative methods in Greek texts as well, there can be little doubt that their awareness of the uses of fixed-point iteration was largely (and probably originally) derived from the many examples provided by Indian mathematical astronomy.

## Two-Point Iteration in Indian Astronomy

*Regula Falsi.* Two-point iteration techniques also make their appearance early in the development of siddhāntic astronomy, and in fact are first attested in Sanskrit

13 Actually, Ḥabash prescribes a fixed number of steps for the iteration [Kennedy 1956: 51]; like Ptolemy, and unlike Indian mathematicians, Islamic mathematicians generally set explicit limits for the iteration procedure instead of just directing the user to iterate. As noted by Kennedy, the convergence is reliable and fast, so a few iterations are sufficient.

14 [Kennedy 1956: 50] considers the parameter 24 to refer simply to the Indian value for $\varepsilon$, but I think that in equation 20 it must also serve as the conversion factor between the time units.

15 Whether the present form of Ḥabash's formula itself was originally Indian, or whether it resulted from further development within the Islamic tradition, is unknown; as noted in [Pingree 1996], later *zīj*es generally discarded these approximative methods in favor of more geometrically rigorous procedures after the fashion of Ptolemy.

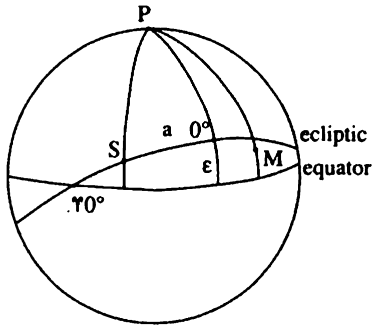


**Figure 6.** Sun and moon at a *mahāpāta*

works. The instance we will examine is the computation of the astrological phenomena known as the *mahāpātas*. These are the disastrous moments when the sum of the solar and lunar longitudes equals 180° or 360° — that is, when the sun and moon are on opposite sides of, and equidistant from, a solstice and an equinox, respectively. This appears to be the original defining condition for these events;[16] but as early as the fifth century, a more complicated version of the condition appears in the *Paitāmahasiddhānta*, requiring that the luminaries' declinations at the time of the *mahāpāta* be equal in magnitude. Since the moon's declination is affected by its latitude, at the time $t_0$ when the bodies are equidistant in longitude from the equinox or solstice, the true *mahāpāta* will not occur. Figure 6 illustrates the situation for the sun $S$ and moon $M$ near a solstice, on which we shall focus in the following discussion, since except for some sign adjustments the computations for the ominous event near an equinox are exactly the same.

What we need to do, then, knowing the time $t_0$ and longitudes $\lambda_{S_0}$ and $\lambda_{M_0}$ when the luminaries are equidistant from the solstice, as well as the way to compute their declinations $\delta_S$ and $\delta_M$ from their longitudes, is to find the time $t$ when the difference between their declinations is zero. We can define the necessary quantities as follows:

$$\begin{gathered}\delta_S(t) = \mathrm{Sin}^{-1}\left(\frac{\mathrm{Sin}\,\lambda_S(t)\cdot\mathrm{Sin}\,\varepsilon}{R}\right),\\ \delta_M(t) = \mathrm{Sin}^{-1}\left(\frac{\mathrm{Sin}\,\lambda_M(t)\cdot\mathrm{Sin}\,\varepsilon}{R}\right) + (\lambda_M(t) - \lambda_N(t))\cdot\frac{4;30}{90},\\ f(t) = \delta_M - \delta_S.\end{gathered} \tag{21}$$

Here, the ecliptic declinations are found from the corresponding longitudes (which are dependent on time) by the standard Indian rule, but the lunar value is modified (somewhat imprecisely) by its latitude, which is considered to vary be-

16 At least, they are so defined in the *Pauliśasiddhānta* of the *Pañcasiddhāntikā* (III: 20–22), as well as by Āryabhaṭa in the late fifth century and his followers up to the time of Lalla.

tween zero and a maximum of 4;30° as the difference between the longitude of the moon and that of its orbital node, $\lambda_N$.[17]

How, then, is the true time $t$ of the *mahāpāta*, at which $f(t) = 0$, to be found? The *Paitāmahasiddhānta* prescribes the following procedure:

> One should divide the difference between [180°] and the sum of [the longitudes of] the Sun and Moon [at a given time] by the sum of their velocities; with the quotient, [one finds the time to pass] until the sum of [the longitudes of] the Sun and Moon will be equal to [180°]. Then [one finds] the declination of the Sun for [that] time of the *pāta* and the true declination of the Moon. ... Thus, by means of the *pāta*[-calculations] of the Sun and Moon at desired times are found the second [approximation] and so on. At these two times of the first and second [approximations] there exists a future or a past *pāta*; their sum or difference is the divisor of the desired time [interval] multiplied by the first [approximation]. Thereby, repeatedly making a second [approximation], the time is corrected by a process of iteration.[18]

The passage is very terse (although the omitted portion goes into some detail about the criteria for determining whether the ominous event is past or future), but its mathematical meaning is certain. First one computes the time $t_0$ when the longitudes sum to 180° and the corresponding longitudes and declinations at that time. Then one calculates the same quantities for a "desired," i.e., arbitrarily chosen, time $t_1$. The amount of the *pāta* or difference between the declinations (our $f$) is computed for both $t_0$ and $t_1$, and the product of the time-difference and the *pāta* at $t_0$ is divided by the difference of the two *pāta*s; the quotient corrects $t_0$. In mathematical notation, this produces a new approximation $t_2$ to the true time $t$ as follows:

$$t_2 = t_0 - \frac{(t_1 - t_0) \cdot (\delta_{M_0} - \delta_{S_0})}{(\delta_{M_1} - \delta_{S_1}) - (\delta_{M_0} - \delta_{S_0})} = t_0 - \frac{(t_1 - t_0) \cdot f(t_0)}{f(t_1) - f(t_0)}, \qquad (22)$$

which precisely matches our definition of the two-point iteration technique Regula Falsi in equation 2. The suitability of this approach is confirmed by considering that the basic problem — finding when a faster and a slower body will reach a specified configuration starting from a known position — is really just an "astronomical problem of pursuit" and therefore, as has been noted elsewhere in this volume [Bréard 2002], is solvable by means of the method of Double False Position. Since the function $f(t)$ is highly nonlinear, to get an accurate answer one must iterate the application of Double False Position, that is, apply Regula Falsi.

Of course, this method can be applied in finding the root of any function; yet its use was restricted by Indian astronomers to the computation of the *mahāpāta*s and to the similar problem involved in computing a particular type of planetary

17 Strictly speaking, the latitude and the ecliptic declination should not be combined as though they were on the same great circle; in practice, however, this is very commonly done by Indian astronomers.

18 *Paitāmahasiddhānta* V, 9 [Pingree 1967/68: 498–499]. I have changed some of the editorial interpolations in the original translation to make my interpretation more clear.

conjunction.[19] Regula Falsi seems never to have experienced the sort of "iteration explosion" that led to the abundance of fixed-point iterative rules in Indian astronomy: it remained a method specific to these few "astronomical problems of pursuit."

Moreover, the evidence for its adoption in any mathematical tradition other than the Indian during this time appears so far to be slight.[20] Al-Bīrūnī describes the Indian use of iterative Regula Falsi for the *mahāpātas* when discussing astrological matters in his *India* [Sachau 1888: 204–210], but as these computations do not form part of Islamic astrology, the procedure may have been a matter of merely ethnological interest to him and his readers. And Levi ben Gerson three centuries later appears to have iterated the Rule of Double False Position to refine approximate values of lunar parameters, but without explaining the procedure.[21]

*A Note on the Non-Iterated Rule of Double False Position.* The iterative technique Regula Falsi bears such a close mathematical relationship to its non-iterated form, the so-called "Rule of Double False Position," that the history of the latter is relevant here; interestingly, it turns out to provide some baffling contradictions. Sanskrit texts, so far as is now known, do not mention the non-iterated Rule of Double False Position, although they do use the so-called "operation with an assumed (quantity)" or Rule of (Single) False Position [Hayashi 1995: 396–398]. The "Double False" Rule has therefore been reconstructed [Chemla 1997; Bréard 2002; Liu 2002] as originating in China sometime before the first century CE, bypassing India entirely in its transmission to the Islamic world by the ninth century, and proceeding thence to Europe in the thirteenth century. We now have to add to this reconstruction the caveat that at least from the fifth century, the iterated form of the rule is well known in India, making it almost incredible that the non-iterated form should have remained totally unknown — but in that case, why does it not appear in the texts? The only (rather unconvincing) explanation I can suggest is that early Indian mathematicians may have become aware that for linear equations, the "Single False" Rule is simpler to apply, while for non-linear equations, the "Double False" Rule should be iterated.

19 As it was used, for example, by Brahmagupta in *Brāhmasphuṭasiddhānta* 9, 22–24. But as we have noted, Regula Falsi was not universally employed even within this limited context: Brahmagupta remarks that the application of this technique to the problem of calculating the *mahāpātas* is unique to the Brāhmapakṣa (*Khaṇḍakhādyaka* U 1, 20 [Chatterjee 1970: II, 192]). His commentator and his editor interpret him to mean that this method originated in the *Brāhmasphuṭasiddhānta* itself; but as mentioned above, it is presented earlier in the *Paitāmahasiddhānta*, whence Brahmagupta doubtless derived it. In fact, there seems to be no occurrence of any two-point iterative method in the Āryapakṣa until the rules for *mahāpātas* and conjunctions are borrowed from Brahmagupta by Lalla in the eighth century (*Śiṣyadhīvṛddhidatantra* 12, 6–9, and 10, 15–20 [Chatterjee 1981: I, 171–172, 152–154]).

20 [Rashed 1994: 113] has noted that a fixed-point rule for *n*th-root extraction given by al-Samaw'al may have been derived from the form of the "Double False" Rule, but it is not itself an example of iterative Regula Falsi.

21 [Mancha 1998], where it is also noted (p. 18) that Regula Falsi is explicitly presented by Cardano under the title "regula aurea" in 1545.

Hence, perhaps, the non-iterated form of the "Double False" Rule dropped out of the Indian tradition.[22]

Increasing the confusion is the fact that a Latin version of an Arabic or Hebrew text on (non-iterated) Double False Position explicitly attributes it to Indian sources: namely, the *Liber augmenti et diminutionis vocatus numeratio divinationis, ex eo quod sapientes Indi posuerunt, quem Abraham compilavit et secundum librum qui Indorum dictus est composuit* [Libri 1838–41, I: 304]. Who this "Abraham" was,[23] and what his justification was for considering this technique to be originally the work of "sapientes Indi," remain mysterious.

*The Secant Method.* The sole instance known to me of the other attested form of iterative approximation, generally known as the secant method and defined in equation 3, appears in the work of Mādhava's student Parameśvara in the fifteenth century. I will not repeat my earlier discussion of this development in [Plofker 1996] except to point out that it apparently represents not only the discovery of a previously unknown iterative technique (although doubtless inspired by the familiar and very similar Regula Falsi), but a groundbreaking experiment in repairing the convergence problems of a fixed-point iteration by selective substitution of a two-point method. However, as far as we now know, these innovations were not subsequently developed further even within Parameśvara's school, much less outside it.

## Conclusion

There is much yet to be learned about iterative approximations in the ancient and medieval world before confident conclusions can be drawn concerning the details of their transmission. However, since I believe that no general overview of their early history at present exists, I venture tentatively to sketch one in the following paragraphs.

*Fixed-Point Iteration.* Although there is reason to believe that this form of approximation may go back as far as Old-Babylonian times, it makes its first positive appearance in Hellenistic Greek astronomical and mathematical works of the first centuries CE. The comparatively few Greek examples we have of such methods provide well-behaved approximations to the solution of analytically intractable problems. Similar (though apparently not Greek-derived) rules appear in Indian mathematical astronomy by the fifth century; their number greatly increases in the second half of the first millennium to include many "superfluous" procedures for solving trigonomet-

22 [Datta & Singh 1962, I: 230], relying on [Smith 1923–25, II: 437], lump the two non-iterated techniques together under the generic name "Rule of False Position" and ascribe to it an Indian origin. While the two methods are indeed more or less equivalent theoretically, they are procedurally so different that this seems inadequate to reconstruct the process of transmission.

23 [Steinschneider 1925: 486–492] discusses the possibility of identifying him with the twelfth-century Abraham ibn Ezra. I am indebted to Professor Tony Lévy for drawing my attention to this work and to [Mancha 1998].

ric problems for which exact solutions already exist. Not all of these "asakṛt" rules prove consistently reliable; there is evidence of some experimental attempts to improve them with empirical investigations of convergence conditions and convergence speeds, particularly within the Kerala school in the mid-second millennium.

Early Islamic *zījes* reveal some borrowing from the Indian plethora of astronomical fixed-point iterations, but their use subsequently declines with the predominance of Greek-inspired models relying on exact spherical trigonometry. Islamic mathematicians continue to employ fixed-point methods providing well-behaved approximations to solutions of analytically intractable problems. It is not yet clear how much of the inspiration for this approach is due to Indian prototypes and how much to the Hellenistic mathematical tradition.

*Two-Point Iteration.* The technique of iterating the Rule of Double False Position, or "Regula Falsi," is invented in India sometime before the fifth century, possibly inspired by an encounter with the non-iterated Rule of Double False Position in a Chinese source. However, the non-iterated "Double False" Rule does not appear in Sanskrit texts, while Regula Falsi is well known but restricted to a few types of astronomical problems. Islamic and Hebrew texts attest to the use of the non-iterated rule but not directly to that of Regula Falsi. A similar iterative technique, the secant method, is developed by Parameśvara of the Kerala school in the fifteenth century, but is not known elsewhere.

The greater development of iterative methods in Indian mathematical science is, I think, easily explained in general as a byproduct of this tradition's focus on computation as much as or more than on proof. Iterative techniques, although procedurally often very simple, are theoretically somewhat involved. They do not lend themselves well to static geometrical representation (and in fact can be difficult to represent adequately even with modern graphing mechanisms); and their potential unreliability in the case of fixed-point rules can make them most difficult to analyze mathematically. Therefore they, like many other numerical methods, could not find a place in the "dianoetic" knowledge of the Graeco-Islamic mathematical tradition beginning with Euclid, though they were brilliantly handled in several cases where that knowledge could not provide the necessary numerical data. In Indian mathematics, however, where deductive rigor was not mandatory in the written tradition and computational ingenuity was highly prized, iterative algorithms could be empirically developed and appreciated even in cases where analytic solutions were known.

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# Anthyphairetic Ratio Theory in Medieval Islamic Mathematics

by JAN P. HOGENDIJK

Al-Māhānī (ca. A.D. 860) and ʿUmar al-Khayyāmī (1048–1131) preferred the so-called anthyphairetic definitions of equal and greater ratios over the definitions that had been given by Euclid in Book V of the *Elements*. Al-Nayrīzī (ca. A.D. 880) also mentions anthyphairetic definitions. In this paper I summarize and compare the work of these three medieval Islamic mathematicians in anthyphairetic ratio theory.

## Introduction

One of the difficult parts of Euclid's *Elements* is the theory of ratios and proportions of general magnitudes in Book V. This theory was developed by Eudoxus, who lived around 350 B.C., half a century before Euclid. In 1933, Oskar Becker argued that Greek mathematicians before Eudoxus used a theory of ratio on the basis of the process of anthyphairesis which will be explained below, and which is mathematically related to modern continued fractions. This anthyphairetic ratio theory has been reconstructed in [Fowler 1999]. In a new analysis, [Vitrac 2002] points out that the very limited Greek evidence does not prove that an anthyphairetic ratio theory of general magnitudes existed in Greek mathematics.

Euclid's *Elements* was translated into Arabic several times, and it soon became one of the basic mathematical texts in the medieval Islamic tradition. Many Arabic commentaries were written to Book V of the *Elements*, and whereas most Islamic commentators followed the theory of Eudoxus, as presented by Euclid, there were at least three authors who considered the anthyphairetic definitions of equal and greater ratio to be more correct than the definitions given by Euclid. I will discuss three texts dealing with anthyphairetic ratio theory:

1. The *Letter on the Difficulty in the Matter of the Ratio* by Abū ʿAbdallāh al-Māhānī. The author made astronomical observations around A.D. 860 in or near Baghdād, and he originated from Māhān in Iran. This text by al-Māhānī had been previously studied by the Dutch historian of science E.B. Plooij in [1950: 50–51], on the basis of the almost illegible Paris manuscript of al-Māhānī's treatise. Plooij summarized a few definitions but he did not mention the fact that there is more in the text. Al-Māhānī's treatise is historically interesting because it is the oldest extant mathematical text on anthyphairetic ratio theory. I have prepared a critical edition with translation of this difficult text.[1] In the present paper I will summarize the treatise and discuss some of its mathematically interesting parts

1 Cf. the note added in proof on p. 202. On al-Māhānī see [Sezgin 1974: 260–262]. In this paper we will be concerned with al-Māhānī's text listed as [Sezgin 1974: 261, no. 1].

in more detail, namely al-Māhānī's anthyphairetic definition of "greater ratio" and one of his propositions related to this concept.

2. The commentary on Book V of Euclid's *Elements* by Abu l-ʿAbbās al-Faḍl ibn Ḥātim al-Nayrīzī. The author worked around A.D. 880 in the Eastern Islamic world, and he originated from Nayrīz in Iran.[2] The Arabic text of this part of the commentary was published with Latin translation in [Junge *et al.* 1932].
3. The *Letter on the Explanation of the Difficulties in the Postulates in Euclid's Book* by ʿUmar al-Khayyāmī (1048–1131).[3] This letter is in three parts, and the second part is about anthyphairetic ratio theory. The Arabic text of the letter was published by [Erani 1936] (not seen), [Sabra 1961], [Humāʾī 1967], [Rezazadeh Malek 1998], and [Vahabzadeh 1999], and the letter has been translated into English in [Amir-Móez 1959] and into French in [Djebbar 1997; Vahabzadeh 1999; Djebbar 2002]. Persian and Russian translations have also appeared.[4]

Another text which should be mentioned here is the *Explanation of the Postulates of Euclid* by Ibn al-Haytham (ca. 965–1041).[5] A facsimile of this text with an introduction by Matthias Schramm has appeared in [Ibn al-Haytham 2000]. In this text Ibn al-Haytham argues that the definitions of equal and greater ratios in the *Elements* are not the "right" definitions, but he does not discuss anthyphairetic ratio theory.

As we will see, the treatment by al-Māhānī is more complicated than, and mathematically superior to, the discussions by al-Nayrīzī and al-Khayyāmī. Below I will present the hypothesis that al-Māhānī's proofs were ultimately derived from a Greek work which has not come down to us. If this is correct, there must have existed an anthyphairetic theory of general magnitudes in Greek mathematics.

## Anthyphairetic Ratio Theory in Greek Mathematics

Anthyphairesis, meaning reciprocal subtraction, is the Greek name of a process which Euclid uses to find the greatest common divisor (gcd) of two integers $a_1$ and $a_2$. His idea is as follows (see *Elements* VII.1–2 [Heath 1956, II: 296–300]). Suppose $a_1 > a_2$. If $a_2$ divides $a_1$, the gcd is $a_2$. If $a_2$ does not divide $a_1$, we subtract $a_2$ from $a_1$ a number of times until the remainder $a_3$ is less than $a_2$. If $a_3$ divides $a_2$, the gcd is $a_3$. If $a_3$ does not divide $a_2$, we subtract $a_3$ from $a_2$ a number of times until the remainder $a_4$ is less than $a_3$, and so on. Eventually we find an integer $a_n$ which divides $a_{n-1}$. This number $a_n$ is the gcd of $a_1$ and $a_2$. In *Elements* X.2–3 [Heath 1956, III: 17–22], Euclid uses the same process for two magnitudes $a_1$ and $a_2$ of the same kind. There are two possibilities: If $a_1$ and $a_2$ are commensurable, we

2 On al-Nayrīzī see [Sezgin 1974: 283–285].

3 On ʿUmar al-Khayyāmī see [Youschkevitch & Rosenfeld 1973].

4 The Persian translations can be found in [Humāʾī 1967; Rezazadeh Malek 1998], for the Russian translations see the references in [Matvievskaya & Rozenfeld 1983, II: 316, no. M3].

5 On Ibn al-Haytham see [Sezgin 1974: 358–374]. The text in question is [Sezgin 1974: 370, no. 28].

will find for some $n$ a magnitude $a_n$ which measures $a_{n-1}$, which means that $a_{n-1}$ is an integer multiple of $a_n$. Now the process stops, and $a_n$ is the greatest common measure of $a_1$ and $a_2$. If $a_1$ and $a_2$ are incommensurable, the process never stops.

In [1933a] Oskar Becker argued that the process of anthyphairesis was the basis of a theory of ratios which was developed before Eudoxus (ca. 350 B.C.). Like Eudoxus' theory, the anthyphairetic ratio theory could be used for rational and irrational ratios. Becker's most important evidence is a definition by "ancient geometers" which is cited by Alexander of Aphrodisias in a commentary on the *Topica* of Aristotle. This definition says that two ratios are equal if they have the same anthyphairesis [Becker 1933a: 312]. Becker interpreted the definition in a way which is as follows in modern notation: Suppose we have two ratios $a_1 : a_2$ and $b_1 : b_2$, where $a_1, a_2$ and $b_1, b_2$ are two pairs of magnitudes of the same kind. Now perform the anthyphairesis on these pairs: let $a_1 = k_1 a_2 + a_3$ with $0 < a_3 < a_2$, and $k_1$ an integer; $a_2 = k_2 a_3 + a_4$ with $0 < a_4 < a_3$, and $k_2$ an integer; and so on, stop this process if $a_n = k_n a_{n+1}$ for an integer $k_n$; and let $b_1 = k'_1 b_2 + b_3$ with $0 < b_3 < b_2$, and $k'_1$ an integer; $b_2 = k'_2 b_3 + b_4$ with $0 < b_4 < b_3$, and $k'_2$ an integer; and so on, stop this process if $b_{n'} = k'_{n'} b_{n'+1}$ for an integer $k'_{n'}$. Then $a_1 : a_2 = b_1 : b_2$ if (1) the two processes continue forever, or stop after the same number of steps ($n = n'$) and (2) for all $i$ for which $k_i$ and $k'_i$ have been defined, $k_i = k'_i$.

The coefficients $k_i$ and $k'_i$ occur in the continued fraction expansions of the real numbers $a_1/a_2$ and $b_1/b_2$ respectively:

$$\frac{a_1}{a_2} = k_1 + \frac{1}{k_2 + \ldots}, \qquad \frac{b_1}{b_2} = k'_1 + \frac{1}{k'_2 + \ldots}.$$

Fowler has shown that it is historically misleading to interpret anthyphairesis in terms of continued fraction expansions, because the ancient Greeks saw anthyphairesis as a process of subtraction, whereas continued fractions are the result of a process of division [Fowler 1999: 30, 313 (n. 13), 366]. I also note that the modern concept of a real number (such as $a_1/a_2$) was not used in ancient Greek and medieval Islamic mathematics. Instead of a "real number" $a_1/a_2$, the mathematicians dealt with a ratio $a_1 : a_2$ between two magnitudes $a_1$ and $a_2$. I will sometimes use a notation such as $a_1/a_2 = [k_1, k_2, \ldots k_n]$ for a continued fraction terminating with $k_n$, and $a_1/a_2 = [k_1, k_2, \ldots]$ for a continued fraction in which there are more than two coefficients $k_1, k_2$. In this notation, $[k_1, k_2, \ldots]$ may terminate with some $k_n$ (in which case $a_1/a_2$ is a rational number, and $k_n > 1$) or continue indefinitely (in which case $a_1/a_2$ is irrational). On the theory of continued fractions the reader may consult [Hardy & Wright 1979: 129–153; Fowler 1999: 304–355].

In Book V of the *Elements*, Euclid presented the theory of ratios which was discovered by Eudoxus, and which is based on the following definition [Heath 1956, II: 114]: $a : b = c : d$ if for all pairs of integer multiples $ma, mc, nb, nd$ we have $ma > nb \Longleftrightarrow mc > nd$, $ma = nb \Longleftrightarrow mc = nd$, and $ma < nb \Longleftrightarrow mc < nd$. From now on I will use the notations $a : b \stackrel{A}{=} c : d$ for the anthyphairetic definition, and $a : b \stackrel{E}{=} c : d$ for the definition of Eudoxus, which was rendered by Euclid in Book V of the *Elements*. The Greek evidence for the older theory is very limited, as is shown by the following list:

- Becker [1933a: 329–330] argues that there were anthyphairetic proofs of theorems of the form $a : b \overset{A}{=} c : d \Longrightarrow a : c \overset{A}{=} b : d$ for different kinds of magnitudes $a, b, c, d$, but we do not know how these or other theorems in anthyphairetic ratio theory were proved;
- Eudoxus has a definition for greater ratio (Def. 7 in Euclid's *Elements* V [Heath 1956, II: 114]), namely $a : b > c : d$ if there are multiples $ma, mc, nb, nd$ for which $ma > nb$ and $mc \leq nd$. From now on I will use the notation $a : b \overset{E}{>} c : d$ for this definition. The ancient sources contain no trace of a definition $a : b \overset{A}{>} c : d$ for the anthyphairetic theory. That this is not a trivial matter can be seen if we try to write out Becker's reconstructed definition [Becker 1933a: 317]: Consider the continued fraction expansions $a/b = [k_1, k_2, \ldots]$ and $c/d = [k'_1, k'_2 \ldots]$. Then $a : b \overset{A}{>} c : d$ if $k_1 > k'_1$, or $k_1 = k'_1, k_2 < k'_2$, or generally $k_1 = k'_1, k_2 = k'_2 \ldots k_{i-1} = k'_{i-1}$ and $k_i > k'_i$ if $i$ is odd, $k_i < k'_i$ if $i$ is even; I leave it to the reader to discuss the case where one of the continued fractions terminates at some $k_n$. The medieval Islamic mathematicians did not have this modern notation, and for them the definition was complicated by the fact that the $k_i$ and $k'_i$ had to be found by anthyphairesis applied to the pairs $(a, b)$ and $(c, d)$ respectively. Al-Māhānī's definition is rather short in comparison with the elaborate (and not completely exhaustive) definition given by al-Khayyāmī.
- The equivalence of the definitions can be proved by ancient methods, and if a Greek anthyphairetic ratio theory for general magnitudes existed, the geometers (or commentators on the *Elements*) would naturally be interested in proofs of $a : b \overset{A}{=} c : d \Longleftrightarrow a : b \overset{E}{=} c : d$ and $a : b \overset{A}{>} c : d \Longleftrightarrow a : b \overset{E}{>} c : d$. Traces of such proofs have not been found in Greek texts.

## Summary of al-Māhānī's Treatise

Al-Māhānī begins with an introduction which I will quote below. He then gives two different (but equivalent) definitions of $a : b \overset{A}{=} c : d$. The introduction and these definitions were quoted by Plooij [1950: 50–51], who does not say anything on the rest of al-Māhānī's treatise.[6] Al-Māhānī then defines $a : b \overset{A}{>} c : d$, see the next section. He then informs the reader that he will use Euclid's propositions *Elements* V.1, 2, 3, 5, 6. These propositions are about multiples of magnitudes and do not rely on the definitions of $a : b \overset{E}{=} c : d$ and $a : b \overset{E}{>} c : d$.

Al-Māhānī then proves four propositions:

1. $a : b \overset{E}{=} c : d \Longrightarrow a : b \overset{A}{=} c : d$.
2. $a : b \overset{E}{>} c : d \Longrightarrow a : b \overset{A}{>} c : d$.

6 This explains the incorrect statement in [Juschkewitsch 1964: 251]: "al-Māhānī, bei dem wir allerdings noch keine logische Analyse des Zusammenhanges zwischen dieser Definition (i.e., the anthyphairetic definition) und der Definition im Buch V der 'Elemente' des Eukleides finden," in the French translation [Youschkevitch 1976: 84]: "al-Māhānī, qui ne donne encore aucune analyse du lien existant entre cette définition et celle du Livre V des Éléments ...."

3. $a : b \overset{A}{=} c : d \Longrightarrow a : b \overset{E}{=} c : d$ and $a : b \overset{A}{>} c : d \Longrightarrow a : b \overset{E}{>} c : d$. The proofs in Proposition 3 are by *reductio ad absurdum*.
4. An alternative proof of $a : b \overset{A}{>} c : d \Longrightarrow a : b \overset{E}{>} c : d$, with an explicit construction of numbers $m, n$ such that $ma > nb$ and $mc \leq nd$. The treatise ends with this construction.

Al-Māhānī says in his introduction:

> This is what I had found in the beginning about the ratio of magnitudes and their proportionality. I have used it in the proof of the multiples which Euclid used in (defining) magnitudes which have the same ratio, and which have a greater ratio, in the preamble to the fifth Book, where Thābit ibn Qurra wrote about this that knowledge of the ratio of magnitudes and of their proportionality according to rules is something which man can acquire from the knowledge of ratios in the way of (whole) numbers and the propositions in the beginning of the tenth book.[7]

The introduction shows that al-Māhānī preferred the anthyphairetic definitions of equal and greater ratio over the Euclidean definitions, and that he thought that the properties in the Euclidean definitions have to be proved. Thābit ibn Qurra (836–901) translated Books V–VII of the *Conics* of Apollonius from Greek into Arabic, and he also revised the translation of Euclid's *Elements* by Isḥāq ibn Ḥunayn.[8] The propositions in the beginning of the tenth book are *Elements* X.2–3 mentioned above. Apparently, Thābit also knew about the anthyphairetic theory of ratios.

Al-Māhānī does not tell us whether the contents of the treatise are his own invention. The Arabic "this is what I have found" (*wajadtu*) does not imply originality; it may also mean that he found the proofs in some source. As I will argue below, I believe that the wording of the proofs is by al-Māhānī, but that he took the substance from another source, probably a Greek work in Arabic translation. As far as I know, there is no evidence that al-Māhānī knew Greek.

## Al-Māhānī's Anthyphairetic Definitions of Equal and Greater Ratio

Al-Māhānī's two definitions of $a : b \overset{A}{=} c : d$ correspond to Becker's reconstruction of the ancient theory. I quote al-Māhānī's last definition here so the reader can get a feeling for the terminology. Al-Māhānī says:

7 Arabic text:
هذا ما كنت وجدته أولًا في نسبة المقادير وتناسبها وعملت عليه في برهان الأضعاف التي عمل عليها أقليدس في المقادير التي نسبتها واحدة وفي الأعظم نسبةً في صدر المقالة الخامسة، عند ما كان ثابت بن قرة كتب به من أن العلم بنسبة المقادير وتناسبها على الأحكام شيء يمكن للإنسان أن يقف عليه من قبل المعرفة بالنسبة على السبيل العددية ثم من الأشكال التي في أول المقالة العاشرة.

8 On Thābit ibn Qurra see [Sezgin 1974: 264–272].

> Magnitudes in the same ratio are those with (the following property): if the first and the third are measured by the second or the fourth, or inversely, they are measured the same number of times, and if there are left over two remainders less than the two lesser magnitudes, and the two lesser magnitudes are measured by them, they are measured the same number of times, and in this way indefinitely.[9]

This means in the notation of the previous section of this paper that $a_1 : a_2 \stackrel{A}{=} b_1 : b_2$ and $a_2 : a_1 \stackrel{A}{=} b_2 : b_1$ if $k_1 = k'_1, k_2 = k'_2, \ldots$ ("they are measured the same number of times").

In all manuscripts of al-Māhānī's treatise which I have seen there are one or more serious scribal errors in the definition of $a : b \stackrel{A}{>} c : d$. What I present here as "al-Māhānī's definition" is my tentative restoration of the text, based on the assumption that the definition of $a_2 : a_1 \stackrel{A}{>} b_2 : b_1$ in al-Māhānī's treatise in its original form was consistent with the way in which he used it in Propositions 2, 3 and 4 of his text. On the basis of the same argument, I have also made a number of emendations elsewhere in my edition of the text.

Here is my reconstructed definition. Al-Māhānī says:

> On being greater in ratio. The ratio of the first (magnitude) to the second is said to be greater than the ratio of the third to the fourth, if the first, or a remainder of it, (which is) not less than the second, or a remainder of the second, corresponding to the remainder of the first, exceeds the second or a remainder of it, corresponding to the remainder of the first (magnitude), any of them (exceeding) the one corresponding to it, but the third (magnitude), or a remainder of it, of the (same) rank of the remainder of the first, and of the same rank as the remainder of the second, does ⟨not⟩[10] exceed the fourth (magnitude) or a remainder of the fourth (magnitude) of the rank of the remainder of the second.[11]

I have divided the following explanation of the definition into steps, to agree with my discussion of Proposition 2 in the next section. For the same reason, I use the notation $a_2 : a_1$ and $b_2 : b_1$ for the two ratios, although it is of course confusing to call al-Māhānī's first, second, third and fourth magnitudes $a_2$, $a_1$, $b_2$, $b_1$ respectively.

9 Arabic text:
المقادير التي نسبتها واحدة هي التي إذا قدر الأول والثالث بالثاني والرابع أو بالعكس كانت مرات تقديرهما متساوية وإن فضل منهما مقداران أقلّ من الأصغرين وقدر الأصغران بهما كانت أيضًا مرات هذا التقدير متساوية وهكذا إلى ما لا نهاية له.

10 The word "not" (Arabic: *lā*) is my addition to restore the mathematical sense.

11 Arabic text:
في الأعظم نسبةً. يقال أن نسبة الأول إلى الثاني أعظم من نسبة الثالث إلى الرابع متى كان الأول أو فضل باقٍ منه ليس بأصغر من الثاني أو فضل يبقى من الثاني نظير للفضل الباقي من الأول يزيد على الثاني أو على فضل باقٍ منه نظير للفضل الباقي من الأول أيهما كان على نظيره، وكان الثالث أو فضل يبقى منه في مرتبة الفضل الذي يبقى من الأول في مثل مرتبة الفضل الباقي من الثاني ⟨ لا ⟩ يزيد على الرابع أو على فضل يبقى من الرابع في مرتبة الفضل الباقي من الثاني.

**Step 0.** We have $a_2 : a_1 \overset{A}{>} b_2 : b_1$ in the following two (overlapping) cases:

- $a_2 \geq a_1,\ b_2 < b_1$,
- $a_2 > a_1,\ b_2 \leq b_1$.

In the definition, al-Māhānī only refers to the second of these two cases. It is possible to make further emendations to the text of the definition so that the first case is included. These emendations are unnecessary if al-Māhānī implicitly assumed $a_2 : a_1 \overset{A}{>} b_2 : b_1 \Longleftrightarrow b_1 : b_2 \overset{A}{>} a_1 : a_2$.

**Step 1.** If $a_2 < a_1,\ b_2 < b_1$, there is an integer $k_1 \geq 1$ such that $a_1 = k_1 a_2 + a_3$, $b_1 = k_1 b_2 + b_3$, and either $0 < a_3 \leq a_2$ or $0 < b_3 \leq b_2$ or both. Al-Māhānī calls $a_3$ and $b_3$ "remainders" of $a_1$ and $b_1$.
We have $a_2 : a_1 \overset{A}{>} b_2 : b_1$ in the two (overlapping) cases:

- $a_2 \geq a_3,\ b_2 < b_3$,
- $a_2 > a_3,\ b_2 \leq b_3$.

**Step 2.** If $a_3 < a_2,\ b_3 < b_2$, we continue in the same way: there is an integer $k_2 \geq 1$ such that $a_2 = k_2 a_3 + a_4$, $b_2 = k_2 b_3 + b_4$, and either $0 < a_4 \leq a_3$ or $0 < b_4 \leq b_3$ or both. Al-Māhānī calls $a_4$ and $b_4$ "remainders" of $a_2$ and $b_2$.
We have $a_2 : a_1 \overset{A}{>} b_2 : b_1$ in the two (overlapping) cases:

- $a_4 \geq a_3,\ b_4 < b_3$,
- $a_4 > a_3,\ b_4 \leq b_3$.

If $a_4 < a_3,\ b_4 < b_3$ we continue with Step 3, analogous to Step 1, and so on. In general, al-Māhānī calls $a_{2n+1},\ b_{2n+1}$ remainders of $a_1,\ b_1$, and $a_{2n},\ b_{2n}$ remainders of $a_2,\ b_2$. Note that $a_2 : a_1 \overset{A}{>} b_2 : b_1 \Longleftrightarrow a_2 : a_3 \overset{A}{>} b_2 : b_3 \Longleftrightarrow a_4 : a_3 \overset{A}{>} b_4 : b_3$ and so on.

Al-Māhānī's definition is based on anthyphairesis, but in his definition the subtractions are done at the same time in both pairs of magnitudes. Thus if $a_2 < a_1$ and $b_2 < b_1$, al-Māhānī subtracts a certain number of times $a_2$ from $a_1$ and $b_2$ from $b_1$ and stops as soon as one of the remainders $a_3, b_3$ satisfies the inequality $a_3 \leq a_2$ or $b_3 \leq b_2$. In his definition of $a_2 : a_1 \overset{A}{>} b_2 : b_1$, the remainders are always positive, although in his definition of $a_2 : a_1 \overset{A}{=} b_2 : b_1$ he also considers the possibility that there may not be a remainder left (in modern terms $a_3 = 0$). I leave it to the reader to verify that al-Māhānī's definition of $a_2 : a_1 \overset{A}{>} b_2 : b_1$ is mathematically equivalent to Becker's reconstructed definition.

Al-Māhānī does not give an explicit definition of $a_2 : a_1 \overset{A}{<} b_2 : b_1$. However, elsewhere in his text he implicitly assumes $a_2 : a_1 \overset{A}{<} b_2 : b_1 \Longleftrightarrow b_2 : b_1 \overset{A}{>} a_2 : a_1$. Thus we can derive his definition for $a_2 : a_1 \overset{A}{<} b_2 : b_1$ by interchanging $a_i$ and $b_i$ in the above argument. Then it is possible to prove that for two arbitrary pairs of magnitudes of the same kind $a_2,\ a_1$ and $b_2,\ b_1$ we have either $a_2 : a_1 \overset{A}{=} b_2 : b_1$ or $a_2 : a_1 \overset{A}{<} b_2 : b_1$ or $a_2 : a_1 \overset{A}{>} b_2 : b_1$. Al-Māhānī assumes this property in his text but does not give a proof of it. Perhaps he considered it as self-evident, or as an axiom. The property is an axiom according to al-Khayyāmī in his treatise which I will discuss below.

## Outline of al-Māhānī's Proposition 2

In Proposition 2 al-Māhānī assumes $a_2 : a_1 \overset{E}{>} b_2 : b_1$. Thus there are integers $m, n$ such that

$$na_2 > ma_1,\ nb_2 \leq mb_1.$$

He wants to prove $a_2 : a_1 \overset{A}{>} b_2 : b_1$.

In modern terms, al-Māhānī's proof boils down to the following, if all errors and omissions are corrected.

**Step 0.** Al-Māhānī supposes $n > m$. Therefore $b_2 < b_1$. If $a_2 \geq a_1$, we have $a_2 : a_1 \overset{A}{>} b_2 : b_1$ by Step 0 of al-Māhānī's definition.

**Step 1.** If $n > m$, $a_2 < a_1$, $b_2 < b_1$, consider the first quotient $k_1$ in the continued fraction expansion of $a_1/a_2 = [k_1, \ldots]$. If $k_1$ is not the same as the first quotient in the two continued fraction expansions of $n/m$ and $b_1/b_2$, or if $a_1 = k_1 a_2$, it is easy to show that $a_2 : a_1 \overset{A}{>} b_2 : b_1$ by Step 1 of the definition.

In all other cases we have

$$n'a_2 > ma_3,\ n'b_2 \leq mb_3,$$

for $n' = n - mk_1$, $a_3 = a_1 - k_1 a_2$, $b_3 = b_1 - k_1 b_2$ with $0 < n' < m$, $0 < a_3 < a_2$, $0 < b_3 < b_2$.

**Step 2.** Now consider the first quotient $k_2$ in the continued fraction expansion of $b_2/b_3$. If $k_2$ is not the same as the first quotient in the two continued fraction expansions of $m/n'$ and of $a_2/a_3$, or if $b_2 = k_2 b_3$, it is easy to show that $a_2 : a_1 \overset{A}{>} b_2 : b_1$ by Step 2 of the definition. In all other cases we have

$$n'a_4 > m'a_3,\ n'b_4 \leq m'b_3,$$

for $m' = m - n'k_2$, $a_4 = a_2 - k_2 a_3$, $b_4 = b_2 - k_2 b_3$ with $0 < m' < n'$, $0 < a_4 < a_3$, $0 < b_4 < b_3$.

If one continues this process it will stop eventually, since the integers $n, m, n'$, $m' \ldots$ decrease.

Al-Māhānī does not inform the reader what exactly should be done if the process does not stop with Step 2. In this case one should repeat Step 1, with $a_4, a_3, b_4, b_3, n', m'$ instead of $a_2, a_1, b_2, b_1, n, m$. Al-Māhānī also fails to give instructions for the case where $m > n$ at the beginning. In this case, one can proceed immediately to Step 2 with $n' = n$, $a_3 = a_1$. These substitutions are easy in modern notation but they are by no means self-explanatory for an ancient Greek or medieval Islamic geometer.

The mathematical errors in all available manuscripts of al-Māhānī's text can of course be corrected, but they cannot be explained as the result of mechanical scribal errors. Thus the flaws were, in all likelihood, in the original text that al-Māhānī wrote.

In Proposition 4, al-Māhānī assumes $a_2 : a_1 \overset{A}{>} b_2 : b_1$. Then he uses the process of Proposition 2 in reverse order in the construction of numbers $m, n$ such that $na_2 > ma_1$, $nb_2 \leq mb_1$. He concludes $a_2 : a_1 \overset{E}{>} b_2 : b_1$. This proof also contains mathematical flaws which can be corrected but which cannot be explained as scribal errors.

## Al-Nayrīzī on Anthyphairetic Ratios

In the commentary on the definitions of Book V of the *Elements*, al-Nayrīzī uses some anthyphairetic ratios. The first relevant passage is occasioned by Euclid's Definition 3, "a ratio is a sort of relation in respect of size between two magnitudes of the same kind" [Heath 1956, II: 114]. In his commentary on this definition, al-Nayrīzī discusses anthyphairesis for two commensurable and for two incommensurable magnitudes [Junge *et al.* 1932: 4–6]. Al-Nayrīzī then interpolates the following definition, which he ascribes to Euclid, and which I will call Definition $3\frac{1}{2}$: "Proportionality is equality of ratios, and it (only) exists for at least three magnitudes." Definition $3\frac{1}{2}$ is a combination of Euclid's Definitions 6 and 8. In his commentary on Definition $3\frac{1}{2}$, al-Nayrīzī explains what "equality of ratios" is. He gives a general anthyphairetic definition of $a : b \overset{A}{=} b : c$, followed by an example, and he then explains by an example the meaning of $a : b \overset{A}{=} c : d$, without giving a general definition [Junge *et al.* 1932: 8–12]. Below we will see that his understanding of $a : b \overset{A}{=} c : d$ was less than perfect. He then says that "If the situation of the magnitudes, in their equality between them, is not this situation, they are not proportional" [Junge *et al.* 1932: 12–13], and he gives the following incorrect definition of $a : b \overset{A}{>} c : d$:

> If the first (magnitude) measures the second by a number less than the number by which the third measures the fourth, *and* if from the second a remainder is left which measures the first magnitude by a number which is also *less* than the number by which the remainder of the fourth magnitude measures the third, and if also two remainders from the first and the third are left over which have the same situation with respect to the first remainders of the second and the fourth, and this situation between the interchanging remainders continues indefinitely, then this situation does not fall under proportionality, but it is the situation where the ratio of the first to the second is greater than the ratio of the third to the fourth [Junge *et al.* 1932: 12–13].

In my notation of the previous section, al-Nayrīzī seems to say that if $b/a = [k_1, \ldots]$ and $d/c = [k'_1, \ldots]$, we have $a : b \overset{A}{>} c : d$ if $k_1 < k'_1$ and if $k_1 = k'_1, k_2 < k'_2, \ldots$. In reality, $a : b \overset{A}{>} c : d$ if $k_1 < k'_1$ or $k_1 = k'_1, k_2 > k'_2$, etc. To make mathematical sense here, the italicized words *and* and *less* in the quoted passage should be changed to *or* and *greater*. Al-Nayrīzī then gives a definition of a ratio smaller than another ratio, which we obtain from the quoted passage by changing all words "less" to "greater." This definition is also incorrect. Thus it appears that al-Nayrīzī probably did not understand the mathematics.[12]

12 This was pointed out in [Junge *et al.* 1932: 12, n. 1].

If the mathematical errors are corrected, the definitions are reminiscent of the reconstructed definition of $a : b \overset{A}{>} c : d$ and $a : b \overset{A}{<} c : d$ in [Becker 1933a: 312] and the correct definition by al-Khayyāmī, but al-Nayrīzī's definitions are unlike al-Māhānī's definition of $a : b \overset{A}{>} c : d$.

Al-Nayrīzī next discusses Euclid's Definition 4,[13] and then turns to the problematic Definition 5: $a : b \overset{E}{=} c : d$ if for all pairs of integer multiples $ma, mc, nb, nd$ we have: $ma > nb \Longleftrightarrow mc > nd$, $ma = nb \Longleftrightarrow mc = nd$, and $ma < nb \Longleftrightarrow mc < nd$. Al-Nayrīzī first explains that Euclid did not mean to compare the multiples $m, n$ but the quantities $ma, nb, mc, nd$. He continues with the following, somewhat obscure, sentences, into which I have inserted the numbers (1) and (2). Words that I have restored to the text are in angular brackets ⟨ ⟩, and all italics are mine.

> He (Euclid) means that the excess is similar to the multiples of the common number by which the first (magnitude) measures the second and the third measures the fourth, and this is when the first is commensurable with the second, and the third is commensurable with the fourth. If they are not commensurable but incommensurable, then (1) the number of the magnitudes by which the *multiples of the* first measures the *multiples of the* second are similar to the number of the magnitudes by which the *multiples of the* third measures the *multiples of the* fourth; and (2) the number of the magnitudes by which the remainder of the second measures the first, is equal to the number of magnitudes by which ⟨ the remainder of the fourth measures the third, and the number of the magnitudes by which the remainder of the first measures the remainder of the second, is equal to the number of magnitudes by which ⟩ the remainder of the third measures the remainder of the fourth, and this then continues indefinitely.[14]

One can make mathematical sense of the passage by deleting the four italicized occurrences of the phrase *multiples of the*. What is left is a general definition of $a_2 : a_1 \overset{A}{=} b_2 : b_1$. I conclude that al-Nayrīzī had access to this definition but did not really understand it well enough to avoid confusion with the Euclidean definition.

13 "Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another."

14 Here is the Arabic text of the last part of the passage:

فيكون عدد المقادير التي تعدّ أضعاف الأول لأضعاف الثاني مساويًا لعدد المقادير التي تقدر أضعاف الثالث لأضعاف الرابع ويكون عدد المقادير التي تقدر فضلة الثاني للأول مساويًا لعدد المقادير التي تقدر ⟨ فضلة الرابع للثالث ويكون عدد المقادير التي تقدر فضلة الأول لفضلة الثاني مساويًا لعدد المقادير التي تقدر ⟩ فضلة الثالث لفضلة الرابع ثمّ لا يزال كذلك إلى غير نهاية.

Junge [1932: 16] emended the text in a different way: he did not include the passage in angular brackets but changed the last فضلة الثالث لفضلة الرابع to فضلة الرابع الثالث, which is less plausible. I have checked the text in Junge's edition (based on the Leiden manuscript) with f. 99b of the newly found manuscript of al-Nayrīzī's commentary in the Marashi Library in Qom; this manuscript agrees with the Leiden manuscript but not with Junge's emendation.

Al-Nayrīzī continues:

> Euclid did not mean anything else than this. As for those who wanted to present proofs of this and other (related things), this is a deviation, since it forces them to propositions whose subjects are more advanced, and if their purpose had been the truth itself, they would have known that this thing does not need a proof, because it belongs to the first principles for someone who has reached this place, since every treatise has principles which are in accordance with the level of that treatise [Junge *et al.* 1932: 16–17].

Al-Nayrīzī probably means that Euclid's definition is equivalent to the anthyphairetic definition, but apparently he found it unnecessary to prove the statement $a : b \overset{E}{=} c : d \Longrightarrow a : b \overset{A}{=} c : d$ in his commentary on this elementary work.

I conclude that if al-Nayrīzī had seen al-Māhānī's treatise, he was probably not impressed by it. Since al-Nayrīzī's anthyphairetic definitions of ratios greater or less than other ratios cannot have been derived from al-Māhānī's treatise, there must have been more literature on the subject available around A.D. 900, when al-Nayrīzī wrote his commentary.

## Al-Khayyāmī on Anthyphairetic Ratio Theory

I will now present a brief analysis of the second part of the *Letter on the Explanation of the Difficulties in the Postulates of Euclid* by al-Khayyāmī (1048–1131), on which see also [Vitrac 2002]. This second part is devoted to the relation between Euclidean and anthyphairetic ratio theory. According to al-Khayyāmī, the anthyphairetic definitions are the "true" definitions, whereas the Euclidean definitions are only "well-known." The purpose of the second part is to present proofs of the equivalence of the two definitions of equal ratios and greater ratios. Thus al-Khayyāmī proves the same theorems as al-Māhānī, but the proofs are very different.

Al-Khayyāmī has a separate discussion of the case where the ratios in question are ratios between rational numbers. This case will not concern us here.

In his discussion of irrational ratios, al-Khayyāmī assumes as an axiom that for two pairs of magnitudes $a, b$ and $c, d$ of the same kind, either $a : b \overset{A}{>} c : d$ or $a : b \overset{A}{=} c : d$ or $a : b \overset{A}{<} c : d$. For two ratios $a : b$ and $c : d$ which satisfy the Euclidean definition $a : b \overset{E}{=} c : d$, al-Khayyāmī assumes that $a : b$ and $c : d$ also satisfy all properties of ratios proved by Euclid in Book V of the *Elements*.

Al-Khayyāmī assumes as an axiom the existence of a fourth proportional in a way which is in modern notation as follows [Djebbar 1997: 48; Vahabzadeh 1999: 292–293, 350–351; Djebbar 2002: 113]:

If $a, b$ are two magnitudes of the same kind, and $c$ is another magnitude, then

1. there is a magnitude $f$ of the same kind as $c$ such that $a : b \overset{A}{=} c : f$,
2. there is a magnitude $f'$ of the same kind as $c$ such that $a : b \overset{E}{=} c : f'$.

Al-Khayyāmī does not make a clear distinction between $f$ and $f'$, but in his proofs, he does not make the tacit assumption that $f$ and $f'$ are necessarily the same.

The existence of a fourth proportional $f$ or $f'$ is a problematic assumption because it implies the solution of numerous problems, for example the quadrature of the circle.[15]

Al-Khayyāmī first proves $a : b \overset{E}{=} c : d \Longrightarrow a : b \overset{A}{=} c : d$, by a method which differs from that of al-Māhānī. The remaining three proofs in al-Khayyāmī's text are based on two lemmas:

1. Lemma 1: $c : d \overset{A}{=} c : f \Longrightarrow d = f$ [Vahabzadeh 1999: 295, lemme 2.3],
2. Lemma 2: $c : d \overset{A}{>} c : f \Longrightarrow f > d$ [Vahabzadeh 1999: 296, lemme 2.4].

These lemmas are straightforward consequences of the anthyphairetic definitions. They do not require a complicated proof as in al-Māhānī's Proposition 2.

The three other proofs are now simple:

- Suppose $a : b \overset{A}{=} c : d$. There is a magnitude $f$ such that $a : b \overset{E}{=} c : f$, hence by the first theorem $a : b \overset{A}{=} c : f$, so, by Lemma 1, $d = f$, whence $a : b \overset{E}{=} c : d$.
- Suppose $a : b \overset{E}{>} c : d$. If $a : b \overset{A}{=} c : d$, we have by the previous theorem $a : b \overset{E}{=} c : d$, contradiction. If $a : b \overset{A}{<} c : d$, there is a magnitude $f$ such that $a : b \overset{A}{=} c : f$, so $c : f \overset{A}{<} c : d$, hence, by Lemma 2, $f > d$. By the previous theorem $a : b \overset{E}{=} c : f$, hence $c : f \overset{E}{>} c : d$, so, by Euclid's *Elements* V.10, $f < d$, contradiction. Thus $a : b \overset{A}{>} c : d$.
- Suppose $a : b \overset{A}{>} c : d$. In the previous proof, we interchange $E$ with $A$, and *Elements* V.10 with Lemma 2. We conclude $a : b \overset{E}{>} c : d$.

I finish my brief analysis of al-Khayyāmī's text by a discussion of his definition of greater ratio, which consists of an incomplete list of cases presented in a confusing order. I will discuss al-Khayyāmī's definition here in much greater detail than [Vahabzadeh 1999: 291–292], not only because it may have been influenced by al-Māhānī, but also to illustrate the difficulty of the subject to a medieval Islamic mathematician. For sake of clarity I have inserted the numbers (1) to (8) in the definition. Al-Khayyāmī says:

> As for the geometrical (definition of $\overset{A}{>}$): If all multiples of the first are subtracted from the second and a remainder is left, and all multiples of the third are subtracted from the fourth and a remainder is left, and (1) the number of multiples of the first is less than the number of multiples of the third, or (2) this number is equal to that number, but all multiples of the remainder of the second are subtracted from the first until a remainder is left, and all multiples of the remainder of the fourth are subtracted from the third until a remainder is left, and the number of multiples of the remainder of the second is greater than the number of multiples of the remainder of the fourth, or (3) this number is also equal to that number, but if all multiples of the remainder of the first are subtracted from the remainder of the second, and all multiples of the remainder of the third are subtracted from the remainder of the fourth, and the number of multiples of the remainder of the first is less, (4) or no

15 If the existence of a fourth proportional is assumed, we can find the quadrature of a circle with radius $r$ as follows. Take $a$ the square with side $r$, $b$ the circle with radius $r$, $c$ an arbitrary line segment, and let $f$ be the line segment such that $a : b = c : f$. If $g$ is a mean proportional between $c$ and $f$, the square with side $g$ is equal to the circle with radius $r$. On the history of the fourth proportional in Greek geometry see [Becker 1933b].

remainder is left of the remainder of the second or of the second, and a remainder is left of the remainder of the fourth or of the fourth, then the ratio of the first to the second is necessarily greater than the ratio of the third to the fourth, in reality (i.e., according to the anthyphairetic theory).

In general, in this kind (of ratio): if (5) either no remainder is left of the second or of its remainders, (6) or its remainders are less in number,[16] (7) or a remainder is left of the first or of its remainders, and no remainder is left of the third or of its remainders, (8) or the remainders of the first are greater (in number)[17] than the remainders of the third, then the ratio of the first to the second is greater than the ratio of the third to the fourth.

For this notion there is a longer distinction of cases than this, and you can know it by this rule which I have taught you, so understand it (my own translation of the Arabic text in [Vahabzadeh 1999: 349/8–25]; compare the French translations in [Djebbar 1997: 47–48; Vahabzadeh 1999: 348; Djebbar 2002: 111–112] and the English translation in [Vahabzadeh 1997: 252]).

Here is a paraphrase of al-Khayyāmī's definition in my standard notation.

His first, second, third and fourth magnitudes are my $a_2, a_1, b_2$ and $b_1$ respectively. Al-Khayyāmī assumes, just like al-Māhānī, $a_2 < a_1$ and $b_2 < b_1$. He then defines the "remainder of the second" (my $a_3$) and the "remainder of the fourth" (my $b_3$) by $a_1 = k_1 a_2 + a_3$, $b_1 = k'_1 b_2 + b_3$ for integers $k_1$, $k'_1$, such that $0 \leq a_3 < a_2$, $0 \leq b_3 < b_2$. Then $a_2 : a_1 \overset{A}{>} b_2 : b_1$ if $k_1 < k'_1$ (Case 1).

If $k_1 = k'_1$, he tacitly assumes $a_3 \neq 0$, $b_3 \neq 0$ and proceeds to define his "remainder of the first" (my $a_4$) and "remainder of the third" (my $b_4$) by $a_2 = k_2 a_3 + a_4$, $b_2 = k'_2 b_3 + b_4$ for integers $k_2$, $k'_2$, such that $0 \leq a_4 < a_3$, $0 \leq b_4 < b_3$. Then $a_2 : a_1 \overset{A}{>} b_2 : b_1$ if $k_2 > k'_2$ (Case 2).

If $k'_2 = k_2$, he tacitly assumes $a_4 \neq 0$, $b_4 \neq 0$ and then defines $k_3$ and $k'_3$ in the same way by $a_3 = k_3 a_4 + a_5$, $b_3 = k'_3 b_4 + b_5$ with $0 \leq a_5 < a_4$, $0 \leq b_5 < b_4$. Then $a_2 : a_1 \overset{A}{>} b_2 : b_1$ if $k_3 < k'_3$ (Case 3).

Having come thus far, he realizes some of his tacit assumptions and says that $a_2 : a_1 \overset{A}{>} b_2 : b_1$ also if ($k_3 = k'_3$) $a_5 = 0$, $b_5 > 0$, or $a_3 = 0$, $b_3 > 0$ (Case 4).

He then lists four more general cases where $a_2 : a_1 \overset{A}{>} b_2 : b_1$ (I render his tacit assumptions in parentheses and take $n$ as an arbitrary integer $\geq 0$.)

- ($k_1 = k'_1$), $a_3 = 0$, $b_3 > 0$, and in general ($k_1 = k'_1$, $k_2 = k'_2$, ... $k_{2n-1} = k'_{2n-1}$), $a_{2n+1} = 0$, $b_{2n+1} > 0$ (Case 5);
- ($k_1 = k'_1$), $k_2 < k'_2$ or ($k_1 = k'_1$, $k_2 = k'_2$, ... $k_{2n-1} = k'_{2n-1}$), $k_{2n} < k'_{2n}$ (Case 6);

16 Vahabzadeh supplied the passage "than the remainders of the fourth." The word "petits" in his translation "ses restes sont plus petits que les restes de la quatrième," is imprecise. Al-Khayyāmī uses *aqall*, meaning: less in number, so he means $k_2 < k'_2$, $k_{2n} < k'_{2n}$. For $a_3 < b_3$, $a_{2n+1} < b_{2n+1}$, he would have used the word *asghar*, because $a_3$ and $b_3$ are magnitudes but not numbers. Djebbar translates correctly "inférieurs en nombre."

17 Vahabzadeh's translation: "les restes de la première sont plus grands que les restes de la troisième," is inaccurate. Al-Khayyāmī uses *aʿẓam* for magnitudes and *akthar* for numbers. Djebbar translates correctly "(en nombre) supérieurs."

- $(k_1 = k'_1, k_2 = k'_2)$, $a_4 > 0$, $b_4 = 0$, and in general $(k_1 = k'_1, k_2 = k'_2, \ldots k_{2n} = k'_{2n})$, $a_{2n+2} > 0$, $b_{2n+2} = 0$ (Case 7);
- $(k_1 = k'_1, k_2 = k'_2)$, $k_3 > k'_3$ (Case 8).

Case 8 is incomplete, and Cases 6 and 8 are incorrect and the opposite of the correct Cases 1 and 2. The tacit assumptions at the beginning of al-Khayyāmī's definition resemble similar assumptions in the definition by al-Māhānī, who never works with remainders equal to zero.

## Conclusion

Anthyphairetic ratio theory may have been investigated by more Islamic mathematicians than have been mentioned in this paper. My impression is that al-Māhānī and al-Khayyāmī understood the definition of $a : b \overset{A}{=} c : d$ correctly. Al-Nayrīzī did not understand the definition of $a : b \overset{A}{>} c : d$; al-Khayyāmī stated the definition of $a : b \overset{A}{>} c : d$ in a longwinded and incorrect way but he was able to apply it in a simple situation; only in al-Māhānī's text do we find more profound applications of $a : b \overset{A}{>} c : d$, such as in his Proposition 2. My tentative conclusion is that a real tradition of *working* with anthyphairetic ratios did not exist in Islamic mathematics in a way which went beyond a study of the definitions.

Mathematically, the proofs in al-Māhānī's text are more interesting than al-Khayyāmī's proofs because al-Māhānī does not assume the existence of a fourth proportional and dislikes proofs by *reductio ad absurdum*. Al-Māhānī's text is problematic because there are many flaws in it that cannot be the result of scribal errors, and that must therefore have been in the original version which al-Māhānī wrote. Because the proofs are very complex and can be easily corrected, I believe that they were ultimately derived from correct proofs, that had been transmitted to al-Māhānī in incomplete or mutilated ways. If the author of these correct proofs was an Islamic mathematician, he must have been a near contemporary of al-Māhānī, so an incomplete transmission would be very unlikely. It is much more probable that the proofs were in a Greek work that had been transmitted into Arabic in an incorrect way. This work may have been a lost commentary on Euclid's *Elements* or a text such as Menelaus' lost *Geometrical Elements* which was translated into Arabic by Thābit ibn Qurra.[18] The Greek text may have contained a proof which was obscurely worded but correct, or the proof may have been incorrect but adapted from an even more ancient Greek source which stated the proof correctly. Al-Māhānī rephrased the text and he did his best to correct the mathematical errors and infelicities, but he was not completely succesful.[19] Thus al-Māhānī's text may tell us as much about Greek mathematics as about mathematics in Islamic civilization. My hypothesis of a Greek origin of al-Māhānī's text is preliminary, and will have to be the subject of further research.

18 On this work see [Hogendijk 2000].

19 In a similar way, al-Māhānī tried to make sense of the incorrect version of the *Spherics* of Menelaus that was available to him, see [Krause 1936: 26].

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Dr. Ahmed Djebbar (Paris) made manuscripts and his own preliminary edition of al-Māhānī's treatise available to me. I thank Dr. Djebbar, Dr. David Fowler (Warwick), Dr. Bernard Vitrac (Paris) and the editors Dr. Joe Dauben, Dr. Benno van Dalen and Dr. Yvonne Dold for their comments on preliminary versions of this paper.

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Yushkevitch, Adolf P. — *see* Juschkewitsch, Youschkevitch.

*Note added in proof.* After the typesetting and proofreading of this book had been completed, I received B. Vahabzadeh's new article: Al-Māhānī's Commentary on the Concept of Ratio, *Arabic Sciences and Philosophy* **12** (2002): 9–52, which includes a very good edition of al-Māhānī's treatise with English translation. Compare my footnotes 7, 9, 10, 11 with Vahabzadeh's Arabic text on pp. 44/5–10, 45/9–12, 18, 13–19 and my translations on pages 191 and 192 with his translations on pp. 31/7-17, 32/29-35, and 33/1-13. Vahabzadeh's mathematical and historical commentaries are different from mine.

# An Early Link of the Arabic Tradition of Practical Arithmetic: The *Kitāb al-Tadhkira bi-uṣūl al-ḥisāb wa 'l-farā'iḍ wa-ʿawlihā wa-taṣḥīḥihā*[1]

by ULRICH REBSTOCK

With his *Tadhkira*, an abridged selection of his larger (but unknown) *Kitāb al-Maʿūna*, the hitherto unknown Syrian mathematician Abū 'l-Ḥasan ʿAlī b. al-Khiḍr al-Qurashī (d. 450/1067) offers to interested laymen and students of *ḥisāb* (arithmetic) a series of techniques of basic arithmetical rules applicable to the solution of quite different everyday problems. Various quotations from classical sources and comparative reports on the local methods of calculation as well as on the "hard facts" (measures and weights etc.) they had to cope with turn the *Tadhkira* into an authentic source for the practitioners' milieu of the Middle East in the eleventh century.

## The Book

The *Tadhkira* does not hold the promise of its title. It is divided into two parts (§6–75, §76–131) both of which fall short of being serious treatments of the *principles* of arithmetic or the calculation of inheritance-shares. The keyword "*Tadhkira*," however, which could be rendered as "Memorandum," could explain this defect. It most probably refers to the *opus magnum* of the author, the *Kitāb al-Maʿūna* (The Book of Sustenance), which is mentioned nine times in the *Tadhkira*. The citations are dispersed all over the text and even allow the tentative reconstruction of the contents of the latter. Apparently, the *Kitāb al-Maʿūna*, of which no further bibliographical trace can be detected, must be regarded as the "complete" folio in the back-ground against which the *Tadhkira* only appears as an outline. The markedly unsystematic and erratic structure of both of its parts (*maqāla*) indicate that it was composed to give a summary of the *Kitāb al-Maʿūna* and to call selected parts back to mind.

1 The following remarks are extracted from the recent publication [Rebstock 2001]. For the sake of identification the numbering of the paragraphs was adhered to here. — Translation of the title: "The Book of Memorandum of the Principles of Arithmetic and Inheritance-Shares, of 'Proportional Reduction' and its Adjustment." "Proportional reduction": *ʿawl*, "supplying with", originally applies in *ḥisāb al-farā'iḍ* when the inheritance is over-subscribed and the *portions* must be reduced; hence, in the context of *muʿāmalāt*-fractions, *ʿawl* came to mean the increase of the denominator.

## The Author

Just as his books were not prominent, neither was the author a prominent figure in the history of Arabic literature, let alone in its mathematical circles. The particular silence about the *Tadhkira* requires a specific research effort to reveal the circumstances of its composition and the life of its author. Abū 'l-Ḥasan ʿAlī b. al-Khiḍr b. al-Ḥasan al-ʿUthmānī (al-Qurashī)[2] was born in Rajab 421 / June 1030, presumably in Damascus. He composed books on arithmetic (*ḥisāb*) and bore the nickname "the arithmetician" (*al-ḥāsib*). After a short life — at the age of 37 — he died on Shawwāl 25, 459 / September 10, 1067 in his home-town which he evidently never left. Ibn ʿAsākir (d. 571 / 1176), the famous chronologist of Damascus and earliest source for the existence of our author, made use of the record of the family history by Abū 'l-Ḥasan's brother, al-Ḥasan.[3] With respect to his mathematical tendencies, no more was known about Abū 'l-Ḥasan until ʿUmar Kaḥḥāla came across the manuscript of the *Tadhkira* in 1972 at the ʿArif Ḥikmat-Library in Medina.[4] The present investigation is based on the microfilm copy of this hitherto unique manuscript (written 668 / 1271) produced for the Institute for the History of Arabic Science in Aleppo.

Without any further testimony of the mathematical reputation of our author throughout the last 800 years, the professional quality and seriousness of the *Tadhkira* is difficult to assess. Scattered biographical information, however, allow us to assume — before embarking on a study of the text itself — that Abū 'l-Ḥasan undeservedly slipped through the fingers of the Arabic (and non-Arabic[5]) historians of mathematics. To sum up, this information yields with relative certainty that:

- Abū 'l-Ḥasan was a respected figure in the intellectual life of Damascus during the fifties of the 11th century.
- to a certain degree he enjoyed a mathematical formation. Among his teachers — mainly traditionalists — is mentioned Abū 'l-Qāsim as-Sumaisāṭī (d. 453 / 1061 at the age of 80 in Damascus) who besides *ḥadīth* taught Geometry (*handasa*) and Astronomy (*haiʾa*).
- his striking familiarity with "classical" mathematical texts cannot be traced to a local mathematical school. Far and wide no prominent representative can be identified.

2 al-Khaṭīb al-Baġdādī, who knew Abū 'l-Ḥasan, was the first to call him also "al-Qurashī," see [Ibn ʿAsākir n.d.: 79/13].

3 [Ibn ʿAsākir n.d.: 79/23, 80/7].

4 [ʿUmar Kaḥḥāla 1972: 36].

5 No entry in [Brockelmann 1937–49; Suter 1900; Matvievskaya & Rozenfeld 1983].

## The Text and its Sources

The *Tadhkira* runs over 149 folios at 9 lines each. Rare corrections in the margin as well as several omissions and mistakes point to the limited interest of the copyist in the topic. Numerals or otherwise abbreviated forms of numbers are not used.[6]

Besides the *Kitāb al-Maʿūna*[7] several texts are mentioned: *al-Arithmāṭīqī*, perhaps identical with the *Risālat al-Arithmāṭīqī* of Abū 'l-Wafāʾ al-Būzjānī,[8] the *Kitāb al-Ījāz* of Muḥammad ibn ʿAbd Allāh ibn al-Ḥasan Abū 'l-Ḥusain Ibn al-Labbān,[9] the *Kitāb Tamām al-jabr wa-kamālihī* of Abū Kāmil Shujāʿ b. Aslam,[10] a *Kitāb al-Misāḥa* which could refer to the corresponding chapter of the *Kitāb al-Maʿūna*,[11] the *Kitāb al-M(u)njiḥ al-Ṭabarānī* of a certain Abū 'l-Fatḥ b. M(u)njiḥ[12] and the *Uṣūl*, the *Elements*, of Ūqlīdis.[13]

In addition to the authors of the books cited above, other names are mentioned. Most frequent are references to Abū 'l-Wafāʾ al-Būzjānī, though without adding the title of the *Kitāb fīmā yaḥtāj ilaihi 'l-kuttāb wa 'l-ʿummāl* where most of the citations could be identified.[14] The second most frequently referenced is Abū Bakr Muḥammad b. Mūsā al-Khuwārizmī[15] with one clear contribution of his *Algebra*. To a certain Muḥammad Abū 'l-Ḥasan al-Ḥarrānī al-Thaqafī is attributed the "second best" (after Abū 'l-Wafāʾ's) approximative decomposition of $\frac{1}{11}$ into principal fractions.[16] Leaving aside that "Abū 'l-Ḥasan al-Ḥarrānī" could stand for Thābit b. Qurra, this "known" mathematician could not be identified. The same must be admitted with respect to Muḥammad b. Ḥ-s-m-h al-Ṭarābulsī who treated the prime numbers "in his book" and evidently made use of the *Risālat al-Arithmāṭīqī* of Abū 'l-Wafāʾ.[17] The two experts on the calculation of inheritance (*ḥisāb al-farāʾiḍ*), Abū ʿAbdallāh[18] and Ayyūb b. Sulaimān al-Baṣrī,[19] who wrote a book on the subject, were also ignored by the biographers.

6 With one exception in §54 = fol. 66: 400 is depicted like a pair of pliers.

7 §6: *Kitāb al-Maʿūna ʿalā 'l-ḥisāb*, §34: *Kitāb al-Maʿūna fī 'l-ḥisāb*.

8 §69, cf. [Sezgin 1974: 324, no. 5]; the title could also point to the *Kitāb al-Arithmāṭīqī* of Nicomachos [*ibid.*: 165–166]. The unclear relation of the two books is complicated by the similarly entitled *al-Madkhal al-ḥifẓī ilā ṣināʿat al-arithmāṭīqī* of al-Būzjānī, see [Rebstock 1992: 83, 107, and *passim*].

9 §125, a jurist and *faraḍī* from Baṣra, died 402/1011, cf. [al-Ziriklī n.d., VII: 101].

10 §16 (without title) and §33, no reference in [Sezgin 1974: 277 ff.] or [Matvievskaya & Rozenfeld 1983, II: 112–114]; Ibn an-Nadīm [Tajaddud 1971: 339/–3] mentions a *Kitāb al-Kifāya*.

11 §115.

12 §68, not identified; but see [Sezgin 1967: 195–197], the traditionalist Sulaimān al-Ṭabarānī (of Tiberias), died 360/971.

13 §5, 32, 38, 40, 48, 77, 79.

14 §36, 47, 55, 58, 60, 83.

15 §5–6, 12.

16 §67, 69.

17 §69.

18 §125.

19 §3, 125.

## The Content: Part Two

Abū 'l-Ḥasan concludes Part One with the words: "We have now come to the end with the exposition of many principles that are needed for '*al-ḥisāb al-muwallad*' (approximately: practical arithmetics) and useful for 'buying and selling' (*baiʿ wa-shirāʾ*)".[20] The second part, indeed, deals with the so-called *muʿāmalāt*-problems. They are partly based on the four "proportional numbers" (*al-aʿdād al-mutanāsiba*) and occur whenever unknown magnitudes (of objects) that can be dry measured (*makīl*), weighed (*mauzūn*), measured out (*madhrūʿ*) or counted (*maʿdūd*) are wanted. Both definition and terminology closely resemble the relative passage in *Ḥisāb al-muʿāmalāt* of Ibn al-Haytham (written before 417 / 1027).[21] Frequent references to the *Kitāb al-Maʿūna* make clear that the author is not inclined to go into depth. He rather concentrates on two areas of application: "law" (*fiqh*) and inheritance-shares (*farāʾiḍ*).

By "law", in particular, the conversion of currencies is meant. The juridical aspects of a "lawful" conversion are evident and need not be set forth here. Neither are the mathematical aspects significant. The author skillfully operates with interrelated systems based on 6, 12, 24, 60 and 90. The rates of exchange are expressed in the form of (mixed) fractions. General rules are not demonstrated. The historical aspects, however, are striking. Mathematically organized — i.e., starting with complete tables and continuing with key values — the author gives a systematic surview of the rates of cash values, measures and weights between Iran, Egypt and Yemen. On almost 30 folios nearly 500 equivalents are listed. Compared to the known sources of the socio-economic conditions during the first half of the 11th century, the *Tadhkira* contains material that requires a thorough revision of the data hitherto accepted. The insertion of multiplication tables in which amounts of money are either calculated within one or between two or more currencies, into *ḥisāb*-texts belongs to the Arabic tradition since Abū 'l-Wafāʾ al-Būzjānī.[22] The earliest appearance of this type of "applied" multiplication in authentic European treatises seems to be contained in the *Tractatus algorismi* of Jacopo da Firenze[23] written about 1307. The multiplications operate with *solidi*, *libri*, and *denari*. Jacopo's borrowing of algebraic elements from the Arabic world were closely inspected elesewhere.[24]

20 §75.

21 §77–79 and cf. [Rebstock 1995/96: 65, 111]. J. Sesiano identified the underlying text (Istanbul MS Feyzullah Ef. 1365, fol. 74ʳ/6–76ᵛ/8) as insertion into "ein(es) Werk(s) von Ibn al-Haytham (ca. 965–1040) über Handelsrechnung (*muʿāmalāt*)" [Sesiano 1989: 53]. Closer inspection of the MS copy proved that the anonymous author not only knew of the *muʿāmalāt*-treatise of Ibn al-Haytham but also borrowed complete passages from it, cf. p. 63/–5 ff. (fol. 76ᵛ, 9–13) and [Rebstock 1995/96: 94/3–7]. The dating of this clear repercussion of Ibn al-Haytham's text on the Egyptian milieu — his later home — remains vague: [Sezgin 1974: 368, no. 18] (incipit of the text in [Sezgin 1979: 411–412]): "9th century H.", and [Sesiano 1989: 55].

22 Cf. [Saidan 1971: 178 ff.] (*manzila* II, *bāb* 3, *fuṣūl* 3 f.) and [Rebstock 1992: 253–254].

23 [Høyrup 1993b: 16–18].

24 See [Høyrup 1999a: 23–31].

The paragraphs on the *farā'iḍ*, a keyword after all, are attached to Part Two. Their treatment within the frame of a *ḥisāb*-text represents one of the earliest examples of the literary fusion of the two disciplines. The paragraphs are concise and specific and refer to three peculiarities, one of which is obviously copied from the second part of al-Khuwārizmī's *Algebra*: the calculation of (restricted) bequests (*waṣāyā*). Again, no general rules are quoted. The algebraic terminology explained in the first part[25] is not taken up. Abū 'l-Ḥasan concentrates on the solution of a dynamic inheritance case where the restricting factor of the bequest increasingly complicates the affair. The procedure used to solve the case clearly follows the technique put forth by al-Khuwārizmī and others.

The remaining two peculiarities betray the original perspective of the author; both involve general validity. The first one deals with *ʿawl*, the *pro-rata* reduction of the six basic Qurʾānic portions.[26] The author demonstrates not only that the occurrence of *ʿawl* is limited to a defined class of cases, but also that the corresponding increase of the denominator may not — according to the arithmetical rules — transgress a certain upper limit. The demonstration operates with the methods to find the smallest common denominator. The distinction between the number of portions (*sihām*, *ajzā'*) and the number of "heads" (*ru'ūs*,[27] i. e., heirs) leads to a proportion in which the two (or more) numbers relate to each other either as *mushtarik* (common divisor) or *mutabāyin* or *mukhtalif* (no common divisor); a *mushtarik* proportion may be *mutawāfiq*,[28] *mutamāthil*,[29] or *mutadākhil*.[30] The same terminology is found in the Arabic tradition of al-Nairīzi's commentary on the *Elements* but was so far not known to have been used by the *ḥisāb al-farā'iḍ* before the Yemenite al-Ṣardafī (died around 500/1105) used it to structure his *Kāfī fī 'l-farā'iḍ*.[31]

Abū 'l-Ḥasan is not really interested either in the method itself or in an exhaustive demonstration of the calculation of shares. His intention is rather to point to the different local traditions to solve the cases. For the first time we learn that the experts of Kufa preferred arithmetical methods different from their colleagues' of Baṣra. The difference in calculating the final shares of three or four different *mushtarik*-portions is valued by effectiveness: the Kufians omit one step and are therefore quicker than the Basrians.[32] Evidently, the adoption of arithmetical methods into the hybrid discipline of *ḥisāb al-farā'iḍ* had already advanced to an extent that allowed for local differences and for their supra-regional comparison.

25 §31.

26 §119–120.

27 The same term is used by Abū 'l-Wafā' for fractions of the type $\frac{1}{n}$.

28 I. e., having a *tawāfuq*, agreement, between $a$ and $b$ such as when $a'$ and $b'$ (integers) then $a = na'$ and $b = nb'$.

29 I. e., having a *tamāthul*, similarity, between $a$ and $b$ when $a = b$.

30 I. e., having a *tadākhul*, mutual penetration, between $a$ and $b$ when $a = n \cdot b$ or $b = m \cdot a$ ($n$, $m$ integers).

31 [Rebstock 1992: 223 ff.]; slightly less differentiated by [Saidan 1978: 416] who ascribes "the [clear] idea of the *highest* common factor" to al-Ṭūsī (d. 1274).

32 §124 contains a concise verbal description of the two methods, but no numerical example.

## Part One

Much as Part Two, the first *maqāla* lacks an organized structure. Its 69 paragraphs take up 85 folios. If the second *maqāla* could be described — with good reasons — as "practical," the first one does not really deserve to be opposed to it in the sense of being qualified as theoretical. No proofs are given. Even procedures are rarely demonstrated. The main stress is laid on the results and — ever so often — on the attached crude rules by which these results can be obtained. Clearly, Abū 'l-Ḥasan does not address himself to professional colleagues but to a broad range of practitioners unfamiliar with the basics of arithmetic. The subjects treated across the paragraphs can be grouped into: multiplication of whole numbers (*ḍarb al-ṣiḥāḥ*),[33] multiplication of fractions (*kusūr*),[34] short remarks on *al-jabr wa 'l-muqābala*,[35] remarks on even/odd and prime numbers (*aṣamm*),[36] division of whole numbers and fractions,[37] proportion (*nisba*),[38] various lists of the transformation (*naql*) of fractions into sexagesimal fractions and *vice versa*,[39] and the operation with fractions with prime denominator.[40] Most of what is told in these paragraphs can be traced down to the first and second *manzila* of the *Kitāb fīmā yaḥtāj ilaihī 'l-kuttāb wa 'l-ʿummāl* of Abū 'l-Wafā'.[41]

Several quotations of the *Elements* can be identified.[42] Two aspects, however, deserve further consideration.

**1.** Like in the second part the author focuses his efforts on the presentation of selected subject-matters that are useful in a two-fold sense: they help to solve everyday problems and they can easily be looked up. Two examples may demonstrate this intention.

"Now there is something the ancients arranged for beginners as mnemonic aid. They completed it up to 100 and omitted the primes. The portion (*juz'*) [of 1] is:

$$12 \rightarrow \genfrac{}{}{0pt}{}{1}{\frac{2}{6}};\ 14 \rightarrow \genfrac{}{}{0pt}{}{1}{\frac{2}{7}};\ 15 \rightarrow \genfrac{}{}{0pt}{}{1}{\frac{3}{5}};\ 16 \rightarrow \genfrac{}{}{0pt}{}{1}{\frac{2}{8}};\ 18 \rightarrow \genfrac{}{}{0pt}{}{1}{\frac{2}{9}};\ 20 \rightarrow \genfrac{}{}{0pt}{}{1}{\frac{2}{10}};\ 21 \rightarrow \genfrac{}{}{0pt}{}{1}{\frac{3}{7}};\ 24 \rightarrow \genfrac{}{}{0pt}{}{1}{\frac{3}{8}};$$

$$25 \rightarrow \genfrac{}{}{0pt}{}{1}{\frac{5}{5}};\ 27 \rightarrow \genfrac{}{}{0pt}{}{1}{\frac{3}{9}};\ 28 \rightarrow \genfrac{}{}{0pt}{}{1}{\frac{4}{7}};\ 30 \rightarrow \genfrac{}{}{0pt}{}{1}{\frac{3}{10}}\ \text{(p. 43)};\ 32 \rightarrow \genfrac{}{}{0pt}{}{1}{\frac{4}{8}};\ 35 \rightarrow \genfrac{}{}{0pt}{}{1}{\frac{5}{7}};\ 36 \rightarrow \genfrac{}{}{0pt}{}{1}{\frac{4}{9}};$$

33 §6–21.

34 §22–29.

35 §31: *shay'*, *māl*, *kaʿb* for the exponents of $x^n$ ($n = 1, 2, 3$), and *ḍilʿ* / *musaṭṭaḥ* and *murabbaʿ* for square numbers and their roots are explained.

36 §32–34, primes > 7.

37 §37–46. §47: division by prime numbers.

38 §48–49, 53.

39 §54–65.

40 §66–75, dealt with at length. Apparently, fractions with prime denominators were suspected to carry the bacillus of fraud into financial transactions.

41 See [Saidan 1971: 64–201].

42 §30–34: Books VII 11, 13, 15–16, VIII 28–29, according to the German translation [Thaer 1933–37]. I have not been able to spot the underlying Arabic version.

$40 \to \frac{\frac{1}{4}}{10}$; $42 \to \frac{\frac{1}{6}}{7}$; $45 \to \frac{\frac{1}{5}}{9}$; $48 \to \frac{\frac{1}{6}}{8}$; $49 \to \frac{\frac{1}{7}}{7}$; $50 \to \frac{\frac{1}{5}}{10}$; $54 \to \frac{\frac{1}{6}}{9}$; $56 \to \frac{\frac{1}{8}}{7}$;[43]
$60 \to \frac{\frac{1}{6}}{10}$; $63 \to \frac{\frac{1}{7}}{9}$; $64 \to \frac{\frac{1}{8}}{8}$; $72 \to \frac{\frac{1}{8}}{9}$; $75 \to \frac{\frac{\frac{1}{3}}{5}}{5}$; $80 \to \frac{\frac{1}{8}}{10}$; $81 \to \frac{\frac{1}{9}}{9}$; $84 \to \frac{\frac{\frac{1}{2}}{6}}{7}$;
$90 \to \frac{\frac{1}{9}}{10}$; $96 \to \frac{\frac{\frac{1}{2}}{6}}{8}$; $98 \to \frac{\frac{\frac{1}{2}}{7}}{7}$; $100 \to \frac{\frac{1}{10}}{10}$" [§40 = fol. 42–43].[44]

"The [sexagesimal] ratio of $\frac{1}{3}$ [to 60] is: $\frac{1}{3}$ [related] alone gives $\frac{\frac{\frac{1}{2}}{9}}{10}$;[45] $\frac{2}{3} \to \frac{\frac{1}{9}}{10}$;
$\frac{2}{3} + 1 \to \frac{\frac{1}{4}}{9}$; $\frac{1}{3} + 1 \to \frac{\frac{1}{5}}{9}$; $\frac{2}{3} + 2 \to \frac{\frac{2}{5}}{9}$; $[\frac{2}{3}] + 6 \to \frac{1}{9}$; $\frac{1}{3} + 3 \to \frac{\frac{1}{2}}{9}$; $\frac{2}{3} + 12 \to \frac{1}{9} + \frac{1}{10}$;
$\frac{1}{3} + 4 + \frac{1}{9} \to \frac{\frac{2}{3}}{9}$; $\frac{2}{3}$ (p. 69) $+ \frac{1}{4} + \frac{\frac{1}{6}}{8} \to \frac{\frac{1}{8}}{8}$; $\frac{2}{3} + \frac{\frac{2}{3}}{9} \to \frac{\frac{1}{9}}{9}$;

$\frac{2}{3} + \frac{\frac{1}{3}}{7}$ sum up to $\frac{5}{7} \to \frac{\frac{\frac{1}{2}}{6}}{7}$; $\frac{5}{7} + 5 \to \frac{\frac{2}{3}}{7}$; $\frac{2}{3} + \frac{2}{7} \to \frac{\frac{1}{7}}{9}$; $\frac{1}{3} + \frac{1}{7} \to \frac{\frac{\frac{1}{2}}{7}}{9}$;
$\frac{\frac{1}{3}}{9} + \frac{1}{3} \to \frac{\frac{\frac{1}{2}}{9}}{9}$; $\frac{\frac{2}{3}}{7} + \frac{1}{3} = \frac{3}{7} \to \frac{\frac{\frac{1}{2}}{7}}{10}$; $\frac{3}{7} + 1 \to \frac{\frac{1}{6}}{7}$; $\frac{2}{3} + \frac{\frac{1}{3}}{7} + 1 \to \frac{\frac{1}{5}}{7}$;
$\frac{1}{3} + \frac{1}{9} \to \frac{\frac{\frac{2}{3}}{9}}{10}$.

The relation of $\frac{1}{4}$ [to 60] is: $\frac{1}{4}$ [related] alone gives $\frac{\frac{\frac{1}{3}}{8}}{10}$; $[\frac{1}{4}] + 1 \to \frac{\frac{1}{6}}{8}$;
$[\frac{1}{4}] + \frac{1}{8} \to \frac{\frac{\frac{1}{2}}{8}}{10}$; $\frac{1}{4} + \frac{\frac{1}{4}}{7} = \frac{2}{7} \to \frac{\frac{\frac{1}{3}}{7}}{10}$; $[\frac{1}{4} + \frac{\frac{1}{4}}{7}] + 4 \to \frac{\frac{1}{2}}{7}$; $[\frac{1}{4}] + \frac{\frac{1}{2}}{8} \to \frac{\frac{\frac{1}{3}}{8}}{8}$; $[\frac{1}{4}] + \frac{1}{6} \to$
$\frac{\frac{\frac{1}{2}}{8}}{9}$; $3 + \frac{1}{2} + \frac{1}{4} \to \frac{\frac{1}{2}}{8}$; $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \to \frac{\frac{1}{4}}{8}$" [§56 = fol. 68–69].[46]

**2.** Among the — more or less — simplified and fragmented reproductions of the so-called A-type *ḥisāb*[47] of Abū 'l-Wafā' (and others), paragraphs appear that contain curious and original matters. The following ones deserve to be pointed out.

- Abū 'l-Ḥasan claims that "the multiplication of *U*nits and *T*ens with units and tens $(U + T) \cdot (U + T)$ renders only eleven 'kinds' (*anwāʿ*)" [§15 = fol. 13]. Since $(U + T)^2 = T \cdot T + U \cdot T + T \cdot U + U \cdot U$ ("four multiplications"), four "places" (*manāzil*) may be produced: $U$, $T$, $H$(undreds), $Th$(ousands). $U \cdot U$

43 Mind the sequence of the factors!
44 Below, the table is completed up to 200. The decomposition of fractions of the type $\frac{1}{k}$ $(2 \le k \le 200)$ is consequently realized as the product of two, three or four fractions $\frac{1}{m}, \frac{1}{n}, \frac{1}{o}, \frac{1}{p}$ $(2 \le m, n, o, p \le 10; m \le n \le o \le p)$.
45 Sample: $\frac{1}{3} : 60 = \frac{1}{2} \cdot \frac{1}{9} \cdot \frac{1}{10}$.
46 The complete table encloses the ratio of $\frac{1}{2}, \frac{1}{3} \ldots \frac{1}{9}$ to 60.
47 Following the classification in [Saidan 1978: 19 f.].

yields $U, U+T, T$, and $T \cdot T$ yields $H, Th+H, Th$. "Combined with each other" (*fa-idhā ijtamaʿā*), $3 \cdot 3 = 9$ combinations are possible. But since $[2 \cdot] U \cdot T$ may render $H$ and $Th+H$ which cannot be produced by the above combination, two solutions ($3 \cdot 3 + 2 = 11$) must be added.

- When explaining the binomial formulas $(a \pm b)^2$ Abū 'l-Ḥasan explicitly calls numbers "increasing" (*zāʾid*) and "decreasing" (*nāqiṣ*). For example: "Make the [product of the] subtracted [factor] multiplied by the increasing one [always] a decreasing one [...] and the subtracted one multiplied by itself [always] an increasing one if [the decreasing one] is on both sides." Later on, he demonstrates the rule with $(20+5) \cdot (20-4)$ and continues after $400 - 80 + 100$: "then multiply $-4$ (*arbaʿa nāqiṣa*) by $+5$ (*khamsa zāʾida*)." I do not know a similarly explicit notation of and distinction between the positive and negative sign of numbers — not operations[48] — before Ibn Fallūs (d. 1252).[49] Quite obviously, though, the existence of the negative value of $-|80|$ is only recognized *within* an operation that yields a *positive* result [§17–20 = fol. 15–16].
- Remark on the custom of arithmeticians from Iraq, Mosul and the border regions of Syria to reduce (complicated) fractions on the denominator of 6000. In addition, al-Thaqafī, the above-mentioned al-Ḥarrānī, is recorded to have reduced [all] different fractions to the denominator 2520 in order to incorporate the "difficult" factors $7 \cdot 9 = 63$ and (the remaining) 5 and 8 ($5 \cdot 7 \cdot 8 \cdot 9 = 2520$) [§51 = fol. 61f.].
- Terminology: The division by primes (§47) is achieved by "counting up" (*tawakhkhin*). The remainder is called "heap" (*kudy*).

  The sexagesimal transformation of $a < 60$ is called "feeding" (*aurada*), of $a > 60$ "settling" (*anzala*) (§56).

  The verb "receive" (*qabila*) is used (§62) if any fraction is to be "extended" (*basaṭa*) to 60, for example $\frac{1}{2} \rightarrow \frac{1}{2} \cdot 60$ "receives" 30. Calling the adjustment of the numerator *basaṭa* indicates the adoption of "Indian" methods.[50] Shortly later (§68), the "Indian practice" (*al-ʿamal bi-l-hindī*) is explicitly mentioned when calculating (sexagesimally) the value of $\frac{a}{11}$: "Relate $5595 \cdot \frac{a}{11}$ to 6000. This expression is correct but cumbersome and needs the work with Indian [numerals] on the board (*takht*). The affair requires too much time for those who are in a hurry and want to be finished quickly."
- Abū 'l-Ḥasan is constantly concerned about the esthetical preference of fractions set forth by different authors or regional schools. His viewpoint remains non-dogmatic throughout the text: The "better" (*aḥsan*) they are (for example $\frac{\frac{1}{m}}{n}$, with $m < n$) the easier they are to handle.

48 Already al-Uqlīdisī used *nuqṣān* and *ziyāda* for subtraction and addition, see [Saidan 1978: 375].

49 *Kitāb Iʿdād al-asrār* (MS Berlin Nr. 5970), fol. 18a/1, 20a/–5.

50 Cf. [Saidan 1978: 416].

## Summary

The *Tadhkira* represents a low-level, but highly specialized and selective treatise on *ḥisāb al-muʿāmalāt* adressed to laymen ("jurist, secretary and *faraḍī*"[51]). Correspondingly, literary repercussions stayed away. On the other hand, its author displays a remarkable familiarity with literary mathematics. For a good deal, his citations are extracted from known and partly "classical" texts. Others originate from authors and texts that have left no traces in the history of *ḥisāb*. Therefore, the *Tadhkira* could be regarded as the lower end of the "literarisation" of mathematics. Its composition signals a diffusion of mathematical knowledge within the Islamic orbit that makes it difficult to uphold a sharp distinction between "scientific" and "non-" or "sub-scientific"[52] mathematics. The diffusion carries multi-dimensional features: "Classical" problems were not only inserted downwards into "practical" texts but were also the object of a fruitful communication between semi-professional authors from distant provinces.

With respect to the *Tadhkira* and its date of composition — let alone the one of the *Kitāb al-Maʿūna* prior to and backed by it — two major results can be verified: the early vertical diffusion of the arithmetical standards introduced by Abū 'l-Wafāʾ, and the stubborn (partial) rejection of "Indian" techniques introduced by al-Uqlīdisī two generations before in Damascus. In addition, the author's supra-regional viewpoint discloses not only valuable historical information on measures and weights, etc., but also a variety of co-existing local techniques and standards. The vocabulary and terminology are still in a state of flux. The selection of subject matter, too, seems to have been made rather fortuitiously.[53] The ensemble of the text is only roughly structured. Formally, as well as with regards to its contents, the *Tadhkira* must be regarded as an (accidently preserved) forerunner of a genre of *ḥisāb*-texts that hitherto was assumed to have visibly flourished only from the 13th century onwards.

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51 §3.

52 [Høyrup 1990] must be credited for the first serious (and sceptical) discussion of this traditional distinction.

53 Besides the areas mentioned, there are paragraphs on the effect of the different Oriental systems of calendar (§109–110: Christian, Jewish, Coptic, Islamic, Persian), on the calculation of wages, and on the surface calculation of the basic geometrical figures (§116). In both cases the author refers to the extensive treatment of the matter in his *Kitāb al-Maʿūna*.

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# La circulation des mathématiques entre l'Orient et l'Occident musulmans: Interrogations anciennes et éléments nouveaux

par AHMED DJEBBAR

L'étude traite, dans une première partie, certains aspects de la circulation des mathématiques médiévales à l'intérieur des frontières de l'empire musulman, en partant de questions qui ont déjà fait l'objet de discussions parmi les spécialistes: l'algèbre d'arpentage, la circulation de la terminologie mathématique, l'emprunt ou le plagiat. La seconde partie de l'étude concerne le problème de non circulation d'une partie importante des pratiques et des résultats mathématiques arabes de l'Orient vers l'Occident. Dans la troisième et dernière partie est évoquée la circulation des écrits mathématiques andalous et maghrébins vers l'Orient musulman.

Certain aspects of the circulation of medieval mathematics within the borders of the Moslem Empire are treated in the first part of this study, starting with questions already discussed by specialists: algebra of measuring, circulation of mathematical terminology, borrowing or plagiarism. The second part of the study concerns the problem of the non circulation of an important part of Arabic mathematical practices and results from the Orient to the Occident. The third and last part concerns the transmission of Andalusian and Maghrebinian mathematical writings to the Moslem Orient.

L'étude de la circulation des différents aspects des mathématiques des pays d'Islam (méthodes, outils, concepts, problèmes, terminologie, etc.) a souvent été associée au phénomène de traduction en Europe des ouvrages grecs et arabes qui a commencé au début du XII$^{e}$ siècle et qui s'est prolongé jusqu'au XV$^{e}$ [Steinschneider 1893; 1904–05]. Les nombreux travaux qui ont été réalisés dans ce domaine ont permis de mieux cerner le contenu et les diverses formes de l'appropriation de la tradition scientifique gréco-arabe [d'Alverny 1982: 421–445]. Mais ils ont également révélé un certain nombre de questions dont certaines sont encore, jusqu'à aujourd'hui, sans réponse. Parmi ces questions, il y a celle qui concerne la très faible part prise par l'Orient dans le transfert des écrits scientifiques vers l'Occident [Burnett 2000: 1–78]. Il y a aussi l'absence de références explicites, dans les ouvrages traduits, à des écrits mathématiques et astronomiques très importants produits en Orient entre le XI$^{e}$ et le XIII$^{e}$ siècle. Sur un plan régional, il y a le silence, encore inexpliqué, au sujet des ouvrages algébriques produits en Andalus entre le IX$^{e}$ et le XII$^{e}$ siècle. Ce silence est en effet observé à la fois par des bio-bibliographes andalous, des mathématiciens maghrébins et par des auteurs latins qui ont abondamment puisé dans le corpus de l'algèbre des transactions [Sesiano 1988: 69–98]. Il y a enfin la faible circulation, et parfois la disparition pure et simple, de la production géométrique de l'Andalus du XI$^{e}$ siècle, et en particulier celles d'Ibn as-Samḥ, d'al-Mu'taman, d'Ibn Sayyid et d'Ibn Mu'ādh, pour ne citer que les auteurs les plus importants et dont il nous est parvenu des informations précises sur le contenu de leurs contributions en géométrie.

Dans cette étude, nous nous proposons d'aborder certains aspects de la circulation des mathématiques médiévales en nous limitant aux frontières de l'empire musulman et à quelques unes des questions qui ont fait l'objet de discussions parmi les spécialistes. Parmi ces questions, il y a celles qui ont été suggérées par l'étude de l'algèbre d'arpentage à travers les fameux textes traduits en latin au XII$^{e}$ siècle. Il y a également la question relative à la circulation de la terminologie que nous traiterons à travers un seul exemple, celui des différentes utilisations des deux termes *jam'* et *tafrīq*. Nous évoquerons aussi une forme de circulation non encore étudiée, celle de l'emprunt ou du plagiat, plus ou moins reconnu, que nous illustrerons par deux exemples significatifs. En ce qui concerne le phénomène de non circulation, de l'Orient vers l'Occident musulman, d'une partie importante des résultats et des pratiques mathématiques arabes, nous fournirons quelques éléments nouveaux qui sont des réponses encore très partielles à la question. En conclusion, nous évoquerons la circulation des écrits mathématiques andalous et maghrébins vers l'Orient musulman, en nous basant sur des informations bibliographiques, sur des témoignages de certains mathématiciens et sur des manuscrits nouvellement étudiés.

## De *L'épître sur le mesurage* d'Ibn 'Abdūn au *Liber Mensurationum* d'Abū Bakr

L'édition critique et l'analyse du *Liber Mensurationum* d'Abū Bakr par H. L. L. Busard [1968: 65–124], complétées par l'analyse comparative de J. Høyrup [1986: 445–484], ont permis de dégager la double filiation de cet ouvrage, c'est à dire la tradition algébrique d'al-Khwārizmī et la tradition de mesurage babylonienne. Il restait à trouver le nom de l'auteur de cet ouvrage et la filiation intermédiaire pouvant rattacher la partie "babylonienne" de son contenu à la tradition arabe de mesurage. En attendant de pouvoir, un jour, compléter le nom d'Abū Bakr, nous allons présenter quelques éléments de réponses concernant ses sources arabes que nous avons puisées dans un écrit inédit du X$^{e}$ siècle, la *Risāla fī t-taksīr* (l'Epître sur le mesurage) d'Ibn 'Abdūn (m. après 976) [Djebbar 2002].

Il s'agit d'un manuel qui présente certaines ressemblances avec le *Liber mensurationum*, en particulier dans l'agencement des chapitres, dans les énoncés de nombreux problèmes et la formulation de leurs solutions, ainsi que dans la terminologie géométrique. Au niveau de la structure interne, on constate que le même ordre d'exposition est adopté pour les quadrilatères et les triangles et pour leurs différents types.[1] Certains énoncés utilisent les mêmes expressions et, surtout, les

1 Dans l'épître d'Ibn 'Abdūn, les figures géométriques sont classées et étudiées selon l'ordre suivant: carrés, rectangles, quadrilatères, losanges, parallélogrammes, trapèzes isocèles, triangles, trapèzes quelconques, solides rectilignes, figures circulaires, solides arrondis. Dans le *Liber mensurationum*, l'étude des trapèzes est regroupée dans un seul chapitre et les figures planes circulaires précèdent les solides.

mêmes valeurs numériques, comme on peut le voir sur l'exemple du problème n° 6 (correspondant au n° 7 d'Ibn ʿAbdūn) qui est donné en encadré.[2]

| **Problème n° 6 d'Abū Bakr** | **Problème n° 7 d'Ibn ʿAbdūn** |
|---|---|
| *Si quelqu'un te dit: j'ai soustrait les côtés du carré de la surface et il reste 60, combien est chaque côté ?*<br><br>La méthode sera que tu diviseras en deux les côtés, soit deux. Tu les multiplies par elles-mêmes et tu les ajoutes à 60 et tu prends la racine de la somme, qui est huit, ensuite tu y ajoutes la moitié du nombre des côtés et le résultat est dix, et c'est le côté.<br><br>[Busard 1968: 70] | *Si quelqu'un te dit: j'ai soustrait ses côtés de sa surface et il reste soixante, combien est chaque côté ?*<br><br>La méthode de sa résolution est que tu additionnes le nombre des côtés du carré, et c'est quatre; tu prends alors sa moitié, et c'est deux; tu les multiplies par elles-mêmes, ce sera quatre; tu les ajoutes à soixante et tu prends la racine de la somme, qui est huit; tu l'ajoutes à la moitié de quatre, et c'est dix. C'est ce qu'il y a dans chacun de ses côtés.<br>[Djebbar 2002] |

Quant au procédé de résolution des problèmes qui aboutissent à une équation, il correspond exclusivement à la première méthode d'Abū Bakr, c'est à dire à celle qui ne fait pas référence à la terminologie Khwārizmīenne (*shay'*, *māl*, *jabr*, *muqābala*) et que l'on rattache désormais à la tradition babylonienne du mesurage [Høyrup 1990: 32–33; 1994: 100–103].

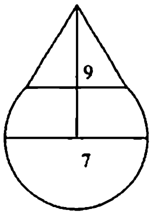
Le dernier élément qui renforce la filiation entre les deux écrits concerne la terminologie géométrique et plus particulièrement celle que l'on ne trouve pas dans les écrits orientaux connus de géométrie élémentaire. Cette terminologie est suffisamment typique pour confirmer l'existence d'une tradition distincte. On trouve en effet, dans les deux écrits et, plus tard, dans un poème géométrique de l'andalou Ibn Liyyūn (m. 1346) et dans son commentaire en prose du maghrébin Ibn al-Qāḍī (m. 1630), les termes suivants: *fanīqa* (en latin *faneche*),[3] *ḥūt aṭ-ṭaʿām* (*piscis*)[4] et *ʿurmat aṭ-ṭaʿām* (*cumulus tritici*),[5] qui désignent trois figures géométriques dont les formes ont suggéré ces appellations imagées [Ibn Liyyūn; Ibn al-Qāḍī: 77–87].

2 Voici quelques correspondances qui illustrent ce fait (nous avons noté $A_n$ le problème d'Ibn ʿAbdūn et $B_n$ le problème équivalent d'Abū Bakr): $A_6 = B_4$; $A_7 = B_6$; $A_{12} = B_{21}$; $A_{14} = B_{22}$; $A_{15} = B_{26}$; $A_{16} = B_{27}$; $A_{20} = B_{30}$; $A_{74} = B_{117}$; $A_{76} = B_{120}$; $A_{77} = B_{141}$; $A_{78} = B_{134}$.

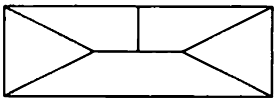
3 *Fanīqa* = sac dans lequel on transporte de la terre. En latin, le titre du chapitre *Capitulum aree solidi similis faneche.*

4 *ḥūt aṭ-ṭaʿām* = poisson comestible. En latin, le titre du chapitre est *Capitulum solidi similis cuidam pisci triangulo chaburi nomine.* Le terme *chaburi* n'est pas utilisé par Ibn ʿAbdūn, mais on le retrouve dans le poème géométrique d'Ibn Liyyūn.

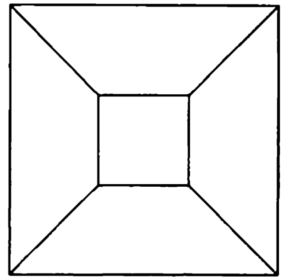
5 *ʿurmat aṭ-ṭaʿām* = tas de grains. En latin, on lit: "*Area corporee piramidis que est similis cumulo tritici*".




*'urmat aṭ-ṭa'ām*



*ḥūt aṭ-ṭa'ām*



*fanīqa*

Une autre partie de la terminologie de l'épître d'Ibn 'Abdūn présente un certain intérêt dans la mesure où son "archaïsme", et sa disparition ultérieure de la plupart des manuels mathématiques, pourraient confirmer son lien direct avec des traditions pré-khwārizmiennes du mesurage (tradition locale et tradition d'origine babylonienne). Parmi les éléments remarquables de cette terminologie, il y a des noms de figures ou d'éléments géométriques classiques, comme *'arīḍat ar-ra's* (⟨figure⟩ à large sommet) pour désigner le trapèze, *murabba' mustaṭīl* (quadrilatère oblong) pour le rectangle, *jānib* (flanc) pour le côté, *zujj* (pointe) pour le sommet. Il y a aussi des termes qui désignent des opérations, comme *injabara* (se restaurer) et *injibār* (restauration), qui seront remplacés plus tard par *ajbara* et *jabr*. Il y a enfin le mot *shaṭr* qui est synonyme de *niṣf* et qui intervient une douzaine de fois pour exprimer la notion de *moitié* (d'un nombre ou d'une grandeur géométrique). Il est à noter que l'auteur utilise, indifféremment, l'un ou l'autre terme pour désigner *la moitié* d'un nombre, alors que seul le mot *niṣf* lui sert à désigner *un demi.*

## *Jam'* et *tafrīq*: signification et circulation

Dans les textes mathématiques arabes du moyen âge, un certain nombre de couples de mots interviennent, la plupart du temps, ensemble. C'est le cas par exemple des couples suivants: *taḥlīl-tarkīb* (analyse-synthèse), *jam'-ṭarḥ* (addition-soustraction), *ḍarb-qisma* (multiplication-division), *jabr-muqābala* (restauration-comparaison), *jabr-ḥaṭṭ* (restauration-réduction), *ziyāda-nuqṣān* (augmentation-diminution), *ishtirāk-tabāyun* (commensurabilité-incommensurabilité), *munṭaq-aṣamm* (rationnel-irrationnel), etc. Comme d'autres termes arabes utilisés dans la pratique scientifique, ces couples de mots ne sont pas restés figés dans leur sens littéraire originel. Ils se sont enrichis de nouveaux sens, parce qu'ils ont été sollicités, au cours du temps, pour nommer des notions, des opérations et parfois même des chapitres mathématiques.[6] Ce sont donc des éléments précieux dans l'étude de la circulation des concepts et des outils scientifiques. C'est ce que nous voudrions

6 *Taḥlīl-tarkīb* signifie aussi: composition-décomposition; *munṭaq-aṣamm* signifie littéralement: exprimable-sourd.

montrer, à travers l'exemple du couple de mots *jam'-tafrīq* qui a été évoqué, pour la première fois semble-t-il, par F. Woepcke [1863: 446] et qui a suscité, au cours des deux dernières décennies, un certain nombre d'interrogations dépendantes les unes des autres [Høyrup 1986: 445, 470–471; Allard 1992, I; Folkerts 1997: 169].

La première question concerne l'utilisation de ce couple de mots par les bio-bibliographes arabes et par des mathématiciens, lorsqu'ils évoquent les écrits de certains auteurs et plus particulièrement ceux d'al-Khwārizmī (m. 850). En effet, des ouvrages, portant dans leur titre les mots *jam'* et *tafrīq*, sont attribués respectivement à al-Khwārizmī, ad-Dīnawarī (m. 895), aṣ-Ṣaydanānī (X$^{e}$ s.), Abū Kāmil (m. 930), an-Nihāwandī (X$^{e}$ s.) et Sinān Ibn al-Fatḥ (X$^{e}$ s.) [Ibn an-Nadīm 1971: 86, 334, 338–340]. Aucun de ces écrits ne nous est parvenu, mais des témoignages précis nous permettent de penser que leur contenu est différent de celui des manuels qui contiennent, dans leurs titres, l'expression *al-ḥisāb al-hindī* (le calcul indien). Un premier argument en faveur de cette hypothèse est fourni par Ibn an-Nadīm lui-même qui évoque deux commentaires au *Kitāb al-jam' wa t-tafrīq* d'al-Khwārizmī: celui d'Ibn al-Fatḥ et celui d'aṣ-Ṣaydanānī [Ibn an-Nadīm 1971: 338, 340]. Le second argument, se trouve à deux endroits du *Livre sur le calcul indien* d'al-Khwārizmī où ce dernier évoque, en ces termes, un autre livre de calcul qu'il aurait publié (et qui pourrait donc être le *Kitāb al-jam' wa t-tafrīq*):

> J'ai déjà expliqué dans le Livre de l'algèbre et de la muqābala (...) que tout nombre est composé (...). Et c'est ce qui est dit dans *un autre livre d'arithmétique*. (...). J'ai déjà expliqué *dans un livre* qu'il est nécessaire pour tout nombre, qu'on multiplie par un autre quelconque, que l'un soit multiplié selon les unités de l'autre. (...) [Allard 1992, I: 9].

Un troisième témoignage, déjà signalé par A.S. Saïdan [1971: 48–49], est donné par Ibn Ṭāhir (XI$^{e}$ s.) dans son *Kitāb at-takmila* (Livre de la complétion) [Saïdan 1987: 439–440]. Cet auteur se réfère explicitement au livre perdu d'al-Khwārizmī et à un problème qui n'est pas traité dans son livre sur le calcul indien. Voici le passage en question:

> Pour le calcul de l'acquittement de l'aumône légale sur les dirhams pour les années passées, c'est, selon ce propos, deux méthodes: l'une d'elles est selon le ⟨procédé⟩ originel, sans multiplication ni division et, la seconde, selon le procédé de la simplification par le produit et la division. Evocation du procédé originel: Si nous voulons déterminer l'aumône légale de sept mille cinq cent quatre vingt six dirhams, pour trois années, nous les posons ainsi: 7586. Et ceci est l'exemple indiqué par Muḥammad ibn Mūsā al-Khwārizmī (...). Muḥammad ibn Mūsā al-Khwārizmī a indiqué, dans cet exemple, dans son *Kitāb al-jam' wa t-tafrīq*, que chaque fois que l'on ajoute à ces fractions restantes leur moitié, elles deviennent des minutes, des secondes et des tierces [Ibn Ṭāhir 1985: 272–273].

Un dernier témoignage nous est donné par Abū Kāmil à la fin de son traité d'algèbre. Il y évoque "le chapitre des doublements des nombres et de leur sommation dans leur ordre successif, comme les doublements des cases de l'échiquier". A cette occasion, il fait référence à la contribution d'al-Khwārizmī dans ce domaine sans préciser le titre de l'ouvrage qui contient cette contribution. Mais cela ne peut être que le *Kitāb al-jam' wa t-tafrīq* puisque les deux autres ne traitent pas de cette question [Abū Kāmil 1986: 218–219].[7]

La seconde question, concernant le couple *jam'-tafrīq*, est étroitement liée à la précédente dans la mesure où elle vise à expliciter les différentes significations de ces deux mots et leur circulation dans les écrits mathématiques postérieurs à ceux d'al-Khwārizmī.

Etymologiquement, *jam'* signifie réunion et *tafrīq* signifie séparation.[8] C'est, semble-t-il, avec ces significations larges que le couple a d'abord désigné la science du calcul, sans distinction entre les différentes traditions locales qui l'ont alimenté. Mais, avec l'avènement du calcul basé sur le système décimal positionnel, le couple de mot a laissé place, dans certains titres d'ouvrages, à l'expression de "calcul indien". Il est alors possible que, dès le IXe siècle, les livres de *jam'* et *tafrīq*, aient été consacrés exclusivement aux traditions de calcul non indiennes, c'est à dire à celles du calcul digital et mental.[9] Il n'est pas possible de dire plus, à l'heure actuelle, sur cette tradition de calcul puisque aucun livre ne nous est parvenu. Cela dit, des témoignages, dispersés dans des écrits mathématiques postérieures ou dans d'autres ouvrages, nous permettent de dire quelques mots sur les différentes significations du couple *jam'-tafrīq* et sur ce qui a continué à circuler comme traditions du calcul malgré le triomphe du calcul indien.

C'est ainsi qu'au Xe siècle, les Ikhwān aṣ-Ṣafā' définissaient le calcul comme la science de "la réunion du nombre et de sa séparation" [Ikhwān aṣ-Ṣafā', I: 50]. Des formulations semblables sont également données par des mathématiciens, comme Ibn al-Bannā (m. 1321) [Aballagh 1988: 250] et par des encyclopédistes, comme Ibn al-Hindī (XIIe s.) [Ibn al-Hindī 1985: 9; Brentjes 1987/88: 37–38] et at-Tahānawī (XVIIe s.) [Tahānawī 1984, I: 41]. Ibn Sīnā, quant à lui, parle de "la réunion et de la séparation indienne" au moment où il définit le domaine appliqué des nombres [Ibn Sīnā 1975: 69; 1984: 147]. Plus précis que ses prédécesseurs, l'encyclopédiste du XIVe siècle Ibn al-Akfānī rattache les opérations de *jam'* et *tafrīq* à une tradition, celle du *Ḥisāb maftūḥ* (calcul ouvert), c'est à dire mental, par opposition au *Ḥisāb at-takht* (calcul de la tablette) qui est le calcul indien [Ibn al-Akfānī 1998: 84].

7 Abū Kāmil dit:

> Quant au chapitre des doublés ⟨successifs⟩ des nombres, de leur sommation, de leur succession, de leur agencement comme les doublés ⟨successifs⟩ des cases de l'échiquier et d'autres (.. ), c'est comme ce qu'a dit Muḥammad ibn Mūsā al-Khwārizmī — que la miséricorde de Dieu soit sur lui —, et qui est que: le second excède le premier de un. (...). Muḥammad ibn Mūsā, que Dieu l'agrée, a simplifié cela et l'a facilité en disant: 'Nous considérons le premier ⟨égal à⟩ deux'. Et il a considéré le premier ⟨égal à⟩ deux pour se dispenser d'ajouter le un.

8 Ils peuvent exprimer, respectivement, les idées d'*agrégation* et de *dissociation*. Dans ce qui suit, nous adopterons, comme traduction, les termes de *réunion* et de *séparation*.

9 Le mathématicien du Xe siècle, al-Uqlīdisī, utilise l'expression de "calcul arabe" et de "calcul byzantin" pour distinguer cette tradition de celle du "calcul indien" [al-Uqlīdisī 1985: 118; Saīdan 1978: 7, 10].

Comme on le voit, et quelle que soit les différences dans leurs formulations, les auteurs cités donnent tous, implicitement, aux mots *jam'* et *tafrīq* un sens très large pouvant englober différentes opérations arithmétiques. Ce fait est confirmé par d'autres mathématiciens qui ont explicité tous les sens que pouvaient renfermer ces deux termes à leur époque. Leurs témoignages informent en même temps de la circulation, d'est en ouest, de la terminologie mathématique des IX$^e$-X$^e$ siècles. Un des plus anciens écrits que nous avons pu trouver, mais qui témoigne d'une circulation bien antérieure, est le *Liber Mahamalet*, une compilation latine du XII$^e$ siècle, à partir d'écrits mathématiques d'al-Andalus. Son auteur dit, en parlant du calcul:

> Cette science d'application prend divers aspects, comme le calcul proprement dit avec ses opérations de *réunion* ou de *séparation* des nombres; (...) Mais, parmi les opérations de *réunion*, *l'addition* est la première. Elle apparaît dans la *multiplication* (...). Il faut donc traiter de *l'addition* en premier. Mais puisque la plupart des Arabes commencent par la *multiplication*, nous aussi ⟨nous⟩ les suivrons en commençant par elle [Sesiano 1988: 73–74].

Parmi les auteurs polygraphes connus qui ont évoqué notre sujet, c'est Ibn Khaldūn (m. 1406) qui nous fournit la définition la plus explicite et la plus complète des deux termes. En effet, on lit, dans sa classification des sciences que

> ... parmi les branches de la science du nombre, il y a l'art du calcul. C'est un art pratique pour le calcul des nombres à l'aide de la *réunion* et de la *séparation*. La *réunion* a lieu dans les nombres par l'individualisation, et c'est *l'addition*, et par la multiplication — c'est à dire qu'on multiplie un nombre avec les unités d'un autre nombre —, et c'est cela la *multiplication*. La *séparation* a lieu également dans les nombres, soit par l'individualisation — comme le retranchement d'un nombre d'un ⟨autre⟩ nombre et la détermination du reste —, et c'est la *soustraction*, soit ⟨comme⟩ la décomposition d'un nombre en parties égales dont le nombre est donné, et c'est la *division*. Cette *réunion* et cette *séparation* ont lieu dans les nombres entiers et fractionnaires [Ibn Khaldūn 1983, II: 896].[10]

C'est cette définition qui est reprise par un certain nombre de mathématiciens contemporain d'Ibn Khaldūn ou postérieur à lui, mais avec des variantes importantes qui pourraient se rattacher à différentes traditions du calcul. C est ainsi que le maghrébin al-Ghurbī (XIV$^e$ s.) donne une définition semblable à celle d'Ibn Khaldūn mais en substituant au couple *jam'-tafrīq*, le couple *takthīr-taqlīl* (augmentation-diminution) [al-Ghurbī: 238].[11] Quant à Ibn Sammāk (XIV$^e$ s.), il préfère

10 Il faut signaler qu'Ibn Khaldūn a substitué le mot *ḍamm* au mot *jam'*, pour réserver ce dernier à l'opération d'*addition*.

11 Cette même terminologie sera reprise, bien plus tard, par Muḥammad aṭ-Ṭfayyash, dans son commentaire du *Kashf al-asrār* (Le dévoilement des secrets) d'al-Qalaṣādī. Dans ce commentaire, on trouve aussi, en marge, une autre définition, tirée d'une source maghrébine inconnue, pour laquelle "ce qui se ramène à la *séparation*, c'est comme la *division*, la *réduction*, la *soustration*, *l'extration de la racine carrée* et *cubique*. Et ce qui se ramène à la *réunion*, c'est comme la *multiplication* parce que la *multiplication* c'est la *réunion* des nombres semblables" [Ṭfayyash: 29].

parler de *taḥlīl* (composition), au lieu de *jam'*, et de *tarkīb* (décomposition), au lieu de *tafrīq*. De plus, il considère que le calcul ne se limite pas à ces deux opérations mais qu'il englobe leur combinaison qui fournit le *jabr* (restauration), le *ḥaṭṭ* (réduction), le *ṣarf* (conversion) et le *tanāsub* (proportionnalité) [Ibn Sammāk 1981: 23].

Par manque d'informations sur l'histoire de la pratique du calcul en pays d'Islam, nous ne pouvons pas encore étudier dans le détail le processus qui a permis l'évolution du sens mathématique du couple *jam'-tafrīq*. Nous nous contenterons de constater la persistance de deux niveaux dans l'utilisation de ce mot. Dans les classifications des sciences et dans les définitions de la science du calcul, ils continuent de désigner, chez un certain nombre de mathématiciens, plusieurs opérations arithmétiques: pour le *jam'*, celles qui aboutissent à une augmentation et, pour le *tafrīq*, celles qui aboutissent à une diminution.[12] Mais, dans la pratique du calcul, le destin des deux mots a été différent, si on en juge par les témoignages qui nous sont parvenus: le *jam'* continue à être utilisé pour désigner uniquement l'opération d'*addition.*[13] Le *tafrīq* disparaît du vocabulaire mathématique des manuels de l'Occident musulman, remplacé par le mot *ṭarḥ* dans le sens de *soustraction*. En Orient, son utilisation a persisté chez certains auteurs, comme Ibn Ṭāhir et al-Kāshī (m. 1437), qui l'utilisent pour désigner uniquement la *soustraction* [Ibn Ṭāhir 1985: 30, 39–41; Kāshī 1967: 47].

## Le rôle de l'emprunt dans la circulation mathématique

En pays d'Islam, comme d'ailleurs dans les autres traditions scientifiques, la circulation des objets, des outils, des concepts et des résultats mathématiques s'est faite, parfois, selon des voies qui ont provoqué des polémiques et qui ont abouti à des condamnations de la part de toute la communauté scientifique de l'époque ou de certains de ses membres. Pour l'Orient, nous avons deux exemples célèbres de polémiques liées à l'établissement de résultats nouveaux et aux accusations de plagiats qui en ont découlé. Le premier concerne l'inscription de l'heptagone régulier dans un cercle à l'aide des sections coniques [Anbouba 1977: 73–105; Hogendijk 1984] et le second, amplement décrit par al-Bīrūnī, est lié à la démonstration du théorème du sinus [Bīrūnī 1985: 92–94]. Pour l'Occident musulman, il nous est parvenu également une polémique concernant des emprunts conséquents au

12 Voici, à titre d'exemple, ce que dit as-Samaw'al (m. 1175) dans son livre *al-Qiwāmī* à propos du *tafrīq*:

> La seizième section du cinquième chapitre sur l'exposé d'un seul principe à l'aide duquel on détermine l'ensemble des opérations de séparation (*tafrīq*) qui sont la division, l'extraction de la racine carrée et l'extraction de la racine de toutes les positions [Samaw'al: f. 111b].

13 Mais ce mot n'a pas été le seul à désigner l'*addition*, comme le mot *ṭarḥ* n'a pas été également le seul à désigner la *soustraction*. On trouve par exemple chez les deux mathématiciens du Xe siècle, al-Uqlīdisī et Kūshyār ibn Labbān les mots *ziyāda* et *nuqṣān* pour désigner, respectivement, l'*addition* et la *soustraction* [Uqlīdisī 1985: 249; Kūshyār ibn Labbān 1988: 71–72].

traité d'algèbre d'al-Qurashī (m. 1128) faits par le mathématicien Ibn al-Bannā, au moment de la rédaction de son propre livre d'algèbre [Aballagh et Djebbar 2001: 40–42]. Ce sont deux exemples semblables, relatifs à la tradition mathématique de l'Occident musulman, que nous voulons présenter ici. Leur intérêt ne tient pas seulement à l'absence de polémique de la part de la communauté scientifique de l'époque, ce qui est étonnant en soi car, comme on va le voir, les auteurs concernés avaient une certaine notoriété. Il tient aussi et surtout aux informations précises qu'ils nous fournissent sur la circulation de certains ouvrages de l'Orient vers l'Occident et de l'Andalus vers le Maghreb.

Le premier exemple concerne les emprunts faits par Ibn al-Bannā à un important ouvrage de calcul d'Ibn al-Yāsamīn (m. 1204), le *Talqīḥ al-afkār* (La fécondation des esprits). On constate en effet que des chapitres entiers de ce livre se retrouvent dans le fameux manuel intitulé *Talkhīṣ a'māl al-ḥisāb* (L'abrégé des opérations du calcul). Il s'agit des chapitres de l'addition, de la multiplication et de la division. Les seules différences que l'on décèle entre les deux contenus sont dans la modification d'un ou deux mots par paragraphe, le résumé, par Ibn al-Bannā, de 23 lignes du premier chapitre et l'ajout, dans le second chapitre, de deux procédés de multiplication appelés "procédé par excédent" et "procédé par quadrature" [Ibn al-Bannā 1969: 41–42, 47–53; Ibn al-Yāsamīn 1993: 114–116, 119, 131–132, 136].

Il est intéressant de remarquer que ces emprunts ont été également évoqués, par al-Qallūsī qui avait accusé Ibn al-Bannā d'avoir plagié le livre d'al-Qurashī. On lit en effet dans l'ouvrage d'un auteur du XIV$^{e}$ siècle, ash-Shāṭibī, qu'al-Qallūsī "lui aurait raconté que le Talkhīṣ d'Ibn al-Bannā est également tiré d'un livre de mathématique d'une ⟨autre⟩ personne" [Shāṭibī 1983: 164–165]. Au vu des résultats que nous avons exposés, cette accusation est manifestement exagérée. Mais elle nous pousse à réexaminer les liens entre le livre d'algèbre d'Ibn al-Bannā et celui d'al-Qurashī. Malheureusement l'ouvrage de ce dernier n'a pas encore été retrouvé et les quelques citations que nous avons découvertes chez Ibn Zakariyyā' al-Gharnāṭī ne nous permettent pas de trancher [Ibn Zakariyyā': ff. 146a–155b].

Le second exemple d'emprunt est de même nature que le premier puisqu'il s'agit de la reprise, parfois mot à mot, de problèmes tirés d'un ouvrage plus ancien. Mais son intérêt vient du fait qu'il concerne, cette fois, un auteur et un livre d'Orient. On s'est interrogé, depuis longtemps, sur l'absence d'informations relatives à la circulation éventuelle, vers l'Occident musulman, d'ouvrages mathématiques de savants aussi importants qu'al-Bīrūnī, al-Khayyām, al-Karajī et as-Samaw'al [Richter-Bernburg 1987: 373–401; Samsó 1980: 65–66]. Mais aucune preuve matérielle attestant de la présence de tel ou tel écrit de ces auteurs, en Andalus ou au Maghreb, n'a pu être exhibée jusqu'à ce jour. L'exemple que nous présentons ici nous semble être un élément en faveur de l'hypothèse d'une circulation mathématique plus importante que ne le laisse supposer les sources accessibles.

Dans le *Talqīḥ al-afkār* d'Ibn al-Yāsamīn, que nous avons évoqué au paragraphe précédent, le cinquième et dernier chapitre est intitulé *Sur des choses dont on a besoin en algèbre et en muqābala*. C'est dans la neuvième section de ce chapitre, qui traite des figures planes, que l'on trouve de nombreux passages correspondant, à la virgule près, à des sections entières du *Kitāb al-Kāfī* (Le livre suffisant) d'al-Karajī. Cette section contient, dans sa première partie, les définitions des figures géométriques planes élémentaires, classées selon le nombre de leurs côtés et la forme de leurs périmètres (triangles, quadrilatères, cercles et portions de cercles). La seconde partie présente "la manière de faire le mesurage ⟨des éléments⟩ de ces surfaces", c'est à dire un ensemble de formules, illustrées par des exemples, donnant les aires ou les grandeurs des éléments particuliers de chaque figure (hauteur, diagonale, diamètre, périmètre). Une troisième et dernière partie traite à nouveau des figures planes, puis des problèmes de détermination de nombres pensés, de calcul de l'aumône et de répartition d'héritage [Ibn al-Yāsamīn 1993: 288–299]. Les sept sous-sections de cette dernière partie concernent la géométrie et elles correspondent aux sections 46 à 51 et à la section 69, selon la numérotation du livre d'al-Karajī [Karajī 1986: 141–155, 201–209].[14] Le contenu des six premières regroupe les formules donnant l'aire ou des éléments de chaque figure plane élémentaire. Ces procédures sont illustrées parfois par des exemples numériques, mais aucune démonstration n'est donnée pour en justifier la validité. La dernière section regroupe une série d'exercices qui visent à illustrer à la fois les sujets de géométrie déjà exposées et l'utilisation des outils de l'algèbre. On y retrouve, en particulier, le problème de la division d'un champs en trois parties séparées par un chemin, et celui du roseau qui se penche sous le souffle du vent. Mais, bizarrement, le dernier problème de la section n'est pas repris dans le *Talqīḥ*.[15]

Comme on le voit, il s'agit de formules et de problèmes appartenant à un fonds commun relativement ancien qui a continué à circuler parallèlement au chapitre de la géométrie euclidienne plane. Voici d'ailleurs ce qu'en dit al-Karajī lui-même à la fin du *Kāfī*:

> J'ai trouvé ces problèmes qui sont évoqués à la fin de ce livre, en circulation parmi la plupart des auteurs. Et ils avaient un faible pour leur calcul et pour la connaissance de leur résolution. Que personne ne s'étonne donc que j'aie suivi les gens de mon époque dans leur choix [Karajī 1986: 209].

Mais d'autres éléments de comparaison sont en faveur d'un emprunt direct au livre d'al-Karajī. On constate d'abord que, dans les deux ouvrages, les sections portent exactement les mêmes titres et que l'auteur de l'emprunt n'a pas toujours supprimé les phrases non mathématiques qui pouvaient renseigner sur le style de l'auteur ou sur son origine géographique. On retrouve en effet dans le *Talqīḥ* la phrase suivante d'al-Karajī: "ceci *a été dit* par les Anciens. Et ce que *je* dis sur la surface latérale de la sphère ..." [Karajī 1986: 147; Ibn al-Yāsamīn 1993: 291].

14 La section 52 du *Kāfī* traite de l'arpentage d'un terrain pour connaître ses différentes dénivellations, en vue du creusement d'un canal, par exemple. Elle a peut-être été abandonnée parce qu'elle décrit des manipulations d'instruments topographiques. Les autres sections manquantes traitent de l'algèbre, sujet amplement exposé dans les chapitres précédents du *Talqīḥ*.

15 Il s'agit du problème classique des deux oiseaux qui, pour attraper un poisson, doivent se lancer de deux palmiers plantés sur chacune des deux berges d'une rivière.

Mais, d'un autre côté, le passage évoquant les unités de mesure de Bagdad n'est pas repris [Karajī 1986: 202].[16] Il y a enfin, à un endroit de l'exposé, une référence explicite au véritable auteur à l'aide de l'expression "al-Ḥājj Abū Bakr a dit", suivie d'une citation qui correspond, exactement, au propos d'al-Karajī dans la cinquantième section de son livre.[17]

Cela dit, et compte tenu de la structure de la dernière partie de l'ouvrage d'Ibn al-Yāsamīn que nous avons déjà décrite, on peut aussi supposer que la partie qui contient l'emprunt à été rajoutée par une autre personne. On lit en effet, à la fin de la seconde partie, les deux phrases suivantes:

> Avec cela, nous avons évoqué ce qui, nous l'espérons, est suffisant, si Dieu le très haut le veut. Puis nous le faisons suivre par un propos sur les figures ⟨déjà⟩ évoquées sans les dessiner [Ibn al-Yāsamīn 1993: 287].

Les copies de l'ouvrage d'Ibn al-Yāsamīn qui nous sont parvenues ne nous permettent pas de trancher ni de dater l'insertion des extraits du *Kāfī*. Mais elles nous autorisent à affirmer que ces extraits ont bien été tirés de l'ouvrage d'al-Karajī et que leur insertion dans le livre d'Ibn al-Yāsamīn a bien eu lieu au Maghreb avant 1825, date de la copie d'Alexandrie.[18]

## La circulation mathématique à travers les domaines appliqués

En pays d'Islam, il y a plusieurs domaines d'application qui ont permis l'intervention des mathématiques, et plus particulièrement des outils du calcul, mais qui n'ont pas encore fait l'objet d'études approfondies et systématiques. Il s'agit des transactions commerciales [Djebbar 2001], des jeux de société, de la répartition des héritages — avec, en particulier, son chapitre des donations — ainsi que de la résolution de certains problèmes posés par le Droit musulman, tels que l'acquittement de l'aumône légale et l'indemnisation des victimes de blessures. Comme il n'est pas possible, dans le cadre de cette modeste étude, de faire une présentation détaillée des problèmes liés à ces différents domaines et de leurs résolutions, nous nous contenterons d'en évoquer quelques uns à travers les seuls aspects qui permettent d'illustrer le phénomène de circulation qui nous préoccupe ici.

Parmi les problèmes récréatifs qui ont le plus circulé d'Orient vers l'Occident musulman et même vers l'Europe, il y a ceux qui sont regroupés habituellement sous le titre de "nombres dissimulés" mais qui ne s'expriment pas uniquement par

16 Voici ce passage:

> Sache que le mesurage dans la plupart des endroits ⟨se fait à l'aide⟩ d'un roseau de longueur six coudées, selon la coudée hachémite (...) et cette coudée ⟨vaut⟩ huit poignées à Bagdad.

17 Le nom complet de ce mathématicien est Abū Bakr Muḥammad ibn al-Ḥasan al-Karajī (ou al-Karkhī, à cause de la similitude de la graphie des lettres *jīm* et *khā* en l'absence de point sur l'une d'elle).

18 Le manuscrit d'Alexandrie semble être une copie de celle de Rabat (Bibliothèque Générale 222 K) laquelle est malheureusement non datée.

des nombres.[19] Un des plus anciens écrits connu sur le sujet est celui d'al-Kindī (m. 870) [Kindī: ff. 81a–86a]. Certains des problèmes qu'il expose et même certaines solutions se retrouvent, parfois avec des variantes ou des généralisations, dans des écrits d'Orient de différentes époques, comme ceux d'al-Anṭākī (X$^{e}$ s.) [Anṭākī: ff. 35b–36b], d'Ibn Ṭāhir (XI$^{e}$ s.) [Ibn Ṭāhir 1985: 290–292] et d'az-Zanjānī (XII$^{e}$ s.) [az-Zanjānī: ff. 88a–91a]. On les retrouve aussi dans des écrits du Maghreb. C'est le cas par exemple du procédé de détermination d'un nombre entier *n*, à l'aide de la congruence modulo 9. Ibn al-Yāsamīn l'expose dans son *Talqīḥ* et il le généralise à des nombres fractionnaires [Ibn al-Yāsamīn 1993: 313–314]. Cet auteur reprend également, dans des formulations parfois différentes, les problèmes suivants de l'épître d'al-Kindī: recherche de trois nombres pensés, d'un objet caché, d'une personne malade parmi deux personnes.

Parmi les procédés de détermination des nombres dissimulés, il y a la méthode des restes qui ne restera pas confinée dans le chapitre des problèmes récréatifs puisqu'elle sera reprise, plus tard, sous forme d'un problème de théorie des nombres dont la résolution théorique sera donnée par Ibn al-Haytham [Wiedemann 1970: 529–531; Rashed 1980: 305–321]. Il s'agit de trouver un nombre dont la division par des nombres $p_j$ fournit des restes $r_j$. Un premier procédé de résolution de ce type de problèmes utilise un tableau de nombres dont la manipulation est équivalente aux opérations arithmétiques. Al-Anṭākī est le seul auteur connu à avoir utilisé cette technique qui ne semble pas avoir circulé puisque les auteurs postérieurs n'évoquent même pas son existence. Quant au procédé arithmétique, on le trouve d'abord chez Ibn Ṭāhir qui est le seul à en faire un long développement avec une tentative de justification. Il est également traité, plus rapidement mais dans le même esprit par az-Zanjānī, par Ibn al-Hā'im (m. 1453) et par al-'Āmilī (m. 1622) [Zanjānī: f. 89a; Ibn al-Hā'im: ff. 62a–63b; 'Āmilī 1981: 192–195]. Au Maghreb, on le retrouve chez Ibn al-Yāsamīn, exposé comme un procédé de recherche d'un nombre dissimulé.[20]

Une autre catégorie de problème appartient au domaine des partages successoraux et du calcul des donations testamentaires. Les procédés de résolutions qui leurs sont associés ont circulé parallèlement à travers deux types d'écrits corres-

19 Il s'agit également de déterminer un nom, un mois de l'année ou un signe du zodiaque, de trouver une bague cachée, le doigt qui porte la bague, une personne parmi d'autres, etc. Pour les problèmes récréatifs en général, cf. [Hermelink 1976: 44–52].

20 Voici la formulation d'Ibn al-Yāsamīn:

> Tu dis à une personne de prendre ce qu'elle veut comme nombre, qu'il soit soixante ou cinquante ou ce qui est inférieur, jusqu'à sept. Puis, tu lui demandes d'ôter du nombre qu'il a pris trois ⟨successivement⟩. S'il y a un reste, tu l'interroges sur sa valeur, et tu le multiplies par soixante-dix et tu conserves le résultat du produit. Puis, tu lui demandes d'en ôter cinq ⟨successivement⟩. S'il lui reste quelque chose, tu l'interroges sur sa valeur et tu le multiplies par vingt-et-un et tu l'ajoutes à celui que tu as conservé. Puis, tu lui demandes d'en ôter sept ⟨successivement⟩. S'il en reste quelque chose, tu l'interroges sur sa valeur et tu la multiplies par quinze et tu l'ajoutes a celui que tu as conservé. Et si tu lui a ôté trois, cinq, sept, ⟨successivement⟩ et qu'il ne reste rien de chacun, le nombre ⟨pensé⟩ est cent cinq [Ibn al-Yāsamīn 1993: 313].

pondants à deux traditions bien distinctes, celle des juristes et celles des mathématiciens. En mathématique, l'ouvrage le plus ancien qui a traité de ces deux sujets est le livre d'algèbre d'al-Khwārizmī [Khwārizmī 1968; Gandz 1938]. Mais, par son intermédiaire, seuls les procédés de résolution algébriques ont circulé dans les différents foyers scientifiques des pays d'Islam. Les autres procédés se retrouvent essentiellement chez des auteurs connus pour être des spécialistes des héritages et qui ont consacré des traités à ces types de problèmes. Au X$^e$ siècle, il y a l'exemple d'al-Ḥubūbī qui a exposé, dans son *Kitāb al-istiqṣā' wa t-tajnīs* (Livre de l'investigation et de la classification), des méthodes de résolution non algébriques: celle du *bāb*, du *dinar* et du *dirham*, de la *géométrie* et des *deux erreurs* [Laabid 1990: 67–79]. Il est à noter que seules la méthode du *bāb* et celle des *deux erreurs* ont, semble-t-il, continué à être utilisées dans les écrits de cette tradition, bénéficiant ainsi d'un canal de diffusion important. Il n'est donc pas étonnant de les retrouver dans les traités andalous et maghrébins. Un des plus ancien écrits de cette région est le *Mukhtaṣar* d'al-Ḥūfī (m. 1192) dans lequel la méthode de fausse position, dite ici *méthode des deux plateaux*, et la méthode du *bāb*, rebaptisée *méthode du nombre*, sont utilisées pour résoudre certains types de problèmes d'héritage contenant des donations [Laabid 2001].[21] Compte tenu du succès de cet ouvrage dans le milieu des mathématiciens juristes, qui lui ont consacré un certain nombre de commentaires, comme le *Sharḥ al-Mukhtaṣar* (Commentaire de l'Abrégé) d'al-'Uqbānī (m. 1408) [Zerrouki 2000], cela a permis aux méthodes de calcul en question de continuer à être utilisées pendant plusieurs siècles au Maghreb. Cela explique aussi la présence des méthodes de simple ou de double fausse position dans de nombreux manuels de calcul, depuis l'époque d'Ibn al-Bannā (m. 1321) jusqu'à celle d'Ibn Ghāzī (m. 1513) [Ibn al-Bannā 1969: 70–71, 88–90; Ibn Ghāzī 1985: 208–227].

Le troisième domaine non mathématique qui a été, lui aussi, un vecteur de la circulation des techniques du calcul des fractions concerne l'indemnisation des blessures. Les problèmes de ce type constituent un chapitre important des traités de droit musulman, mais leurs liens avec la science du calcul restent peu connus, probablement parce que ces problèmes n'ont pas eu, comme les problèmes d'héritage, la possibilité d'être intégrés systématiquement dans un chapitre des manuels de calcul. Pour nous limiter au Maghreb, nous savons que, déjà au IX$^e$ siècle, le juriste malékite Ibn Abī Zayd al-Qayrawānī réservait, dans son *Epître*, un certain nombre de paragraphes où il décrit, d'une manière non exhaustive, les blessures susceptibles d'indemnisation en précisant, à chaque fois, le prix à payer à la victime [Ibn Abī Zayd 1975: 241–249, 263–265]. Depuis cette époque, les textes juridiques et les ouvrages traitant de cas de jurisprudence ont régulièrement évoqué cet aspect des problèmes d'indemnisation [Ibn Rushd 1982, II: 405–409]. Mais, nous n'avions pas connaissance d'un manuel mathématique contenant un chapitre

21 Voici un de ces problèmes (que l'on trouve déjà chez al-Khwārizmī):

> Un homme, ayant quatre fils, lègue à un autre homme une part égale à celle de l'un de ses fils moins un tiers de ce qui reste du tiers de la succession après en avoir ôté la part de l'un des fils.

sur ce sujet. D'où le premier intérêt de l'épître que nous allons brièvement présenter et qui est attribuée au mathématicien et juriste maghrébin du XIIIe siècle al-Jiṭālī [Jiṭālī 1887: 2–96].[22] Son second intérêt provient de la nature des problèmes et des procédés de la science du calcul qui y sont exposés.

Contrairement aux autres ouvrages de calcul, cette épître commence par un chapitre intitulé *Sur le calcul des mesures des blessures*. L'auteur y donne les définitions des différentes plaies du corps et leur classification en fonction de leur gravité et de leur emplacement.[23] Puis il décrit l'unité de mesure des blessures en précisant qu'il y a, dans le rite ibadite,[24] deux traditions dans ce domaine: celle des Maghrébins et celle des gens d'Oman (qui pratiquent le même rite).[25] Ces deux sections sont suivies par onze autres consacrées aux opérations sur les fractions qui peuvent servir dans la détermination de la mesure d'une blessure. Quant au reste de l'ouvrage, il traite des thèmes classiques des manuels de calcul de la tradition arabe (par opposition au calcul indien). Ce sont, la plupart du temps, des thèmes rencontrés dans le *Kāfī* d'al-Karajī, dans la *Takmila* d'Ibn Ṭāhir et dans le *Talqīḥ* d'Ibn al-Yāsamīn pour nous limiter aux écrits que nous avons déjà longuement évoqués dans cette étude. Une première partie contient en effet des problèmes de vente et d'achat, d'héritages et de donations, de rencontre, de répartition de bénéfices, de remplissage de bassin, de courriers, etc. Les deux dernières parties sont consacrées, respectivement, à des procédés de calcul (sommation d'entiers, extraction de la racine carrée) et au mesurage des figures planes.

Cela dit, au vu des procédés utilisés dans la résolution des problèmes traités, l'épître est plus proche de la *Takmila* d'Ibn Ṭāhir que des deux autres traités. En effet, l'auteur y résout les problèmes avec la règle de trois et la méthode de fausse position, en distinguant pour cette dernière, entre le procédé "général décrit par l'Inde", pour lequel les deux valeurs initiales peuvent être quelconques, et celui où

22 Pour une biographie succincte d'al-Jiṭālī, cf. [Ziriklī 1980, I: 327–328]. Je tiens à exprimer mes plus vifs remerciements à M. le Conservateur de la Bibliothèque al-Istiqāma de la ville de Banī Yazgan (Algérie) pour avoir mis à ma disposition une copie de l'ouvrage d'al-Jiṭālī.

23 Il distingue 3 degrés de blessures sur la peau, 3 dans la peau, 3 dans la chair et 3 touchant l'os. De plus, chaque partie du corps est affectée d'un coefficient de gravité: les blessures du corps "valent" la moitié de celles de la tête et ces dernières "valent" la moitié de celles portées au visage.

24 Ce rite, qui a été pendant longtemps réprimé ou marginalisé par les pouvoirs politiques et théologiques sunnites, s'est maintenu dans certaines régions des pays d'Islam et en particulier au Maghreb (Région de Nefoussa en Libye, Ile de Djerba en Tunisie et région du Mzab en Algérie). Sur l'histoire de ce rite et sur ses différents aspects, cf. [*Encyclopédie de l'Islam* I: 3–4; II: 957–961; IV: 640–641].

25 Al-Jiṭālī dit, au sujet de la mesure des blessures:

> La manière de procéder avec elle est de deux sortes. L'une des deux est le procédé qui se trouve dans les livres de nos amis parmi les gens d'Orient et cela ⟨consiste⟩ à considérer la longueur de la phalange ⟨du pouce⟩: on marque sur elle douze points, égaux et équidistants et, de même, douze points pour la largeur (...). Quant à l'autre procédé pour la mesure des blessures, c'est le calcul qui se trouve dans les livres des gens du Maghreb et qui est que l'⟨unité de⟩ mesure de la longueur de la blessure et de sa largeur est la phalange du pouce [Jiṭālī 1887: 3–4].

elles doivent être choisies comme multiples des coefficients du problème. Par ailleurs, aucun procédé algébrique n'y est utilisé et, lorsque al-Khwārizmī est évoqué, c'est à propos d'une formule géométrique de son livre d'algèbre qui donne l'aire d'une portion de cercle [Jiṭālī 1887: 87–88].[26]

## La circulation des mathématiques de l'Occident vers l'Orient musulmans

Les informations, concernant la présence et l'utilisation, en Orient, d'écrits mathématiques d'al-Andalus et du Maghreb, sont plutôt rares. Jusqu'à ces deux dernières décennies, on trouvait des éléments de réponses à cette question essentiellement chez les biobibliographes orientaux. Mais cela ne permettait pas toujours d'affirmer la circulation physique des ouvrages des auteurs concernés, dans la mesure où les informations étaient, le plus souvent, puisées dans des ouvrages de l'Occident musulman. On pouvait, tout au plus, conjecturer raisonnablement que dans leur déplacements pour des raisons religieuses, scientifiques ou commerciales, les mathématiciens d'al-Andalus et du Maghreb transportaient avec eux certains de leurs écrits ou ceux de leurs professeurs. C'est probablement ce qu'a pu faire Ibn 'Abdūn au X^e^ siècle, Abū ṣ-Ṣalt au XII^e^, Muḥyī ad-Dīn al-Maghribī et al-Ḥasan al-Murrākushī au XIII^e^.

En conclusion à cette étude, nous allons rassembler les éléments d'informations qui se trouvent dans certains ouvrages biobibliographiques ou encyclopédiques, en y ajoutant les résultats de nos propres investigations dans les écrits mathématiques qu'il nous a été possible de consulter. Mais ce n'est là qu'une introduction à des recherches systématiques qui devraient concerner différents types de textes: diplômes (*ijāzāt*), biographies (*barāmij*), journaux de voyage (*raḥalāt*) et bien évidemment les écrits mathématiques de différentes régions de l'empire musulman. L'étude comparative de ces écrits pourra, en particulier, nous renseigner sur la circulation et l'adoption de certaines éléments particuliers à telle ou telle tradition, comme les algorithmes, le symbolisme et la terminologie.

Nous n'avons pas encore d'informations précises sur une éventuelle circulation d'écrits mathématiques d'ouest vers l'est avant le XII^e^ siècle.[27] Mais l'exemple que nous allons évoquer rend cette circulation possible dès le XI^e^ siècle à la fois pour des raisons internes à la tradition scientifique (comme la valeur intrinsèque de tel ou tel ouvrage), et pour des raisons externes (en particulier les événements politiques) qui ont favorisé les déplacements des scientifiques et même leur changement de résidence.

Le plus ancien ouvrage mathématique connu d'al-Andalus dont on peut décrire l'itinéraire probable est le *Kitāb al-istikmāl* d'al-Mu'taman Ibn Hūd (m. 1085)

26 Al-Jiṭālī y reproduit, à quelques variantes près, la formule d'al-Khwārizmī donnant l'aire d'une portion de cercle qu'il attribue d'ailleurs à Ptolémée.

27 Pour la circulation d'écrits et d'instruments astronomiques (comme l'equatoria et l'astrolabe universel), d'al-Andalus vers l'Orient, cf. [Samsó 1992: 957–966].

[Hogendijk 1986: 43–52]. Ce dernier avait ambitionné de rédiger une monumentale synthèse de ce qui était alors considéré comme le programme de formation de tout futur chercheur dans un des domaines théoriques des mathématiques ou dans l'une des disciplines scientifiques qui en était un prolongement appliqué. C'est la raison pour laquelle le projet initial d'al-Mu'taman devait comprendre deux volumes: le premier consacré à la théorie des nombres et aux différents chapitres de la géométrie de son époque (géométrie euclidienne, archimédienne, apollonienne, avec des prolongements arabes), et le second réservé, essentiellement, à l'astronomie, à l'optique et à la mécanique.[28] Malheureusement, à la mort d'al-Mu'taman, seul le premier volume devait être achevé ou déjà publié. En tout cas c'est la seule partie qui va circuler. On en trouve une copie chez Ibn Mun'im, un mathématicien originaire de Dénia, qui l'a vraisemblablement amenée avec lui à Marrakech. Dans son *Fiqh al-ḥisāb* il recommande l'utilisation de *l'Istikmāl* et il s'y réfère lui-même à différents endroits de son livre. Il est également possible qu'il se soit basé sur le contenu géométrique de l'ouvrage d'al-Mu'taman pour rédiger un de ses traités perdus.[29] Les livres publiés par les élèves d'Ibn Mun'im ne nous sont pas parvenus mais il est raisonnable de penser qu'ils ont enseigné une partie de cet ouvrage. Ce qui expliquerait les références que l'on trouve chez Ibn al-Bannā, puis chez l'un de ses commentateurs, Ibn Haydūr [Djebbar 1990: 21–42]. Cela dit, il faut bien reconnaître que la présence de *l'Istikmāl* à Marrakech et peut-être dans d'autres foyers scientifiques du Maghreb, comme Bougie et Tunis, n'a pas initié ou entretenu une tradition géométrique puissante et durable. En effet, au vu des informations disponibles, et en dehors peut-être de l'ouvrage perdu d'Ibn Mun'im, cette discipline ne semble pas avoir dépassé les limites de la géométrie d'arpentage.

La seconde destination du *Kitāb al-istikmāl* a été l'Egypte. Cet itinéraire a probablement commencé à Cordoue où vivait le penseur et théologien Maïmonide (m. 1204). Ce dernier aurait, selon le biobibliographe Ibn al-Qifṭī, révisé et enseigné le Traité d'al-Mu'taman, dans la ville du Caire [Ibn al-Qifṭī: 210]. Ce qui explique peut-être qu'un siècle plus tard, le mathématicien et encyclopédiste Ibn al-Akfānī (m. 1328) ait pu disposer d'une copie de *l'Istikmāl* et qu'il ait apprécié son contenu [Ibn al-Akfānī 1998: 74].[30] Quant à l'information d'Ibn al-Qifṭī concernant la révision de l'ouvrage, il faudrait la considérer avec précaution dans la mesure où aucun extrait de ce travail ne nous est parvenu. Quant à l'enseignement de son contenu, qui a dû être un enseignement très partiel, compte tenu à la fois du

28 Pour le détail de la table des matières des deux volumes, cf. [Djebbar 1997: 185–192]. Pour l'analyse du contenu des chapitres géométriques de *l'Istikmāl*, cf. [Hogendijk 1991: 207–281]. Pour l'analyse du contenu du chapitre arithmétique du traité, cf. [Djebbar 2000: 589–653].

29 Cf. [Ibn 'Abd al-Malik 1973, I: 59–60] qui le cite sous le titre de *Tajrīd akhbār kutub al-handasa 'alā ikhtilāf maqāṣidihā* (Abstraction des matériaux des livres de géométrie sans distinction de leurs buts ⟨respectifs⟩).

30 On peut y lire le jugement suivant:

> Je n'ai pas encore vu jusqu'à maintenant un livre englobant ces dix parties ⟨de la géométrie⟩. Mais si le livre de *l'Istikmāl* d'al-Mu'taman Ibn Hūd, que la miséricorde de Dieu le très haut soit sur lui, avait été achevé, il aurait été pleinement suffisant.

nombre des propositions (plus de 400) et de leur complexité, il a probablement permis à Ibn ʿAqnīn (m. 1220), un des élèves de Maïmonide, d'apprécier le contenu de *l'Istikmāl* et d'être un relais pour sa diffusion, en particulier lors de son séjour à Bagdad.[31] Entre cette ville et le centre scientifique de Maragha, il n'y a pas une grande distance et c'est peut-être dans cette ville que le mathématicien Ibn Sartāq (m. après 1327) a pris connaissance de *l'Istikmāl* et qu'il a étudié son contenu avant de décider d'en réaliser une nouvelle rédaction. Des copies initiales de *l'Istikmāl*, il n'a été retrouvé, en Orient, que le premier chapitre consacré à la théorie des nombres. Il semble que ce chapitre ait perdu très tôt son identité puisque toutes les copies connues sont anonymes. Par contre, la rédaction d'Ibn Sartāq a résisté au temps et il nous en est parvenu deux copies complètes accompagnées d'une épître du même auteur sur la théorie des proportions [Djebbar 1997].

Pour rester à l'époque d'al-Mu'taman, on doit signaler rapidement la présence en Egypte, au XIV[e] siècle, d'un livre non encore retrouvé de l'andalou Ibn as-Samḥ, *al-Kāmil fī l-ḥisāb al-hawā'ī* (Le livre complet sur le calcul mental) [Ibn al-Akfānī 1998: 84]. Quant aux ouvrages du XII[e] siècle produits en Andalus et au Maghreb, ils ont connu des fortunes diverses. Certains, comme ceux d'Ibn Munʿim n'ont pas dépassé les frontières du Maghreb. D'autres, d'un niveau pourtant plus modeste, finiront par arriver en Egypte mais à des dates qu'il n'est pas possible de préciser. C'est le cas du manuel de calcul d'al-Ḥaṣṣār (XII[e] s.), *Kitāb al-Bayān wa t-tadhkār* (Le livre de la démonstration et de la remémoration) et du poème algébrique d'Ibn al-Yāsamīn. Ce dernier écrit a eu même un succès apparemment inattendu mais qui pourrait s'expliquer par le ralentissement des activités mathématiques créatrices et par l'abaissement graduel du niveau de l'enseignement à la fois en Orient et en Occident musulman.[32]

Après avoir évoqué ces deux auteurs maghrébins et leurs écrits, Ibn al-Akfānī ajoute, à propos des ouvrages de l'Occident musulman qui appartiennent à la tradition du calcul indien, et qui sont parvenus en Orient, qu'ils se distinguent par l'originalité de certains de leurs procédés.[33]

Les derniers ouvrages maghrébins connus qui ont été enseignés ou utilisés par des mathématiciens orientaux sont ceux d'Ibn al-Bannā. Il y a tout d'abord le *Talkhīṣ* qui a bénéficié de commentaires dont le plus important est le *Ḥāwī l-lubāb* (Le recueil de la moelle) de l'égyptien Ibn al-Majdī (m. 1447) [Aballagh et

31 Voici ce qu'il en dit:

> Nous vous conseillons un livre qui a rassemblé toutes les choses utiles de la géométrie qui est concis et qui montre, à travers les démonstrations de ses propositions des sciences cachées dans chacune de ses démonstrations. C'est le livre de *l'Istikmāl* d'al-Mu'taman Ibn Hūd, le roi de Saragosse, que rien ne peut égaler, dont l'expression est concise et les démonstrations de haut niveau [Ibn ʿAqnīn 1873: 28].

Notre traduction est basée sur la transcription arabe faite par [Hogendijk 1986: 49–50].

32 Le poème d'Ibn al-Yāsamīn a été commenté, en Orient, respectivement par Ibn al-Hā'im (m. 1412), Aḥmad al-ʿIrāqī (m. 1423), Sibṭ al-Māridīnī (m. 1506) et al-Sujāʿī (m. 1782).

33 L'auteur précise que "les gens du Maghreb ont des procédés par lesquels ils se distinguent ⟨des autres⟩ dans les opérations particulières. Certains sont accessibles comme ceux d'Ibn al-Yāsamīn. D'autres le sont moins, comme ceux d'al-Ḥaṣṣār."

Djebbar 2001].[34] Cet auteur utilise également dans son livre certains chapitres de deux autres ouvrages d'Ibn al-Bannā, le *Rafʿ al-ḥijāb* (Le lever du voile) et le *Kitāb al-uṣūl wa l-muqaddimāt* (Le livre des fondements et des préliminaires), auxquels ils se réfère parfois explicitement. Mais il introduit des éléments de symbolisme qui ne sont pas dans les trois ouvrages d'Ibn al-Bannā et qu'il a puisés dans d'autres écrits de la tradition maghrébine. On trouve en effet, dans son commentaire l'utilisation des différents symboles à l'aide desquels les auteurs maghrébins écrivent les fractions, ainsi que le symbole permettant de représenter la racine d'un nombre [Ibn al-Majdī: ff. 63a–72a, 94a–105a]. Mais cette utilisation du symbolisme se limite au domaine du calcul car, malgré la présence d'un important chapitre sur les équations, aucun des symboles algébriques utilisés à la même époque au Maghreb n'est utilisé par Ibn al-Majdī, ni même évoqué.

Il y a enfin, dans l'ouvrage d'Ibn al-Majdī, certains paragraphes qui suscitent des interrogations sur leur origine dans la mesure où l'auteur ne se les attribue pas. Ils peuvent provenir de sources orientales ou occidentales plus anciennes qui circulaient encore en Egypte au XIVᵉ siècle. L'un de ces paragraphes traite du dénombrement, selon une démarche inductive, de toutes les équations à *n* monômes. Ce problème a peut-être été inspiré à son auteur par le développement des pratiques combinatoires au Maghreb, à partir du XIIIᵉ siècle. Mais nous n'avons pas d'éléments pour étayer cette hypothèse [Ibn al-Majdī: ff. 193b–194b].

Un autre paragraphe, beaucoup plus court concerne les relations entre les racines d'une équation du second degré et ses coefficients. Après résolu une équation trinôme d'une manière classique, Ibn al-Majdī ajoute:

> Dans la première ⟨équation⟩, il résulte, par augmentation, trente, et par diminution, quatre. Et la somme de cela est trente quatre. *Et c'est la valeur du nombre des racines* [Ibn al-Majdī: ff. 165a].[35]

Il répète cette remarque pour une autre équation du même type, sans développer l'idée et sans l'exploiter même modestement, comme l'a fait un siècle plus tôt le mathématicien du Yémen Ibn al-Jallād (m. 1390). Ce dernier s'attribue en effet l'idée de construire une équation ayant des solutions choisies par avance. Il montre également comment faire pour qu'un nombre donné soit solution de l'une ou l'autre des trois équations canoniques du second degré. En faisant cela, il semble conscient du fait que ces trois équations ne constituent qu'un seul type. Il dit en effet:

> Nous commençons, dans cet art de l'algèbre, par les trois problèmes mixtes. Ils deviennent tous équivalents. Or tu sais qu'ils se divisent en trois parties (...). Si nous voulons obtenir les trois problèmes à partir d'un nombre quelconque, ⟨on procède ainsi⟩ [Jallād: ff. 249b–256a].

34 L'autre commentateur oriental est Ibn al-Ḥanbalī (m. 1409). Un troisième auteur, Ibn al-Hā'im en a fait un résumé.

35 Il conclut son paragraphe ainsi:

> De cela, il apparaît que les résultats des deux racines, la plus grande et la plus petite, ne changent pas par rapport au nombre des racines de l'équation et que *leur somme est le nombre de ces racines.*

Et il donne une série d'exemples où, partant de deux nombres, il détermine les coefficients de chacune des équations par produit, somme et différence de ces deux nombres.[36]

Cette petite incursion dans le domaine des relations entres les coefficients et les racines d'une équation ne semble pas avoir attiré l'attention des mathématiciens postérieurs, peut-être parce que l'initiative a été tardive et qu'elle est apparue à deux endroits et à deux époques où l'intérêt pour la recherche, en mathématique. avait considérablement faibli. Mais peut-être que la raison est liée aussi au ralentissement du phénomène de circulation des idées, des livres et des hommes entre les différents foyers scientifiques de l'empire musulman.

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36 Partant des deux nombres positifs $a$ et $b$ ($a < b$), il calcule $ab$, $a + b$ et $b - a$ et il montre que $a$ est la solution (positive) de $x^2 + (b - a)x = ab$, que $a$ et $b$ sont celles de $x^2 + (ab) = (b + a)x$ et que $b$ est la solution de $x^2 = (b - a)x + (ab)$.

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# Indian Numerals in the Mediterranean Basin in the Twelfth Century, with Special Reference to the "Eastern Forms"

by CHARLES BURNETT

The Arabic numerals which are now used universally are of ultimately Indian origin and were called "Indian" by Arabic, Greek and Latin scholars of the Middle Ages. The time, place and context of their introduction and development in all three language cultures are still obscure. This article seeks to illuminate, in particular, the use of one kind of Indian numeral (the "Eastern forms") which was shared by Arabic, Greek and Latin mathematicians in Italy and the Eastern Mediterranean in the twelfth century. By the early thirteenth century this kind had been displaced in Latin contexts by the "Western forms" which are the ancestors of our present Arabic numerals.

Mathematical notation is independent of language; it is symbolic, and does not represent sounds. Therefore, there is no need for different notations to be used in different languages, even when they are written in different scripts. Nowadays, the same mathematical notation is used and understood throughout the world, by Chinese, Arabic, Russian and American mathematicians. There was a potential for this to happen in the Middle Ages too. The Indians had invented a symbolic notation for the nine digits and the zero, which was taken over by Syrian and Arabic writers and eventually passed to Western Europeans. Scholars writing in Syriac, Arabic, and Latin alike referred to these symbols as "Indian figures," and they all participated in the same, distinctive, method of calculating with them. Thus a common mathematical language was shared by mathematicians in Bath and Baghdad, in Roskilde and Marrakesh. However, unlike the situation in the modern world of global communication in which systematization has become the norm, in the Middle Ages, symbols inevitably changed and diversified as they travelled from place to place, and as a result of the passage of time. One change which had great consequences for the history of Indian numerals was that which took place somewhere on the Western fringe of the Mediterranean World, which resulted in the forms of the numerals that prevailed among Latin scholars and have eventually been adopted universally by mathematicians (our "Arabic numerals"). What I would like to draw attention to in this paper are certain forms of Indian numerals shared for a while by Latin, Greek and Arabic scholars, but which, in the end, were *not* adopted by Western scholars, with the result that there is now a split between the printed forms of numerals used in the Western world and those used in most parts of the Islamic world.[1]

1 For the Arabic side of the story, see Kunitzsch in [Folkerts 1997], and [Kunitzsch 2002]. I am most grateful for Paul Kunitzsch's advice. I am also indebted to Menso Folkerts for the generous and prompt loan of microfilms, to Nigel Wilson for a careful reading of the article.

## The Eastern and Western Forms of Indian Numerals[2]

The most obvious differences between the numerals which became the norm in Latin Europe (henceforth the "W[estern forms]") and forms closer to the printed (Eastern) Arabic shapes (henceforth, the "E[astern forms]") are as follows:[3]

W: 2 and 3 tend to be "upright," giving the impression of being the cursive representation of two or three horizontal lines. Sometimes the vertical orientation of the form is emphasised by being terminated with a straight descender.
E : 2 and 3 look as if they are "on their backs," giving the impression of being the cursive representation of two or three vertical lines.

W: In the case of 4, the Arabic form was made up of a hook and loop. In W, the loop predominates over the hook, which disappears.
E : The hook predominates over the loop, which disappears.

W: 5 is a cup-shape terminating on the right with a vertical descender.
E : 5 resembles a capital "B," sometimes turning into a figure-of-eight, at other times into a circle either crossed by a horizontal line or squeezed by a belt round its waist. In Arabic, the capital "B" is usually reversed.

W: 6 is a circle or spiral terminating on the left with a vertical ascender, which is sometimes bent over into a horizontal plane, or continues the curve of the spiral.
E : 6 is a cup terminating on the right with a vertical descender. Sometimes the curve-and-descender is replaced by a stepped- or zigzag-shape.

and to Jeremy Johns, Michael Matzke, Bernd Michael, Fritz Saaby Pedersen, Julien Veronese and Clare Woods. All the examples of Eastern forms in Latin manuscripts mentioned in this article are given in the Table on pages 265–267, alongside representative examples of other forms, and of forms found in Arabic and Greek manuscripts. Note that the following terminology is used:

*algorism* A text describing how to use Indian numerals in arithmetical calculations deriving ultimately from al-Khwārizmī's *Indian Arithmetic*.

*Arabic* Written in Arabic. Therefore, "Arabic numerals" are numerals (of any kind) written by Arabic scribes, and not "Arabic numerals" written by Latin scribes.

*Indian numerals* The symbols for numerals known now in the West as "Arabic," but referred to by Arabic and Latin scholars as "Indian." They derive from Indian symbols and are characterised by having place value.

*row* The numerals 1 to 9 set out in a line.

"*standard*" The Western forms that became most widespread in the Middle Ages, from the early thirteenth century onwards.

2 Scholars have often referred to these forms respectively as *hindī* ("Indian") and *ghubārī* ("dust"), but the inappropriateness of these terms has been demonstrated in [Kunitzsch 2002]. For the forms of the Indian numerals in Indian scripts see [Renou & Filliozat 1985: 702–708]; their diffusion is described in [Ifrah 1981: 460–490].

3 Occasionally Latin scribes change the *direction* in which the numerals face, but usually the numerals are not turned round in respect to their Arabic forms: see [Burnett 2001a].

W: The two branches of 7 make the shape of a gallows (gnomon) or a lamda.
E : The two branches of 7 make the shape of a "v."

W: 8 is formed from two circles, one on top of the other.
E : 8 is a lamda shape in which the left-hand branch is sometimes tucked in to form a bow or a circle.

W: 9 is a circle extended downwards on the right in either a straight or a curved line.
E : The form is the same as in W except that sometimes the mirror-image is substituted, perhaps to differentiate it from 8.

Both W and E use a small circle for 0, sometimes substituting the astronomical symbol "τ."[4]

It is not the purpose of this article to explore the origin of these differences. Suffice to say that it is more plausible to suppose that all the Western and Eastern forms derive ultimately from the same source, rather than that, in some cases, symbols from a completely different kind of source have been substituted. For example, it has been suggested that the Western form of 5 is simply the adoption of the way of writing the Roman "v" in Visigothic Spain [Lemay 1977: 452–453]. However, already in Arabic contexts the loops of the reversed-"B" form of 5 often fall away from the straight ascender, which becomes abbreviated; if one imagines this process continuing until they fall into a horizontal plane and the last curve becomes straight, the Western form of 5 would result. Also, the spiral of the Western form of 6 could derive from the cup of the Eastern form: the spiral is still attached to a vertical descender in several versions of the abacus numerals of the eleventh and twelfth centuries (see [Folkerts 1970, Table I] and the *Liber Floridus*, facsimile edition, Gent 1968, fol. 85v). But the ease in which purely hypothetical conjectures like this can be made shows the danger of spending too much time speculating on origins.

## Variant Forms Given in al-Khwārizmī's *Indian Arithmetic*

Before looking at the regional development of different numeral forms, it would seem natural to consider the transmission of the essential text on calculating in the Indian way (*al-ḥisāb al-hindī*), which promoted the use of Indian numerals: al-Khwārizmī's *On Indian Calculation* (ca. 820). Although the original Arabic text has been lost, it gave rise to a whole genre of works, in Arabic, Latin and Greek, which convey more or less of the original text, and are known generically as "algorisms," after the name of the first author.

4 Hence zero is called "circulus" in early algorisms (*Dixit Algorismi* and *Liber Alchorismi*) and in the first version of Abū Ma'shar's *On the Great Conjunctions* (see below, p. 242). The term "dā'ira ṣaghīra" ("small circle") is also used for "zero" in Arabic.

Already in the Arabic algorisms different ways of writing certain numerals are mentioned (see [Kunitzsch 2002]). In a copy of an algorism by Ibn al-Yāsamīn of Morocco (d. ca. 1204) a row of Western forms is given first, followed by a row of Eastern forms (Plate 1). Ibn al-Yāsamīn writes:

> These are the shapes which are called "al-ghubār" (followed by the Western forms), and they are also like this (followed by the Eastern forms).[5]

The differences in certain forms are mentioned (but not illustrated) in the Latin text which represents al-Khwārizmī's work most closely, known from its incipit as *Dixit Alchoarizmi* (*DA*):

> Est quoque diversitas inter homines in figuris earum. Fit autem hec diversitas in figura quinte littere ac .VI., .VII. quoque et octave [Folkerts 1997: 28, commentary 111–112].

In the *Liber Alchorismi* (*LA*), a similar passage can be found:

> Est autem in aliquibus figurarum istarum apud multos diversitas. Quidam enim septimam hanc figuram representant ч, alii autem sic ٧, vel sic 7; quidam vero quartam sic ٤ [Allard 1992: 69].

Of these alternatives, ٧ is clearly the Eastern form of 7, while ٤ is the hook-and-loop form of 4 which underlies both the Eastern and Western forms.[6] In one manuscript that contains the hybrid version of the *Liber pulveris* and *Liber Alchorismi* — Vat. Pal. lat. 1393 — the row of Eastern forms are added after the phrase "Quidam etiam sic scribebant figures." In MS Dresden C 80, similarly, two rows of forms are given, and the author seems to be saying that one is more genuinely "Indian" than the other (see p. 257 below); the Eastern forms are clearly recognizable, and 4, consisting of two curves, one on top of the other, like an open cursive "e," resembles that of Vat. Pal. lat. 1393. In another fifteenth-century manuscript of the algorism, Paris, BNF, lat. 10252, the two rows are given, along with the abacus names, but the numerals have been copied so badly that it is difficult to recognise the Western or Eastern forms in either row.

The fact that the numeral forms vary from one algorism to another, and that different numerals are picked out as having alternate forms in different algorisms, suggest that the differences reflect not the original text of al-Khwārizmī, but rather the practices current at the time and in the locality of the author or scribe of the

5 The photograph of MS Rabat, Maktaba al-ʿāmma, k 222 has been reproduced in [Abū Fāris 1973: 232] and [Kunitzsch 2002], and the text is discussed in [Köbert 1975]. This manuscript itself is written in the Eastern Arabic script and is of a later date than Ibn al-Yāsamīn, but it is interesting to note that the 5 in the row of Eastern Numerals has its loops on the right, just as in the Palermitan Arabic sources discussed below (p. 243).

6 Form 7 is similar to our present 7, which is also the shape of the "tironian *et*," and belongs to the tradition of the Western forms. I have not found ч, which is like an upside-down lowercase "h" with a curved ascender, among the Eastern or Western forms, but it is curious to observe that occasionally it appears, in an upright form, for 7, in the fifteenth-century copy of an early manuscript of Raymond of Marseilles's *Liber iudiciorum* in Paris, BNF, lat. 10252 (see fols. 88r and 89v).

algorism. The algorisms did not determine which forms to use, but rather reproduced the forms in use; nor were they the sole means by which Indian numerals were transmitted. It is to the temporal and regional differences in the numerals that we should now turn.

## The Western Forms

A consistency can be observed in the Western forms, from their earliest examples in a Latin manuscript of Isidore of Seville's *Etymologiae* written in the monastery of Albelda in the Rioja in 976, through their stylised representation on the *apices* (counters) of the "Gerbertian" abacus (where they are also used to number the columns of units, tens, hundreds, etc.), to the forms found in most of the twelfth-century texts on the algorism (exceptions have been mentioned above), until, by the time they were used in the early thirteenth-century manuscripts of translations made in Toledo, they became the standard Medieval forms of Indian numerals. The tenth-century examples and the abacus texts already show all the distinctive features of the Western forms, except that the 4 still retains both its hook and its loop.[7] The earliest form of the Latin algorism — *Dixit Alchoarizmi* — preserves forms (for 2, 3 and 4) that are strikingly similar to those of the Albelda MS; the rotation of the forms of some of the numerals (5, 7, 9) in the context of the abacus is attributable to the non-directionality of the *apices* (see [Beaujouan 1948]). To identify a specific time and place for the origin of these distinctive features is more difficult. It is possible that they arose among mathematicians in Islamic Spain, from which they were brought to the monasteries of León and Old Castile by Christian refugees in the ninth and tenth centuries, and they remained in use after the Reconquista.

A Latin manuscript written in Bavaria in the late twelfth century — Munich, Clm 18927 — compares different forms of Indian numerals by setting them out in three rows: the first is described as "toletane f(igure)" ("Toledan figures"), the second as "indice f(igure)" ("Indian figures"), the last is not described, but is that which the scribe of the manuscript uses himself (Plate 3). This last row, of which the most distinctive element is the vertical tails extending below the 2 and 3, is characteristic of some manuscripts written in England, France and Germany in the twelfth century, e.g. Cambridge, Trinity College, O 7 41, Paris, BNF, lat. 14704 and lat. 16208 (both manuscripts of the works of Raymond of Marseilles), and Munich, Clm 13021 (written by Sigisboto of Prüfening (?), between 1163 and 1168). The Munich manuscript also gives 3 in the form of a cleft stick, which occurs too in a multiplication table added to a manuscript written by Wolfger of Prüfening in 1143 (MS Vienna, ÖNB, 275), and in year-values in the Annals of

7 To the examples listed in [Folkerts 1970, Table 1] and [Ifrah 1981: 506], one may add Vat. Reg. lat. 1308, copied in the eleventh century in France by an Italian scribe who numbered the quires with Arabic numerals and their names (see [Bischoff 1990: 23, n. 27]). For an aberrant example see [Gibson & Newton 1995]. See also [King 2001: 309–317].

Regensburg for the period 1152 to 1197 (MS Munich, Clm 14733 written by Hugo von Lerchenfeld).[8] These forms may have been transmitted with the early manuscripts of the *Liber Ysagogarum* and Toledan Tables, both of which are included in Clm 18927 itself.[9]

What is more difficult to ascertain is the significance of the description of the first row of numerals as "toletane f⟨igure⟩." The 2 and 3 do not have the vertical tails. The non-tailed form is normal for the numerals in later manuscripts of translations made in Toledo (see below p. 255), and eventually became standard. That Indian numerals were known to Latin scholars in Toledo would seem to be guaranteed by their presence in the *Liber Alchorismi* of "magister Iohannes" which accompanies Toledan translations and precedes a calendar relating to Toledo and computus tables for 1143–1159 in its best manuscript, Paris, BNF, lat. 15461.[10] However, that Indian numerals were used by the principal translator of mathematical works in Toledo, Gerard of Cremona (1114–1187), has been questioned [Kunitzsch 1990, II: 8–9]. Two pieces of evidence could be significant in assessing this question. In the earliest copy of Gerard's major mathematical translation — Ptolemy's *Almagest* in Paris, BNF, lat. 14738 — the scribe, who is probably French, but using Spanish parchment, is evidently copying from a manuscript which used Indian numerals, for he starts by transcribing them (with obvious difficulty), and then gives up after the first few pages, and substitutes Roman numerals.[11] In the best manuscripts of a revision (possibly by Gerard, and certainly made in Toledo) of the translation of Abū Ma'shar's *On the Great Conjunctions*, two year-values have been copied in a "fossilised" form, presumably because they were not understood by the copyist (Plates 2 and 4).[12] They show the early form of 4 consisting

8 Examples of these German manuscripts are given in [Menninger 1958, II: 238–239; von Fichtenau 1937; Arrighi 1968; Lemay 1977].

9 The earliest Latin versions of the Toledan Tables were not necessarily written at Toledo; for they owe their name to the fact that they derive from tables made for the meridian of Toledo by al-Zarqāllu in ca. 1080; the first evidence of Latin knowledge of them is in Aragon [North 1995] and Marseilles (Raymond of Marseilles's adaptation of the Tables to the meridian of Marseilles in 1141). Associated with this form is the writing of a compendium of .xl. .ꭓ́. for "40" which is found in the works of Raymond and in Clm 18927, and which is of Spanish origin.

10 For other references to the algorism by Toledan scholars see [Allard 1992: xx; Burnett 1994: 428–429].

11 I am grateful to Patricia Stirnemann for her advice concerning the provenance of this manuscript. Further examples from early copies of Gerard's translations are discussed in [Jacquart 2001: 215–219], to which may be added the copy of Gerard's translation of the *Liber ad Almansorem* in Cambridge, University Library, Additional 9213, where 2 has a tail, but 3 has not (as in Bodleian, Digby 51).

12 The passage containing these year-values does not occur in the original translation of *On the Great Conjunctions* (apparently by John of Seville); for the two Latin versions of the text see [Burnett 2001b]. It is possible that a gloss close to these forms (but without a corresponding reference mark in the text) refers to these strange forms: "hanc litteram nec hunc numerum intellexi" ("I have not understood this letter or this number"). In this case, the reviser himself (who was also responsible for the gloss: see [Burnett 2001b: 55–56]) seems to have been confused by the numeral forms.

of a hook and a loop, and a version of the Western form of 6 that occurs only in the earlier Latin tradition and in Arabic.[13] These forms may retain the shapes of the numerals written in the original Arabic manuscript. That the Arabic manuscripts of *On the Great Conjunctions* included Indian numerals is certain, since Abū Ma'shar writes a very high number (of days in a long period of years) first in words, and then "in the Indian form" (*aṣ-ṣūra al-hindīya*, translated "figura indica / figure indice"). It is curious that, aside from in the two cases mentioned above, the Latin manuscripts (none of which are before the very end of the twelfth century) give the Indian numerals in their standard Western form, and the fossilised forms too have been glossed with these standard forms.

Thus, though it is likely that Indian numerals were used in Toledo, even outside the context of the algorism, the fact remains that we cannot yet, as far as I am aware, point to a manuscript definitely written in Toledo in the twelfth century, in which the scribe shows himself familiar with Indian numerals.[14] Moreover, no Arabic manuscript has been found written in Spain (or any part of the North-West African realm of the Almoravides or the Almohades) which uses the Indian numerals, in any form, until after 1284 [Kunitzsch 2002].

## The Palermitan Forms of the Court of Roger II (1130–1154)

For Sicily we are more fortunate. For there is a manuscript written in an Arabic hand in which Indian numerals occur, and which can be dated and located quite accurately. The manuscript is London, British Library, Harley 5786, which consists of a Psalter written in three languages: the Greek text on the left, the Latin text in the centre, and the Arabic text on the right (Plate 5). The scribes of each language are different from each other and are evidently professional. Each psalm is numbered in the numeral system appropriate to the language. Thus alphabetical numeration is used for the Greek, and Roman numerals for the Latin. For the Arabic, Indian numerals are used, and among these we encounter both the Western form 6 with a straight ascender (as in the Toledan translation of Abū Ma'shar's *On the Great Conjunctions*), and a 4 with a hook and loop. Moreover, 8 is written in its Western form, with two circles. It must be noticed, however, that 2, 3, 5 and 7 have the characteristics associated with the Eastern forms.

13 6 with a straight ascender is found in the Albelda manuscript of 976 and in the abacus texts, but is also used in the Arabic text in Harley 5786 discussed in the next section.

14 The best candidate for such a manuscript is Paris, Bibliothèque de l'Arsenal, 1162, written in Spain, with Spanish decoration, in 1143, which contains the copy of the translations of texts on Islam (including the Qur'ān) commissioned for Petrus the Venerable, abbot of Cluny. The last text is a translation by "Peter of Toledo," and the whole collection has been called the "Collectio Toletana," but most of the translations were made by Hermann of Carinthia and Robert of Ketton (see below, p. 248), and Peter the Venerable's itinerary took in only the valley of the Ebro and Old Castile. The Indian numerals, which are used to number the folios on the *verso* on fols. 1–17, are, however, close to the "toletane f(igure)" in form and look later than the text. See [d'Alverny 1948: 77–80, 108–109].

MS Harley 5786 was written before 1153, for this date has been written on the last folio: "anno incarnationis dominice .m.c.liii. ind. ⟨i⟩ mensis ianuarii die octavo die mercurii."[15] The nature of the work makes the most likely place of origin the Palermo of Roger II. For King Roger astutely promoted the interests of the three language communities of his Sicilian kingdom, and set up a separate chancery for Latin, Greek and Arabic documents. Jeremy Johns has shown that his Arabic chancery (or "Dīwān") was formed on the model of that of Fatimid Egypt, and Roger most probably brought in scribes trained in writing the script from Egypt or another East Mediterranean Islamic chancery.[16] The Arabic script of the Psalter is typical of the Eastern Arabic script used in the Royal Dīwān, but concedes to local custom by writing the Arabic letters *fā'* and *qāf* in the *maghribī* way, rather than as the Eastern Arabs did. This may account, too, for the substitution of Western 6 and 8 for the predominantly Eastern forms of the numerals.

There is a striking similarity between the Indian numerals in the Psalter and those used in the tables in one manuscript of the translation of Ptolemy's *Almagest* made in Sicily in ca. 1165 [Haskins 1927: 157–162; Murdoch 1966]. The translation (as we are informed in the preface, written by the unnamed translator), was made from a Greek manuscript which was brought from Constantinople to Palermo as a present from the Greek emperor to the Sicilian king. Although the translation was made from Greek, it also includes some material from the Arabic astronomer, al-Battānī,[17] thus nicely combining Greek, Arabic and Latin culture, as does the Harleian Psalter. The most accurate, and probably the earliest, of the four known manuscripts of this translation is Vatican, Pal. lat. 1371, and it is this manuscript that contains the numerals which are very similar to those of the Harleian Psalter, when allowance is made for the fact that one is written by an Arabic scribe, the other by a Latin one (Plate 6). Particularly striking is the resemblance of the form for 5, which looks like a "B" with the ascender extended upwards and which is reversed in respect to the usual form found in Arabic manuscripts.[18] The only observable differences are in 4, in which the loop in the Arabic has been

15 The date (discussed in [Haskins 1927: 184]) is now very faint (even under ultraviolet light), due to the rubbing of the last folio, and the indiction number has disappeared.

16 In his detailed study [2002] Johns has shown that the script of the Dīwān is found only in the royal palace and court, and differs from the variety of scripts used in private documents from Sicily of the period, which exhibit strong *maghribī* (Western Arabic) features. Roger's "Great Emir" ("magnus ammiratus"), George (d. 1151), who had ultimate charge of the chancery, was from Antioch. I am most grateful to Jeremy Johns for his advice, and for sending me material in advance of its publication.

17 [Lemay 1987: 466] quoting MS Vat. pal. lat. 1371, fol. 67v: "Hoc fuit necesse, ut ait Albategni, tabulas diversitatum apponere."

18 In the Arabic context of the Harleian Psalter it is clear that the scribe, writing from right to left, has executed the form in one stroke, ending with a sweep of the pen upwards. In the Latin context, because the scribe was writing the form in the opposite direction, he uses two strokes of the pen, the first a single downward stroke, the second, the two bows of the "B." The resulting forms, however, are very similar to each other (I owe the confirmation of this to Clare Woods).

simplified to a right-angle, and in 6, in which the straight ascender has been bent over into a vertical plane.

The agreement of the Indian numerals in the Harleian Psalter and a manuscript of a translation of Ptolemy's *Almagest* made in Sicily, suggests that both texts were written in the same milieu. This is probably Palermo, to which the Greek manuscript of the *Almagest* had been brought, and where, according to a marginal note, the translation was made.[19] That Indian numerals had a privileged position in Palermo is indicated by the fact that Roger II himself used them on his Arabic coins, and is the first Western European — or indeed Arabic — ruler known to have done this. From the evidence of the two dated coins surviving,[20] the same forms of Indian numerals are used as in the Harleian Psalter; in particular, both sources give a 5 with the ascender extended upwards and reversed in comparison with the usual Arabic form. It is likely, then, that the translator of the Sicilian *Almagest* himself used the Indian numerals, which would have been familiar to the scholars of his circle who would have been associated with the royal household. These Palermitan forms, however, do not seem to have been known widely amongst Latin scholars, since they have not been identified in any other Latin manuscript. Therefore, when copies of the translation were made for distribution outside this circle, more familiar forms would have been substituted; hence the Roman numerals used in Florence, Biblioteca Nazionale, Conventi Soppressi, A. 5. 2654 (ca. 1300). This suggests that, in this instance, the use of Indian numerals *preceded* that of Roman numerals in the history of the transmission of the text.

## The Eastern Forms: Hugo of Santalla and Hermann of Carinthia

As we have seen above, the second row of numerals in Clm 18927 is described as "indice f(igure)" ("Indian figures"). These are the Eastern forms of the Indian numerals, to which the rest of this article will be devoted.[21] While we are badly informed about the use of Western forms in Arabic manuscripts in the Western part of the Islamic world, there are several examples of the use of Eastern forms

19 Vat. Pal lat. 1371, fol. 41r: "Translatus in urbe Panormi tempore Roggerii per Hermannum de greco in latinum." For the attribution to "Hermann" see below, p. 248.

20 For a "folles" of 533 (= 1138/39) on which the *hijra* date is given in Indian numerals see [Travaini 1995: 53, 284 (no. 193), Pl. 12], in which the illustration is taken from [Spinelli 1844: Pl. VI, no. 32]. Another coin was struck in Messina with the date 543 (= 1148/49; [Travaini 1995: 300–302, no. 247; Grierson 1998: 626]). Travaini concludes (p. 302) that "questi due follari sono forse, a quanto pare, i più antiche esempi nella numismatica di data con l'anno in cifre arabe, e tale aspetto merita ulteriori indagini." I owe these references to Jeremy Johns. See Plates 7 and 8.

21 The only parallel I have found for singling out the Eastern forms as "Indian" is in a commentary by Ḥusayn ibn Muḥammad al-Maḥallī to a work on arithmetic by al-Sakhāwī (14th century), cited in [Ifrah 1981: 502] in which the Western forms are called *ghubārī* and the Eastern, *hindī*; but see also [Kunitzsch 2002].

in Arabic manuscripts written before 1200 (see [Irani 1955/56; Kunitzsch 2002]). What has received less attention is the use of Eastern forms in Latin and Greek manuscripts in the same period. One of the few scholars who have commented on the Latin diffusion of these forms, Richard Lemay, sees Hermann of Carinthia (fl. 1138–1143) as a key figure [Lemay 2000]. While his arguments are based on dubious attributions, it is instructive to start our investigation by looking at the evidence of the copies of works by Hermann of Carinthia and his close associate Hugo of Santalla.[22]

A rare testimony to the origin of an Arabic manuscript used by a Latin translator is provided by Hugo of Santalla (fl. 1145–1179), a *magister* in the service of Michael, bishop of Tarazona (1119–1151) and his successors, who states, in the preface to his translation of the commentary by Ibn al-Muthannā on the *Zīj* (Astronomical Tables) of al-Khwārizmī, that his patron got his manuscript from the "rotense armarium." This has been identified as the library of Rueda de Jalón, the stronghold to which the Banū Hūd kings of Islamic Zaragoza retreated after the fall of their kingdom in 1118 [Haskins 1927: 75]. Two kings of the Banū Hūd dynasty were renowned for their mathematical prowess [Hogendijk 1986], and it is likely that at least part of the library of mathematical texts that they possessed survived the removal to Rueda. Since they were aware of recent mathematical works written in the East, such as the *Optics* of Ibn al-Haytham, it is possible that they were familiar with the Eastern forms of the numerals. No extant Arabic manuscript has been identified as belonging to their library. But it may be possible to pick up hints of what these manuscripts looked like from the earliest manuscripts of Hugo's translations.

Four manuscripts take us close to the activity of the translator himself. Two of these (Oxford, Bodleian, Digby 159 and Cambridge, Caius College, 456 [= C]) belong to what must originally have been a collection of the translations of Hugo, possibly put together within his life-time and written, for the most part, by an English scribe.[23] The other two (Bodleian, Arch. Seld. B 34 [= A] and Digby 50) were also written in the same hand — that of a professional Italian scribe — and are probably a little later than the first two. In all these manuscripts Roman numerals are normally used, but the Eastern forms occur in two contexts. The first is that of two illustrative tables in Ibn al-Muthannā's commentary on al-Khwārizmī (the very text that Hugo mentions as coming from the library of Rueda) in MSS A and C (see Plates 9 and 10). It is clear from both manuscripts that in the exemplars from which the numerals were copied the Eastern forms were used. However, the scribe of C was evidently unfamiliar with the numerals and did not understand

22 I refer to Lemay's attributions to Hermann of the Sicilian translation of the *Almagest* (discussed above p. 244 and below p. 248) and the *Liber Mamonis* (discussed below, p. 251). It is *prima facie* unlikely that Hermann of Carinthia would have translated the *Almagest* from Arabic as he claimed to be intending to do) and from Greek *and* would have written a book — the *Liber Mamonis* — which uses the terminology of yet a third translation of the work, again from Arabic: see [Burnett 2000b: 10–13].

23 At least one further volume must have existed when a copy was made of the collection into MS Bodleian Savile 15, written in the fifteenth century: see [Burnett & Pingree 1997: 9–10].

how they functioned; for he reverses the order of the numerals and the direction they face (thus providing a mirror image of the table), and misinterprets the symbol for 4 as the Roman numeral xxxvi or lxxvi. In MS A, on the other hand, the numerals have been filled in by a hand other than the professional scribe: presumably by a scholar, and a scholar who was familiar with Indian numerals.[24] The forms copied by the scribe of C apparently differed a little from those used by A. The two scribes share a distinctive form of 7 with a loop at the bottom, but A's 5 is a figure-of-eight, whereas B's 5 is the "B" shape. More perplexing is the 4: while A clearly draws the hook-and-loop shape, C evidently misinterpreted quite a complex symbol in his exemplar in writing xxxvi or lxxvi for the single Arabic digit.[25]

That the scribe (as distinct from the annotator) of MS A was unfamiliar with the Eastern forms is indicated by their appearance in the other text copied by him: Oxford, Bodleian, Digby 50, which contains Hugo's translation of an Arabic text on geomancy, the *Ars geomancie*.[26] In a sequence of chapters following Hugo's text (beginning "Notandum est quod Leticia"; fols. 93–106), it is evident that the scribe was copying from a text that used the Eastern forms, but was not himself used to writing these forms (Plate 11). His first attempts (on fol. 94r–v) are rather crude, and include only the numbers 1, 2, 3 and 4. From fols. 99r onwards, however, he gives systematic lists of each of the sixteen geomantic figures in each of the twelve astrological places. Hence we can see what the scribe does as he repeats the numbers 1 to 12 sixteen times. He copies the Eastern forms throughout for the first three times. The first time his numerals are a little clumsy, with the vertical strokes of 3 separated, and the curve of 6 exaggeratedly rounded; for the second and third times the numbers are written more evenly and confidently. For the fourth time, however, he writes Roman numerals for 1, 2 and 3 and (inadvertently?) realises the symbol for 6 as "etiam."[27] The next time all the numerals are Roman except 9; and after this the scribe only uses Roman numerals. It is as if he, having at last understood the significance of the pattern of recurring symbols, transcribed them into the symbols that he knew best, namely Roman numerals.

24 The same hand has provided, in the margins of fol. 20r, a revised translation of a large part of the chapter headed "Quare alhoarizmi maius attadil..." [Millás Vendrell 1963: 111]: he crosses out the passage printed at [Millás Vendrell 1963: 112, line 9–113, line 6] and rewrites it with different terminology and different transliterations of the Arabic words. He also comments, on the same folio, that the geometrical figures are in the wrong order. It is possible that this scholar has filled in the values in the tables from a copy of Abraham ibn Ezra's *Book on the Foundations of the Astronomical Tables*, which quotes this section of Ibn al-Muthannā's text: see [Millás Vallicrosa 1947: 152] and below p. 249.

25 The scribe of MS C fails to fill in the second table; in MS A, on the other hand, almost the same material is provided in two tables, thus giving three tables altogether.

26 For an account of this text see [Charmasson 1980: 95–109].

27 The Eastern form of 6 does, in fact, resemble a common form of the "tironian *et*," though, strictly speaking, this should only be realised as "etiam" if there is a bar on top of it; but bars are often found over Indian numerals; e.g. in MS Harley 3631 of Abū Ma'shar's *On the Great Conjunctions*.

Once again we have an indication that Indian numerals precede Roman numerals in the transmission of a text.

The forms of the numerals that the scribe was attempting to imitate differ in at least two cases from those of the manuscripts of Ibn al-Muthannā: 4 is written as a "z" with its base-line extended in a downwards curve; 5 is the "crossed-zero" version turned through 90 degrees.

A third work with which Hugo of Santalla was associated was the *Book of the Three Judges.*[28] In two out of the three manuscripts that contain this compendium of judicial astrology there is a dedication to "bishop Michael" ("antistes Michael"), Hugo's patron in Tarazona, and the style of writing is that of Hugo.[29] In a third manuscript the dedication is to "mi karissime R." who is likely to be Robert of Ketton, archdeacon of Pamplona, the close friend of Hermann of Carinthia. It is probable that Hermann and Hugo, both of whom were in the valley of the Ebro in the 1140s, collaborated in the compilation of the *Book of the Three Judges.* In the manuscript with the dedication to "mi karissime R." — British Library, Arundel 268 — the Eastern forms are used exclusively (Plate 12). Moreover, only Eastern forms are used throughout the codex which contains the *Book of the Three Judges*, and they can therefore be assumed to be the forms favoured by the scribe himself. Some of the numerals resemble those found in Digby 50: the 8 is identical, while the 6, with its left-hand stroke almost forming a circle, resembles one of the forms of 6 in the Digby manuscript. The 4, however, consists only of the hook, and the rounded form of 9 is used.

An early thirteenth-century copy of another text possibly by Hermann of Carinthia occurs in a manuscript written with the Eastern forms: namely, the translation of the *Centiloquium* of Pseudo-Ptolemy with the incipit "Mundanorum ad hoc et illud mutatio" (which incorporates part of Plato of Tivoli's translation of the same work), whose copyist in MS Berlin, SBB-PK, Hamilton 557, has numbered the *Verba* in the margin away from the body of the text.[30] The presence also of Western and mixed forms in this manuscript may indicate that it was written when the Eastern forms were already defunct.

As we have seen (n. 19 above), a "Hermann" is also named as the translator of the Sicilian *Almagest.* While it is not *prima facie* likely that this is Hermann of Carinthia, it is interesting to note that the same translator was responsible for the major part of a version of Euclid's *Elements* made from Greek [Murdoch 1966; Busard 1987: 2–3]. The scribe of the earliest manuscript — MS Paris, BNF, lat. 7373 (Plate 13) — uses Eastern forms similar to those of Arundel 268, and both scribes wrote all the texts in their manuscripts in the Eastern forms; they were

28 Note that in the *Biblionomia* of Richard of Fournival the *Book of the Nine Judges* (an expanded version of the *Book of the Three Judges*) was in the same manuscript as Hugo's geomancy ("In uno volumine liber IX judicum, geomancia que vocatur Rerum opifex" [Delisle 1868–81, III: 67, no. LVI-17]).

29 The evidence for Hugo's authorship is assessed in [Burnett 1977: 67–70]. To the two manuscripts mentioned in this article should be added Dublin, Trinity College, MS 368 (fols. 43r–137v).

30 Richard Lemay first drew attention to this manuscript; see [Lemay 1995–96, VII: 141].

evidently the symbols they were most used to using. Moreover, the Eastern forms numbering the theorems of Euclid are placed in the margin away from the body of the text in the same way as in MS Berlin, SBB-PK, Hamilton 557. The Paris manuscript was written in Tuscany [Avril 1980–84, I: 56], and the version of Euclid's *Elements* that it contains, as [Busard 1987: 18–20] has shown, was known to Fibonacci who was working in Pisa, who may have added an appendix to the translation. The association of this translator's work with that of "Hermann" may be due to the fact that his translations were copied in the same environment as those of Hermann of Carinthia and Hugo of Santalla. That this environment was Tuscany will be explored below (see p. 251).

## Abraham ibn Ezra

Another scholar who translated the commentary by Ibn al-Muthannā (this time into Hebrew) and came from the same area as Hugo and Hermann, was the Jewish polymath, Abraham ibn Ezra, who was born in Tudela between 1089 and 1092. It is in the twelfth-century Latin works associated with this scholar that we have the most sustained use of the Eastern forms. These occur in four twelfth-century manuscripts of his *Book on the Foundations of the Astronomical Tables* (written in 1154) — Erfurt Q 381, British Library, Cotton Vespasian A II (Plate 15), Cambridge, Fitzwilliam McClean 165 and Oxford, Bodleian, Digby 40 (Plate 16) — and in his text on the astrolabe also contained in the Cotton manuscript. In another manuscript — British Library, Arundel 377 — the astrolabe text accompanies an introduction to the Pisan Tables attributed to Abraham. The two texts in this manuscript are written with the Western forms, but a list of Eastern forms with the Western forms written on top of them is included (Plate 17); this suggests that, at an earlier stage in its diffusion, it was written in the Eastern forms. These Eastern forms occur also in an anonymous text on arithmetic and geometry which has been copied out into the important late-twelfth-century collection of Arabic-Latin translations made by scholars working in North East Spain (especially Plato of Tivoli in Barcelona): Oxford, Bodleian, Digby 51. It has recently been shown, from a comparison with a Hebrew version of the same text, that this work too is by Abraham (see Plate 14 and [Lévy 2001]).[31]

In all the texts explicitly attributed to Abraham the numeral forms are very similar to each other, with 6 acquiring a zigzag form, 8 written as a curve and 9 written with a straight back and facing right; only in the case of 4 is there variation, but the hook with an extended tail (but without a loop) seems to underly all the versions. The forms in Digby 51 diverge a little in that 4 takes the form that we also meet in Digby 50, the manuscript associated with Hugo of Santalla.

31 The presence of the Eastern forms in the earliest manuscript of the *Ysagoge* and *Liber quadripartitus* attributed to Iohannes Hispalensis (Venice, Biblioteca Marciana, Lat. Z. 344) confirms the close relationship of this text with the astrological works of Abraham ibn Ezra, which is the subject of a forthcoming monograph by the present author.

So far, the picture that is building up is of the employment of the Eastern forms in works written by writers on astronomy who can all be associated with the North East of Spain and the South of France. Hugo of Santalla was translating works for Michael, bishop of Tarazona from 1119 to 1151, and was possibly still in the service of one of his successors in 1179. Hermann of Carinthia probably collaborated with Hugo over the compilation of the *Book of the Three Judges* and its sequel, the *Book of the Nine Judges*, and is attested in León (1138), "on the banks of the Ebro" (1141), and in Toulouse and Béziers (both in 1143). But Hermann's principal collaborator and inseparable friend was Robert of Ketton, successively archdeacon of Pamplona, and canon of Tudela (1157). Abraham ibn Ezra was born in Tudela, and this was his base until the early 1140s, when he started to travel. His *Foundations of the Astronomical Tables* and *Astrolabe* in the Cotton manuscript sandwich another work on the astrolabe written in 1144, again in Béziers, by Hermann's pupil, Rudolph of Bruges. Plato of Tivoli worked in Barcelona, and collaborated with Abraham bar Ḥiyya, whose astronomical writings were known to Abraham ibn Ezra. Both Plato and Rudolph addressed texts to a certain "John David," whose mathematical prowess they praise.

Both geographical vicinity and the combinations of their works in manuscripts show that there were close associations between these scholars. It is possible that they had some acquaintance with the Western forms of Indian numerals.[32] But it is equally possible that they worked in an environment in which the Eastern forms were used. This is particularly the implication of the evidence of the numerals in Hugo's translation of Ibn al-Muthannā, whose Arabic text was found in the library of the Banū Hūd. What is more clear is that the Eastern forms were used in Italy. For it is preeminently copies of these authors' works written by *Italian* scribes that exhibit these forms: Digby 50, Arch. Seld. 34, Hamilton 557, and Arundel 268 are all written in Italian hands on Italian parchment. The master scribe of Digby 51 is also Italian; he writes most of the annotations, including a diagram which he labels in a competent Arabic script (fol. 88v), and could be a key player in our story. Moreover, while Abraham ibn Ezra came from Spain, in the early 1140s he was in Lucca, and his *Foundations* and explanation of the use of astronomical tables presuppose the use, specifically, of the tables of Pisa, which he implies he composed himself.[33]

32 The only text in Digby 51 in which the Western forms are used is Plato's translation of al-Battānī, and the master scribe and another hand use the Western forms in their annotations to this text. However, Roman numerals are used in another twelfth-century copy of al-Battānī — Digby 40. The Western forms are also used in Ibn Ezra's *Sefer ha-Mispar* ("book of arithmetic"), but MS Paris, BNF, ébreu 1052, adds the Eastern forms in the margin. However, these manuscripts are of the fifteenth century and it is unwise to draw sharp conclusions from their evidence.

33 The *Foundations* are referred to, by Henry Bate and Nicholas of Cusa, as "De motibus et opere tabularum super Pisas" [Birkenmajer 1950]. Abraham's responsibility for composing the Pisan tables rests on the presumption that "the tables of aṣ-Ṣūfī" and those drawn up for the meridian of Pisa are the same; see [Millás Vallicrosa 1947: 87]: "Proinde omnium aliorum tabulis omissis, tabulas medii cursus Solis secundum Azofi composui ... Et he tabule composite sunt secundum meridiem Pisanorum."

## Pisa and Lucca

The Pisan tables were drawn up in 1149, or soon after, since 1149 is the epoch from which they are calculated. Their earliest manuscript — Berlin, SBB-PK, lat. fol. 307 (Plate 18) — provides the most sustained example of the use of the Eastern forms, employing them throughout, both in the instructions and in the twelve pages of tables. They differ from those found in the manuscripts of Abraham's works in that the "s" form of 4 is used, that 6 is no longer a zigzag, and 8 has a straight back, whilst 9 is curved. In 1160 someone added some annotations to MS British Library, Harley 5402 in Lucca, some 15 miles from Pisa, and gave the date as "1160" in Roman numerals; the only Indian numerals used are in the Eastern form, of which the most distinctive is 6. At about the same time "Stephen the Philosopher," who, for several reasons, should be identified with "Stephen of Pisa," wrote a cosmology which he called the *Liber Mamonis*; in its only surviving manuscript (Cambrai, Médiathèque municipale, 930), the Eastern forms are used for high numbers.[34] In some cases these manuscripts show forms closer to the (supposed) original Arabic than in Abraham's works, such as the "B"-shaped 5 (in Berlin, SBB-PK, lat. fol. 307) and 4 with a hook and a loop (in Cambrai 930).

The Pisan tables are those for which the two Latin works on astronomical tables bearing Abraham ibn Ezra's name (the *Foundations* in several manuscripts, and the introduction to the tables in Arundel 377) were written. But there are at least two more instructions for the use of the tables which do not mention Abraham, and which rather appear to be written by anonymous Christian scholars: those in Berlin, SBB-PK, lat. fol. 307 and Bodleian, Selden Supra 26 (Plate 19). All these instructions use the Eastern forms, as would be natural if they are introducing tables written in those forms (the tables are lacking in all the manuscripts except that of Berlin). The use of Eastern forms in the Latin texts associated with Abraham ibn Ezra is probably due, therefore, not so much to Abraham himself as to his Latin associates, who were using the tables of Pisa. The combined testimony of these manuscripts strongly indicates that the Eastern forms were being used in Pisa and Lucca in the mid-twelfth century. It has already been noted that MS Paris, BNF, lat. 7373 was written in Tuscany and that the text it contains was known to Fibonacci in Pisa (p. 249 above). This manuscript, in fact, shares with the annotator of MS Harley 5402 the unusual spelling "pacta" for "epact."

It would be expected that these forms rather than the Western forms were used in Pisa, since, in the mid-twelfth century the orientation of Pisa was towards the cultural centres of the Eastern Mediterranean, both the Greek/Arabic centre of Antioch (where the Pisans had a quarter, and where Stephen the Philosopher made his translations), and the Greek centre of Constantinople. It is quite plausible that the Pisan tables themselves came directly from an Eastern source, possibly through the agency of Stephen; see [Burnett 2000b: 14–15].

34 In this manuscript the scribe appears to be unfamiliar with the numerals and writes them as the letters that they most nearly resemble; see [Burnett 2000b: 64–65].

## The Greek Connection

A Latin scholar working in the Crusader States could easily have had access to Arabic manuscripts with Indian numerals written in the Eastern form. That these forms were actually adopted by Christian scholars is proved by the Greek evidence. For, already by the twelfth century, Greek mathematicians had begun to use the Eastern forms.[35] These are found in marginal notes to Euclid's *Elements* which, on palaeographical grounds, can be assigned to the twelfth century. Among these are those of MSS Oxford, Bodleian, Auct. F.6.23, d'Orville 301 (see Plates 20 and 21), and Paris, BNF, gr. 2466. The forms found in these manuscripts are remarkably similar to each other, and are written with an ease that suggests that they were in normal use among certain Greek mathematicians. We find both the curved and the straight-backed 9, a 6 sometimes tending towards a zigzag, and a curved 8, like that in Ibn Ezra's Latin works. Most striking is 4, which is an "s" shape as in the Pisan Tables, sometimes with an added terminal line, which is a vestige of the loop.[36] The main difference is 5, which is written as an oval tipped slightly forward, though this form is given a "belt" which squeezes it into a kind of figure-of-eight in an alternative form 5 written above a row of numerals in the d'Orville manuscript.[37] At least one of the Greek manuscripts with Indian numerals in its twelfth-century glosses was possibly known in a Latin context. For MS Bodleian d'Orville 301 has been claimed by John Murdoch to be the very manuscript that the translator of the Sicilian *Almagest* used as his principal text for his translation of Euclid's *Elements*, and from which he incorporated some glosses into his translation [Murdoch 1966: 260–263].[38]

Pisa, in particular, was an important centre of translation from Greek, thanks to its close connections with Constantinople, and the activity of some outstanding local scholars, such as Richard Burgundio, Leo Tuscus and his brother Hugo Etherianus. It is quite plausible that Pisan mathematicians were using the Eastern forms that they found in Greek manuscripts. But what we may rather be seeing in

35 This had been made clear, on palaeographical grounds, in [Wilson 1981], but seems to have been overlooked by historians of mathematics. I am very grateful to Dr. Wilson for sending me an offprint of this article, to which the following paragraph is much indebted. The other manuscripts with notes dating from the twelfth century (aside from those discussed here) are Paris, BNF, gr. 2344 and Vienna, ÖNB, phil. gr. 31. A different style of Eastern forms is used in one manuscript (Vat. gr. 211) of the late-thirteenth-century astronomical tables of Chioniades, translated from the Arabic tables known as the *Zīj al-ʿĀlaʾī* [Pingree 1985–86, Part 2: 11, 15, etc.]: these presumably have been reintroduced directly from the Arabic models, but are sufficiently close to those found in the twelfth-century Euclid glosses to be immediately recognizable. Western forms appear in Greek for the first time in an anonymous Greek algorism written in 1252 — the *Psēphēphoria* — and dependent on Leonardo of Pisa's *Liber abbaci*: see [Wilson 1981; Allard 1976, 1977].

36 This alternative form of 4 is given above the "s" form in the well-known row of Arabic numerals in D'Orville 301, fol. 32v (now thought to be considerably later than the main text of the ninth century, and probably post-twelfth century [Wilson 1981: 401]).

37 See Table III, Greek b, below.

38 However, [Busard 1987: 8–9] expresses reservations about this claim.

the mid-to-late twelfth century is a "common language" of numerical symbols shared by the mathematicians of Greek Byzantium, Latin Tuscany and the Arabic Middle East, and characteristic especially of those places where two or three of these cultures coexisted, such as Pisa, Constantinople, Antioch and Tripoli. Aside from the common use of the Eastern forms in the three language cultures, one can point to another feature shared across these language boundaries: namely, the use of a mixed system, in which lower values (in principle, values which do not exceed 360, the number of degrees of the ecliptic) are represented by letters of the alphabet, the higher values by Indian numerals. This is a feature common to Eastern Arabic astronomical tables and the *Liber Mamonis* of Stephen the Philosopher, and is found in the thirteenth-century Greek tables of Chioniades; see [Burnett 2000b: 65–66].

## Conclusions

The Tables of Pisa had a moderate diffusion in the twelfth century, and were adapted to the meridians of Angers, Winchester and London. The extant instructions (including those of Ibn Ezra) were written in these places rather than in Pisa itself, and the various local adaptations of the numerals in which the tables were written may explain the variation in their forms. The Berlin manuscript of the tables was copied in the Ile de France — perhaps in Paris itself —, MSS Fitzwilliam McClean 165 and Arundel 377 are in English hands, both including copies of Adelard of Bath's *De opere astrolapsus*.[39] The Eastern forms have been expertly copied in these manuscripts, but Arundel 377, and also Digby 40 (whose place of copying is unknown), provide a key to them in the same hand as the manuscripts, which suggests that they were not current forms in the places where they were copied, and needed interpretation. In Italian manuscripts the Eastern forms are used without an accompanying key. Similarly, when a scribe copies a work in a context in which the Eastern forms of the numerals are no longer current, he is likely to use other forms of numerals in other texts: e.g. the same scribe (probably English?) copied Raymond of Marseilles's *Liber iudiciorum* in Erfurt, Amplon. Q 365 and Abraham ibn Ezra's *Foundations* in Erfurt, Amplon. Q 381, but he uses Western forms in the former. The master-scribe of Digby 51 uses the Western forms when annotating the one text which uses these forms (e.g. the work of al-Battānī), and uses Eastern forms only when copying the text by Abraham ibn Ezra. When a scribe copies a variety of texts using the same numeral system it is reasonable to suppose that this is the system that is normal for him: e.g. in Arundel 268, the numbers are always written in their Eastern forms whether the scribe is copying the *Book of the Three Judges* of Arabic provenance or the *Aratea* of the Latin tradition. A similar copying of more than one work in the Eastern forms can be observed in Berlin, lat. fol. 307, Paris, BNF, lat. 7373, and Oxford, Bodleian, Selden supra 26, though

39 An Anglo-Norman hand also accompanies the Italian hand in MS Digby 51.

the last also includes a key. Of these manuscripts, Arundel 268, Paris, BNF, lat. 7373 and (possibly) Selden supra 26 are in Italian hands.

The evidence for the active use of the Eastern forms in Italy is strong. They were used in Pisa and Lucca in the mid-twelfth century. They seem also to have been adapted to local usage in Sicily at the same time, and were used in some glosses in a Greek manuscript which may have been in Italy. The places of writing of the Italian manuscripts of Hugo of Santalla's and Hermann of Carinthia's works, and of the Italian hand in Digby 51, are not known, but the works of these authors are associated with the Latin works of Ibn Ezra, who moved from Spain to Tuscany.[40] If the Eastern forms were used by scholars in Italy, then, *a fortiori*, one would have expected them to have been used by Latin scholars in Byzantium and in the Crusader States, but no manuscripts with Indian numerals have yet been identified as having these provenances. All one can say is that the evidence points to the spread of the numerals to Northern European centres from Pisa and Lucca, together with the Pisan tables, and that eventually they became unintelligible in these areas.[41]

But the manuscript evidence, on its own, may lead to an underestimation of the currency of the Eastern forms. The clearest examples of their current use in both Greek and Latin manuscripts are in glosses and annotations rather than in texts: i.e., in the writing of scholars rather than in the writing of scribes. Among these scholars are the annotators of Harley 5402, Arch. Seld. B 34 and Soest 24,[42] as well as the Greek manuscripts listed by Wilson. We can therefore assume that the Eastern forms are likely to have been more current among scholars, who found them convenient to use and who understood each other's mathematical jargon. Such annotations are not so likely to have survived.

The question to be asked is why did Italian scholars not continue to use the Eastern forms? Part of the answer may lie in the developments of the early thirteenth century, when the principal successors to the Toledan translators of the twelfth century moved to Northern Italy. Among these were the compilers of the most authoritative collection of Toledan mathematical translations, which is comprised in three surviving manuscripts copied by the same scribe in Northern Italy in the early thirteenth century: Paris, BNF, lat. 9335 and 15461 and Vatican City, BAV, Ross. lat. 579, and in another manuscript copied by a closely related hand in British Library, Harley 3631.[43] Paris, BNF, lat. 15461 includes the *Liber Alcho-*

40 For the possibility that the annotator of one of the Italian manuscripts of a translation by Hugo used a text of Ibn Ezra, see n. 24 above.

41 A curious relic of these numerals is in an illustration in a manuscript in the monastery of Heiligenkreuz (MS 226), where they adorn the book of a "clericus" — perhaps intending to convey the obscure symbols used by scholars (see Plate 22).

42 In all three of these manuscripts the Eastern forms appear only in annotations: for the annotations in Soest, in which the Western form of 5 is found among otherwise Eastern forms see the thorough study in [Becker 1995].

43 For these manuscripts see [Burnett 2001b]. [Avril 1980–84, II: 5] assigns the decoration of Paris, BNF, lat. 9335 to the Venice / Padua region. Forms very similar to those found in these manuscripts occur in the the notarial documents of Raniero da Perugia, which are a valuable testimony because they can be dated and located to Perugia between 1184 and 1206: see

*rismi* of "magister Iohannes," the most advanced and complete algorism of the twelfth century, and the numeral forms given in this text are found throughout the collection; they are similar to the "toletane f(igure)" of Clm 18927, and they were to become the standard forms of Medieval Europe.

One of these scholars was Michael Scot, who left Toledo for Bologna sometime between 1217 and 1220. In 1228, Leonard of Pisa (Fibonacci), working in Pisa, dedicated a revised version of his *Liber abbaci* to Michael. As we have already seen (p. 249 above) Fibonacci knew the Greek-Latin version of Euclid's *Elements*, whose earliest manuscript uses the Eastern forms. But Fibonacci chose to use the Western forms of the Arabic numerals in his mathematical works. He would have known these numerals from learning the art of calculation in Bougie (in Algeria) in his youth [Boncompagni 1857–62, I: 1].[44] But his adoption of these forms is also likely to have been due to his association with the successors of Gerard of Cremona,[45] and the eclipsing of other traditions of Arabic-Latin scientific learning by that stemming from Toledo.

## Appendix: Latin Manuscripts Containing the Eastern and Palermitan Forms of Indian Numerals

Included here are all the Latin manuscripts known to contain examples of the Eastern and Palermitan forms of Indian numerals. In each case the following information is provided (where possible): the manuscript number, the relevant folios, and their date and place of writing; the text in which the numerals occur, the place and date of the writing of the original text. Reference is given to the illustrations of the manuscripts and their numerals in the Plates and Table included in this article. Unless the manuscript itself provides a date of copying, the dates of manuscripts can only be approximate. The variations in numeral forms described

[Bartoli Langeli 2000] and Table I, entry o. Raniero inserted the numerals at the beginning of documents, indicating the number of lines that the document contained, perhaps as a kind of private code. The Greek-Latin translators of Northern Italy in the twelfth century, James of Venice and Burgundio of Pisa, were also notaries and legal scholars.

44 One variant of his numerals, taken from the fourteenth-century MS Florence, Biblioteca nazionale centrale, fondo Magliabechiano, conv. sopp. Scaffale C. Palchetto 1, no. 2616 by [Boncompagni 1852: 103], looks remarkably similar to Western *Arabic* versions of the Indian numerals, and could plausibly have been picked up by Fibonacci during his studies in Bougie. But the other variant recorded by Boncompagni, and attributed to Fibonacci in [Allard 1976] is the "standard" Western forms also found in the copies of the Toledan works.

45 The close connection of Fibonacci with the successors of Gerard of Cremona is suggested by his use of Gerard's translation of the *Algebra* of al-Khwārizmī (see [Miura 1981: 59–60]) and his reference to a work well-known to Gerard's *socii* ("students"): the *Liber de proportione et proportionalitate* of Aḥmad ibn Yūsuf ibn al-Dāya ("Ametus filius Iosephi"), which Fibonacci refers to in a way which can only be interpreted as a truncated form of the *Latin* name of the author and his work: "Ametus filius ... in libro quem de proportionibus composuit" [Boncompagni 1857–62, I: 119]; see also [Rashed 1994: 148, 160].

in this article can, themselves, help in determining the date and provenance of a manuscript.

1) Berlin, Staatsbibliothek zu Berlin-Preußischer Kulturbesitz, lat. fol. 307 (Rose 956), s. XII$^{ex}$, Ile de France (cf. fol. 1r: "nos sumus Parisius existentes"). The normal usage of the scribe is the Eastern forms which appear on fol. 1r, as the chapter numbers in his copy of al-Farghānī's *Rudimenta* (fols. 19r–21r), and, most conspicuously, in the Tables of Pisa and their instructions (giving the meridian of Angers) on fols. 27, 30, 28, 31–34 (ancient fols. 103–108). Only in the copy of John of Seville's translation of al-Qabīṣī's *Introduction to Astrology* do the Western forms appear in one table (fol. 29v); in another table, however, Eastern and Western forms are given on alternate rows (fol. 35r), and a third table (also on fol. 35r) uses Eastern forms only. [Rose 1893–1919: 1177–1185; Leonardi 1960: 9–10; Folkerts 1981: 65]. I am very grateful to Bernd Michael for information on this manuscript. See Plate 18 and Table IIIc.
2) Berlin, Staatsbibliothek zu Berlin-Preußischer Kulturbesitz, Hamilton 557, s. XIII (the year 1202 is mentioned on a horoscope on fol. 8r; the year 1221 on another on fol. 15r), Italy, fols. 1–15v. A mixed version of Pseudo-Ptolemy, *Centiloquium*, combining the translations of Plato of Tivoli and Hermann of Carinthia (?). The Eastern forms of the numerals are used to number the 100 *verba* in the margin, and in a table of the planetary terms (fol. 14r). The same scribe, in the text, retains some of these forms (3, 4, and 5) but replaces others with Western forms (2, 6, 7 and 8); see fols. 8r and 14r; in the other major text in the manuscript (Albumasar, *Flores*) he uses Western forms entirely. Cited wrongly as Berlin, Hamilton 16 in [Lemay 1995–96, IV: 141], but correctly in [Lemay 2000: 382–383; Boese 1966: 273–274]. Table IIIn.
3) Cambrai, Médiathèque municipale 930 (829). Stephen the Philosopher, *Liber Mamonis*. The Eastern forms are used for high numbers (on fols. 27v–28r) alongside Roman numerals and Latin alphanumerical notation. [Burnett 2000b]; for illustrations see [Lemay 2000: 391–392; Burnett 2000b: 73]. Table IIIs.
4) Cambridge, Fitzwilliam Museum, McClean 165, s. XII, fols. 48v–49r contain a portion of Abraham ibn Ezra's *Book on the Foundations of the Astronomical Tables* ("Nunc artem de fructu Almagesti sumptam communem trademus … sed scriptore errasse" = [Millás Vallicrosa 1947: 145–147]). The Western forms are used in another introduction to astronomical tables associated with Pisa [Mercier 1991, Text II] on fols. 67r–80v. Table IIId.
5) Cambridge, Gonville and Caius College, 456/394, s. XII$^2$, in St Augustine's Monastery, Canterbury, by the fourteenth century. The Eastern forms occur in tables (fol. 73r) within Hugo of Santalla's translation of Ibn al-Muthannā's commentary on the tables of al-Khwārizmī. [Burnett & Pingree 1997: 9]. Plate 9 and Table IIIm.
6) Dresden, Sächsische Landesbibliothek, C 80, fols. 156v–157r, s. XV. Although texts on the algorism occupy several folios of this manuscript, the Eastern

forms first occur within a chapter headed "De abaci Sarracenici multi(plica)-t(i)on(e)" beginning: "De aliis abacis illud breviter dico, quod quicumque ⟨...?⟩ Sunt autem ibi caracteres alii quam in latino, quos duobus ⟨...?⟩ non enim illos dicunt (?) indeos (?) quos rectius sic indeus (?) " (there follows a row of Eastern forms and a row of Western forms) "ternarius sic etiam fit ʈ. octonarius Ᵽ." The examples of calculation are entirely in the Eastern forms, and the excerpt ends on fol. 157r with the triangle of multiplication accompanying the text of the *Liber Ysagogarum* [Allard 1992: 27]. The manuscript is exceedingly difficult to read, but special photographs are being made for Menso Folkerts who is preparing an edition. Table IIIx.

7) Erfurt, Wissenschaftliche Bibliothek der Stadt, Amploniana Q 381, s. XII$^{ex}$, fols. 1–34 (a separate codex, probably from a larger MS). Abraham ibn Ezra, *Book on the Foundations of the Astronomical Tables*. [Schum 1887: 638–639; 1882 no. 13] and Table IIIh.
8) Heiligenkreuz, Stiftsbibliothek 226, fol. 129r. Burgundy or Lorraine (?), s. XII$^{ex}$.[46] Five numerals in their Eastern forms have been added to the open book being read by the "clericus" on the frontispiece of Hugh of Fouilloy's *Aviarium*. No other manuscripts of the *Aviarium* has these numerals in the book. [Clark 1992: 283 and Fig. 1a]. Plate 22 and Table IIIu.
9) London, British Library, Arundel 268, fols. 75r–103v, s. XII$^{ex}$, a paper manuscript written in Southern Italy (apparently the earliest paper manuscript of a Latin Classical text). The beginning of the *Book of the Three Judges* (fols. 81rb–84rb), an edition of Germanicus's *Aratea* with scholia (85rb–92v and 96r–103v) and a cento from Virgil's *Aeneid* (fols. 93r–95v). [Lemay 2000: 387–390; Burnett 1977: 73, n. 39 (here the hand is erroneously said to be English) and 78–97; Reeve 1980: 511–515 (gives a s. XIII date)]. Plate 12 and Table IIIp.
10) London, British Library, Arundel 377, fol. 36r, s. XII$^{ex}$, Ely. A row of Eastern forms underneath a row of Western forms and some alternative forms, opposite the end of Abraham ibn Ezra's *Introduction to the Pisan Tables* [Mercier 1991: Text III]. Plate 17 and Table IIIe.
11) London, British Library, Cotton Vespasian A II, fols. 27–40 (a separate codex, incompletely copied), s. xii. An acephalous copy of Abraham ibn Ezra, *Book on the Foundations of the Astronomical Tables*; Rudolph of Bruges, *On the Astrolabe*, and Abraham ibn Ezra, *On the Astrolabe*. The two texts by Abraham ibn Ezra use the Eastern forms throughout; Rudolph's text uses only Roman numerals. Plate 15 and Table IIIg.
12) London, British Library, Harley 5402, fols. 60r–v, 1160 A.D., Lucca. Some notes on how to find the position of the Moon in 1160, followed on fol. 80v with instructions from tables made "super civitas (*sic*) luce" in the same hand. Aside from Roman numerals, the notes include the Indian numerals 0 and 6, the last in its Eastern form. [Burnett 2000a] and Table IIIr.

46 I am grateful to Baudouin van den Abeele for drawing my attention to this manuscript.

13) London, British Library, Harley 5786, ca. 1153, Palermo. A Psalter in Greek, Latin and Arabic, in which the Arabic psalms are numbered in Indian numerals which share characteristics of both Eastern and Western forms. The Arabic numerals are written adjacent to the Arabic text, but in the margin (as in Paris, BNF, lat. 7373), at first at the same time as the text, then (from fol. 86r) after the writing of the main text, but probably at the same time as the addition in the margin (in Arabic) of the times for the recitation of the psalms. Plate 5 and Table IIb.
14) Madrid, Biblioteca nacional, 10009, between 1267 and 1286, Italy? (the MS includes the notes of Alvaro of Toledo, who was in Italy with his patron, Gonzalez Gudiel, archbishop of Toledo), fols. 23va–38va. ʻAlī ibn Aḥmad al-ʻImrānī's *Elections* "interpreted" by Abraham bar Ḥiyya. Eastern forms are used throughout this text, except in the date in the colophon (here the scribe writes "1124" where the other manuscripts of the work give "1134," suggesting that the scribe had misinterpreted a numeral, probably because it was written in a form that he was not used to). 2 and 3 face right and left respectively, presumably to avoid confusion between 2 and 6, as in Arundel 268. The same scribe has copied other texts in the manuscript, including Hugo of Santalla's version of the *Centiloquium*, using Western forms. [Millás Vallicrosa 1942: 171–172]. Plate 23 and Table IIIv.
15) Munich, Bayerische Staatsbibliothek, Clm 18927, s. XII$^{2}$, Southern Germany, fol. 1r. Three rows of numerals of which the first is labelled "toletane f⟨igure⟩," the second (the Eastern forms) "indice f⟨igure⟩" and the third is unnamed (a species of the Western forms). The *Liber Ysagogarum* and Toledan Tables in this manuscript are written in the third form. [Lemay 1977, Fig. 1a]. Plate 3 and Table Ic, Im and IIIa.
16) Oxford, Bodleian Library, Arch. Seld. B 34, 13r–63v, s. XIII$^{in}$, written in the same Italian hand as Digby 50. The Eastern forms occur in the tables on fols. 32v and 33r, within Hugo of Santalla's translation of Ibn al-Muthannā's commentary on the tables of al-Khwārizmī (fols. 11–62v). [Millás Vendrell 1963]. Plate 10 and Table IIIl.
17) Oxford, Bodleian Library, Digby 40, fols. 52–88 (a separate codex), s. XII$^{2}$, English? Abraham ibn Ezra, *Book on the Foundations of the Astronomical Tables*. Written throughout with the Eastern forms. A later hand has added Western forms above the Eastern forms, but makes mistakes, e.g. in interpreting the Eastern 8 as a 7. A key to the numerals is given on fol. 88v. [Birkenmajer 1950; Millás Vallicrosa 1947: 32 (illustration), 69]. Plate 16 and Table IIIi.
18) Oxford, Bodleian Library, Digby 50, s. XIII$^{in}$, written in the same Italian hand as Arch. Seld. B 34. Eastern forms used only within the supplement to Hugo of Santalla's *Ars geomancie*, on fols. 94r–101r; from 100v onwards they are progressively replaced by Roman numerals which eventually are used exclusively. Plate 11 and Table IIIo.
19) Oxford, Bodleian Library, Digby 51, s. XII$^{2}$, fols. 38v–42v. The whole manuscript appears to be written on Italian parchment; these folios are written by

the Italian hand which is that of the master-scribe. Abraham ibn Ezra, *Arithmetic and Geometry*. Although the hand of a single master-scribe is found throughout the manuscript, the Eastern forms are used only in this text. Elsewhere are examples of the abacus numerals (fol. 36v) and the Western forms (used by an Anglo-Norman hand and the Italian annotator on fols. 5–9, 12–15, 17). The scribe becomes more confident in using the Eastern forms as he progresses, and starts off with more complex forms: e.g. 5 is a double bow, developing into a circle with a horizontal line through it; 4 is like a "z" with the lower line curved round, but on fol. 38v there is still a vestige of a detached loop to the lower curve; 6 begins like the Arabic 6, with the right-hand line sometimes sharply curved, at other times straight, making it indistinguishable from a tironian *et*; it later becomes a zigzag shape. [Hunt & Watson 1999: 25–26; Lévy 2001]. Plate 14 and Table IIIj.

20) Oxford, Bodleian Library, Selden supra 26, fols. 96–100 (I), 106–121 (II) and 122–129 (III) (original a single codex, written in one hand, with I apparently following II and III and breaking off incomplete), s. XII$^{ex}$–s. XIII$^{in}$, of unknown provenance; brought to St Augustine's Canterbury by William de Clara in 1277, probably immediately from Paris. Fols. 96–121v, *Algorismus* (a hybrid of the *Liber pulveris* and the *Liber Alchorismi*); fols. 122r–129v, Instructions for the use of the Pisan tables ([Mercier 1991, Text II] = Cambridge, Fitzwilliam McClean 165, fols. 68v ff.). The Eastern forms are used throughout, alongside Roman numerals. On fol. 96r a later hand has written the Western forms above the Eastern forms. A key giving the Western forms above the Eastern forms appears on fol. 106r. Bruce Barker-Benfield (letter to Allard); [Allard 1997: xxxviii–xxxix]. Plate 19 and Table IIIf.

21) Paris, Bibliothèque nationale de France, lat. 7373, 1181 A.D., Tuscany. Euclid's *Elements* (the translation from Greek made by the scholar who translated the "Sicilian *Almagest*"; see Vatican, Biblioteca apostolica Vaticana, Pal. lat. 1371 below). The original numbering of the propositions in the first four books is in the Eastern forms written on the extreme edges of the pages. In Books 5–9 Roman numerals take the place of the Eastern forms; thereafter no numerals are visible (in some cases they may have been cut off when the margins were trimmed). The numbering of the theorems is repeated in the Western forms for Books 1–5, and is continued in these forms in Book 6. Fol. 176r is the "figure of rhetoric" from the *Ars notoria*. Fols. 176v–178r consist entirely of computus tables for the church calendar written in the Eastern forms. The headings of the columns on fol. 176v are "luna Ianuarius; feria termini; pentecostes; rogatio⟨nis⟩; termini .xl.; termini .lxx." Those on fol. 177v are: "numerum aureo (*sic*); quantum habet luna in kalendis (?); in qua feria termini fiunt (?); claves terminorum; terminus pentecost.; terminus rogationis; terminus hebreorum; terminus quadrag.; terminus septuag.; pacta; circulus solis; pacta lune; circulus lune." The date "mclxxxi" (1181) is written in the margin of fol. 177r. [Murdoch 1966: 249–250; Avril 1980–84, I: 56, no. 95]; information from Julien Veronese. Plate 13 and Table IIIq.

22) Paris, Bibliothèque nationale de France, lat. 10252, s. xv, fol. 70r (within a miscellany of chapters on the algorism including some — e.g. on fol. 69r — that correspond with chapters in British Library, Egerton, 2261, fols. 226ra–b). The abacus names are followed by two rows of badly copied Indian numerals. The second row (after "vel aliter") was originally, in all likelihood, the Eastern forms, some of which appear to have been rotated: 5 and 9 are upside down, 4 is turned through 90 degrees. This work is immediately followed by Raymond of Marseilles's *Liber iudiciorum*, which has evidently been copied from an early manuscript. [Poulle 1963: 33]. Table IIIw.
23) Soest, Stadtarchiv, 24, s. xii, fol. 32v, between 15 October 1185 and 5 April 1186, North French? A note on how to convert between *anni domini*, the era of Nebuchadnezzar and the *hijra*, using the present date (15 October 1185) as an example, added to a copy of Firmicus Maternus, *Mathesis*, books II–IV. [Becker 1995]. Table IIIt.
24) Vatican City, Biblioteca apostolica Vaticana, Pal. lat. 1371, fols. 41r–97v, s. xii, Italy. Ptolemy's *Almagest*, translated from a copy brought to Palermo before 1160. Roman numerals are used in the text and in the first tables (fols. 46r–47r, 48v). The distinctive "Palermitan" forms are used in the remaining tables (fols. 53v–54, 57v–60v, 66v, 69r–70r, 81v, 86v, 89r–90r) with the exception that, on fol. 63r, the Western forms are used (a later addition? note that some tables are left unfilled: fol. 70r, 75v, 95v). [Lemay 1987: 468–470]. Plate 6 and Table II.
25) Vatican City, Biblioteca apostolica Vaticana, Pal. lat. 1393, s. xiii$^{mid}$, fols. 1–60. *Liber Alchorismi / Liber pulveris* (the same hybrid version as in Selden supra 26), including a row of Eastern forms as an alternate to the Western forms which are otherwise used throughout the text. [Allard 1992: xl]. Table IIIb.
26) Venice, Biblioteca Marciana, Lat. Z. 344 (1878; Valentinelli, Cl. XI, 104), s. xiii$^{in}$, Italian. Fols. 1–30 contain a copy of "Iohannes Hispalensis," *Ysagoge* and *Liber quadripartitus* in which the Eastern forms are used. In the bottom margins of both fols 1r and 2r keys to these forms are given in the form of three rows of numerals: Roman, Western forms and Eastern forms respectively; over the first key the scribe has written "figure algorismi alterius quam utamus" (*sic*!). On the first folio a scribe has written Western forms over erasures; thereafter he occasionally writes the Western forms above the Eastern forms. Table IIIk.

The crossed-zero form of the Eastern "5" has survived in at least two manuscripts of al-Kindī's *De mutatione temporum*, ch. 4 ([Bos & Burnett 2000: 273]; the manuscripts are Paris, BNF, lat. 16204, p. 374, and *ibid.*, lat. 7316). There are also examples where a mistake in a number may have arisen from a misreading of an Eastern form: e.g. al-Kindī, *De mutatione temporum* ("180" read as "150"; [Bos & Burnett 2000: 274]); Hugo of Santalla, *Liber Aomaris* ("138" read as "130"; [Burnett 1977: 91]); and Hermann of Carinthia, *De essentiis*, 70vA ("120" read as "150"; [Burnett 1982: 166]).

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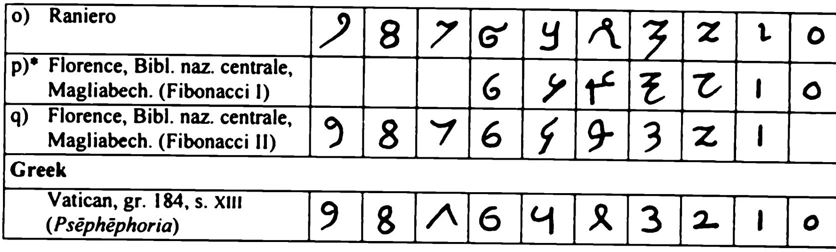
## Table

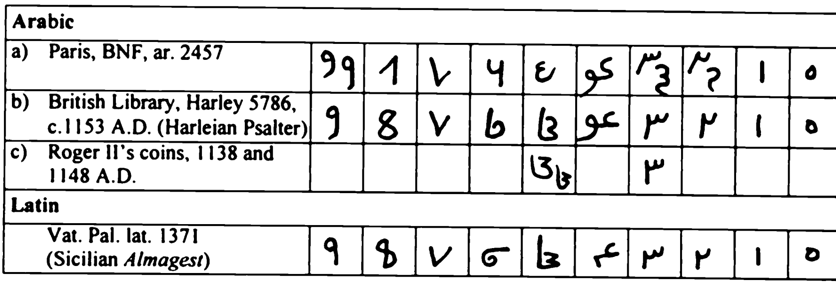
In the following table, numeral forms have been grouped together according to their similarities with each other. All manuscripts are twelfth or early thirteenth century, unless otherwise indicated. While only a representative selection of Western forms is given, a complete list of Eastern forms in Latin is provided. An asterisk (*) indicates that the forms in question are not the forms usually used by the scribe, or have been copied by him with obvious difficulty. A double asterisk (**) indicates that the whole of the relevant manuscript has been written in Eastern forms.

### I Western Forms

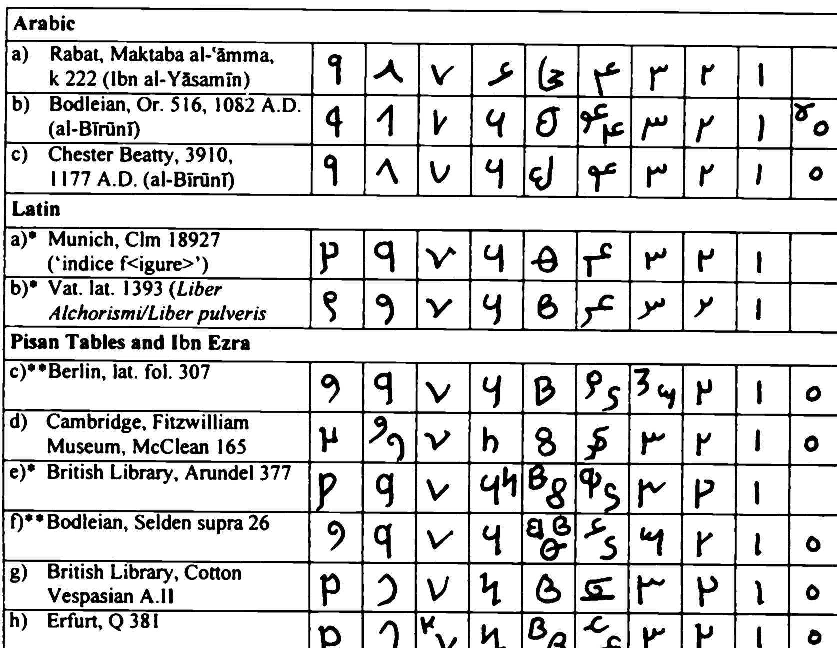
| | |
|---|---|
| **Arabic** | |
| a) Rabat, Maktaba al-'āmma, k 222 (Ibn al-Yāsamīn) | |
| b) Univ. of Tunis, 2043, 1611, (ash-Sharīshī) | |
| **Latin** | |
| a)* Escorial, d.I.2 (from Albelda, 976 A.D.) | |
| b) British Library, Harley 3595, s. XI$^{mid}$ (Abacus forms) | |
| c) Munich Clm 18927 (unnamed row) | |
| d) Cambridge, Trinity O.7.41 | |
| e) Paris, BNF, lat. 16208 (*Liber ysagogarum*) | |
| f) Vienna 275 (*Liber ysagogarum*) | |
| g) Bodleian, Digby 51 (al-Battānī) | |
| h) Hispanic Society of America, HC 397/726 (*Dixit Algorismi*) | |
| i) Erfurt, Q 351 (John of Seville) | |
| j) Erfurt, Q 365 (Raymond of Marseilles) | |
| k)* Paris, BNF, lat. 16204 (*On the Great Conjunctions*) | |
| l)* Paris, BNF, lat. 14738 (Gerard of Cremona) | |
| m)* Munich, Clm 18927 ('Toletane f<igure>') | |
| n) Paris, BNF, lat. 15461 (*Liber Alchorismi*) | |

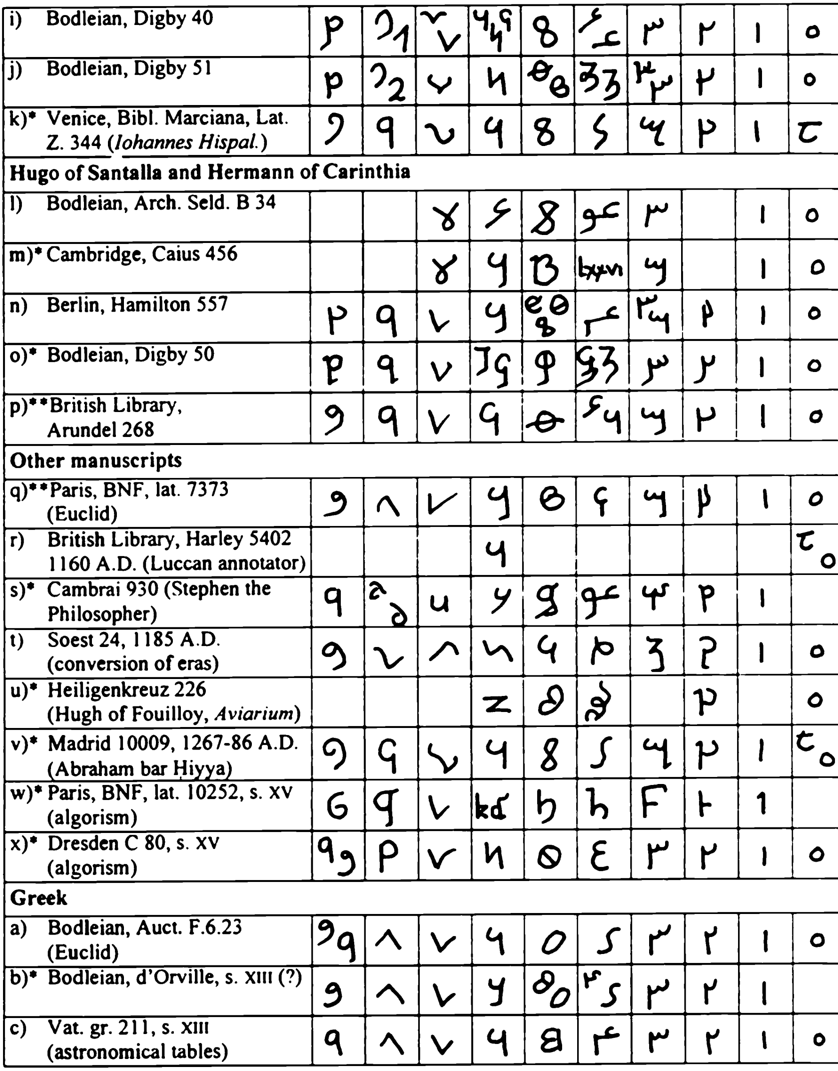



## II Palermitan Forms




## III Eastern Forms






## Sources and comments

### I Western Forms

*Arabic* **a:** see Plate 1; **b:** from [Ifrah 1981: 503].

*Latin* **a:** this "Codex Vigilanus" has been reproduced many times: e.g. [Ifrah 1981: 504]; **b:** [Folkerts 1970, Table I; Ifrah 1981: 506]; **c:** see Plate 3; **d:** from a table of numbers on fol. 62v; **e:** the numeral forms on fols. 67r–69v; **f:** the numeral forms used in a table from the *Liber ysagogarum* (Vienna 275, fol. 27r [Menninger 1958, II: 239]); **g:** forms used in the text and notes to al-Battānī's *Opus astronomicum* on fols. 5–9, 12–15 and 17; **h:** fol. 17r from [Folkerts 1997, Table 1]; **i:** fols. 103–130, John of Seville's translation of al-Farghānī's *Rudimenta*; **j:** a version of Raymond of Marseilles's *Iudicia* on fols. 40v–42r; **k:** see Plates 2 and 4; **l:** the Arabic numerals on the first folios of a copy of the *Almagest*; **m:** see Plate 3; **n:** see [Lemay 1977, Fig. 5]; **o:** for the forms used by the notary Raniero, see [Bartoli Langeli 2000]; **p:** the "Arabic-looking" numerals in the table in the *Liber abbaci* in MS Florence, Biblioteca nazionale centrale, fondo Magliabechiano, conv. sopp. Scaffale C. Palchetto 1, no. 2616, fol. 1r, reproduced in facsimile in [Boncompagni 1852: 103]; **q:** the standard numeral forms in the *Liber abbaci* in the same manuscript [Boncompagni 1852: 103].

*Greek* **a:** the *Psēphēphoria* of 1252 [Allard 1976].

II Palermitan Forms

*Arabic* **a:** a manuscript of the works of al-Sijzī, commonly thought to have been written in Shiraz between 969 and 972, but probably much later (illustration in [Kunitzsch 2002]), given here because of its "mixed" forms of 2s and 3s and similar appearance to the script of the Harleian Psalter; **b:** see Plate 5; **c:** see Plates 7 and 8.

*Latin* see Plate 6.

III Eastern Forms

*Arabic* **a:** see Plate 1; **b:** from [Irani 1955/56: 4]; **c:** from personal inspection.

*Latin* **a:** see Plate 3; **b:** from [Allard 1992: 69]; **c:** see Plate 18; **d:** from personal inspection; **e:** see Plate 17; **f:** see Plate 19; **g:** see Plate 15; **h:** from [Schum 1882, no. 13]; **i:** see Plate 16; **j:** see Plate 14; **k:** from personal inspection; **l:** see Plate 10; **m:** see Plate 9; **n:** from personal inspection; **o:** see Plate 11; **p:** see Plate 12; **q:** see Plate 13; **r:** see [Burnett 2000a: 86]; **s:** see [Lemay 2000: 391–392]; **t:** from [Becker 1995]; **u:** see Plate 22 (the identification of 2 and 9 is not certain); **v:** see Plate 23; **w:** from personal inspection; **x:** from personal inspection.

*Greek* **a:** see Plate 20; **b:** see Plate 21; **c:** these are the numerals in [Pingree 1985–86, Part 2: 11, 15, 17, 20–21, 35, 44].

## Plates

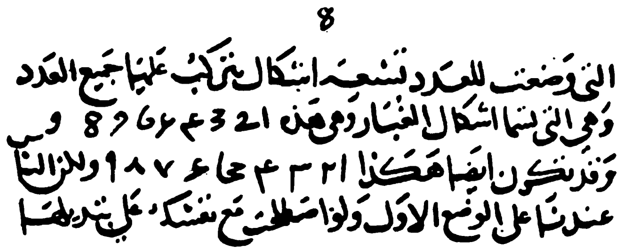


**Plate 1.** The Western and Eastern forms in the algorism of Ibn al-Yāsamīn; MS Rabat, Maktaba al-ʿāmma, k 222, fol. 5a.

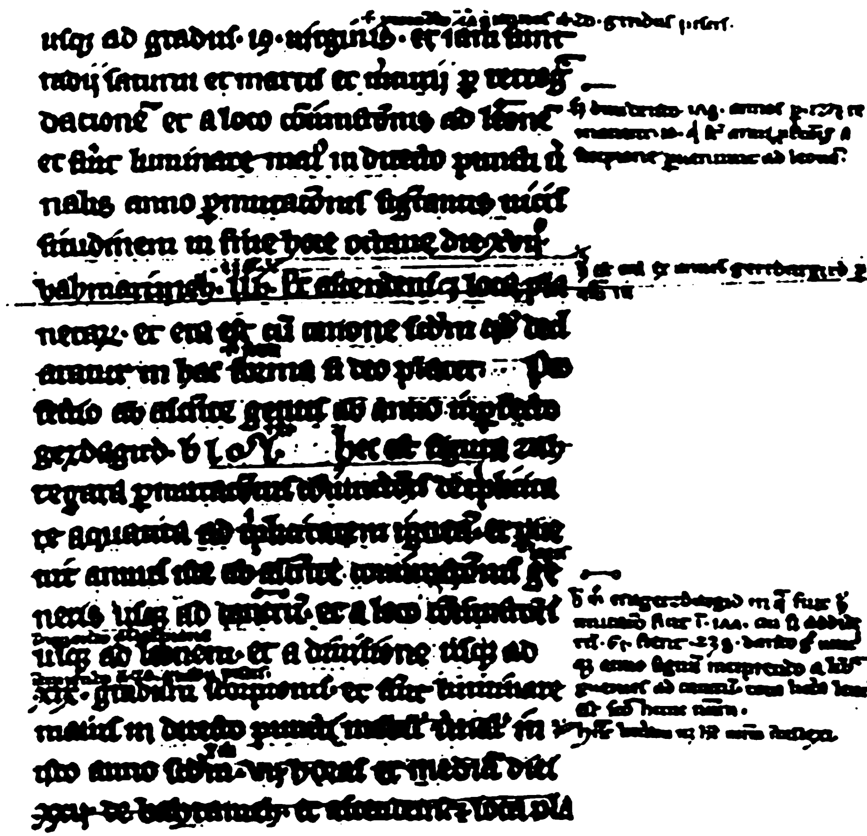


**Plate 2.** The "fossilised" forms of early numerals (116 in line 7; 141 in line 11) in the revised translation of Abū Maʿshar's *On the Great Conjunctions*, in MS Paris, Bibliothèque nationale de France, lat. 16204, p. 299.

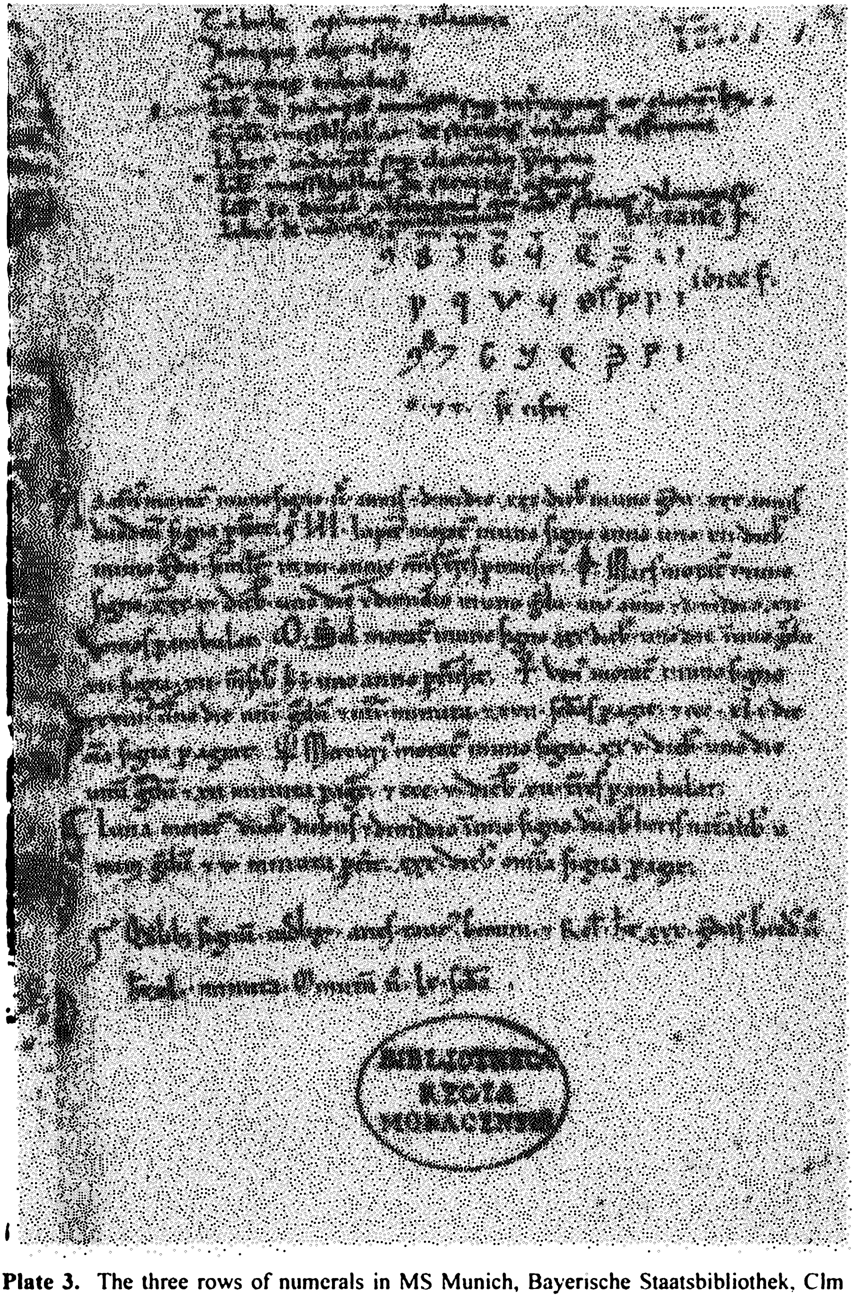


**Plate 3.** The three rows of numerals in MS Munich, Bayerische Staatsbibliothek, Clm 18927, fol. 1r.

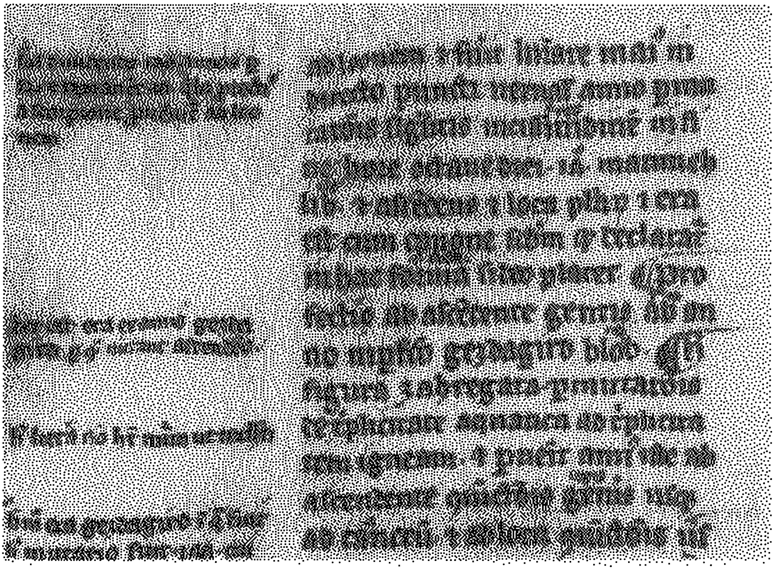


**Plate 4**. The "fossilised" forms of early numerals in MS Cambrai, Médiathèque municipale, 168 (163), fol. 98v (see lines 5 and 9).

**Plate 5.** The number "56" in the Harleian Psalter (note that Psalm 57 in Latin is equivalent to Psalm 55 in Greek and 56 in Arabic); MS British Library, Harley 5786, fol. 74r.

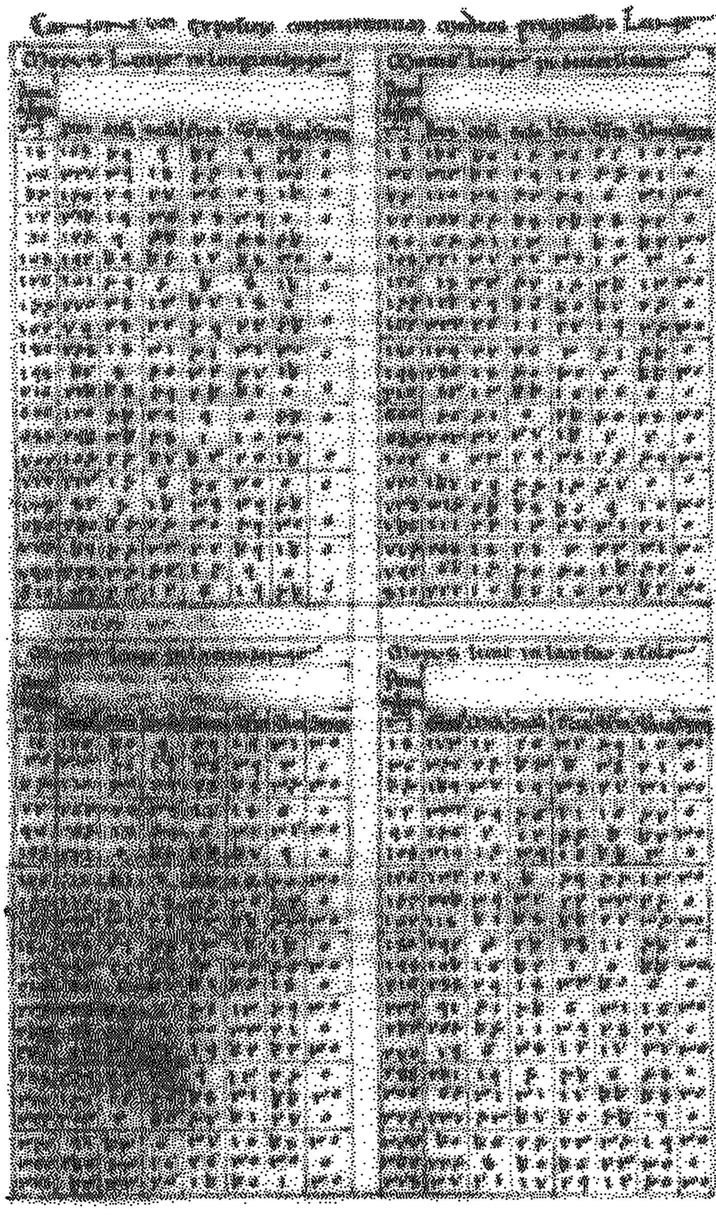


**Plate 6.** The numerals in the Sicilian translation of the *Almagest* in MS Vatican City, Biblioteca apostolica Vaticana, Pal. lat. 1371, fol. 69v.

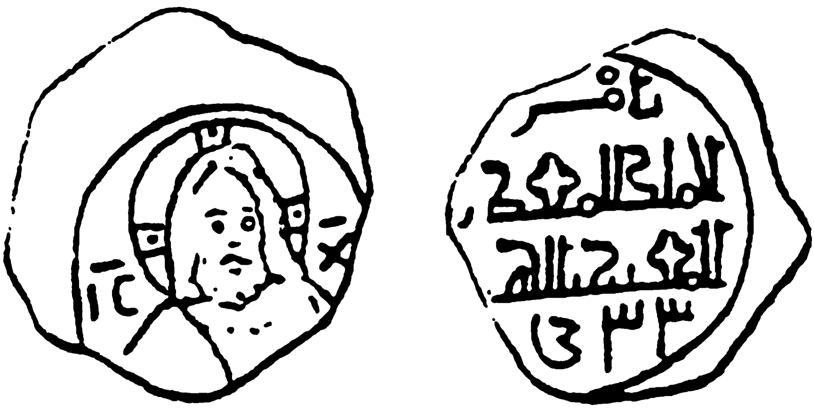


**Plate 7.** A "folles" of Roger II of 1138/39, on which the *hijra* date of 533 is given in Indian numerals; from [Travaini 1995, Pl. 12].

**Plate 8.** A "folles" of Roger II of 1148/49, on which the *hijra* date of 543 is given in Indian numerals; from [Travaini 1995, Table 15].

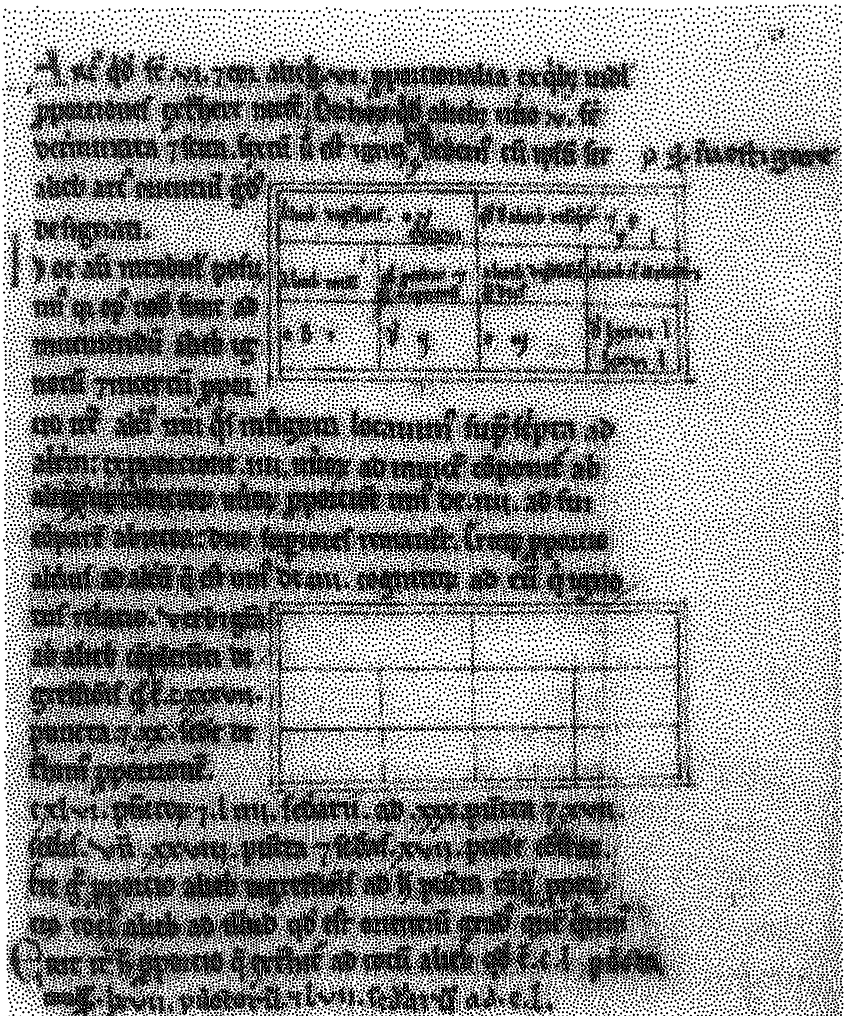


**Plate 9.** Hugo of Santalla's translation of Ibn al-Muthannā's commentary on al-Khwārizmī in MS Cambridge, Gonville and Caius 456/394, fol. 73r. Note the numbers 150, 67, 30 and 147 in the third row of the table.

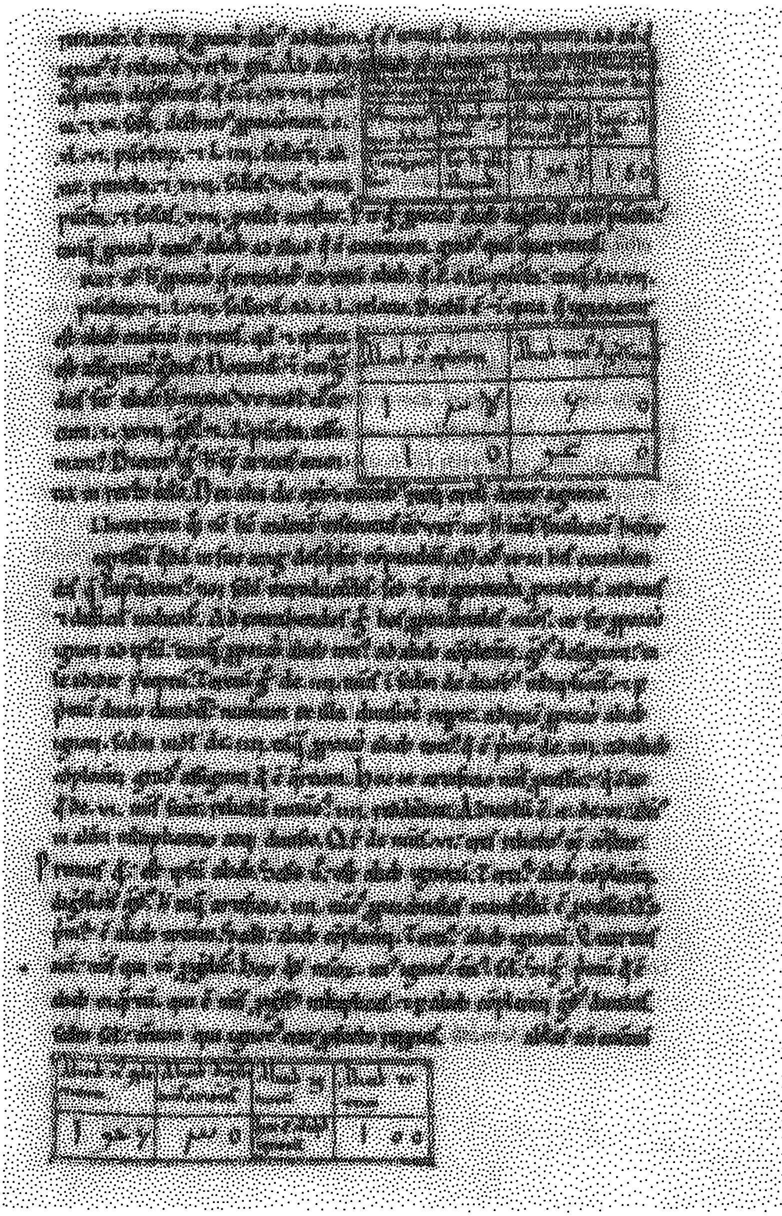


**Plate 10.** The same in MS Oxford, Bodleian Library, Arch. Seld. B 34, fol. 33r. Note the numbers 137, 60, 10 and 40 in the middle table, and 146, 30 and 100 in the lower table.

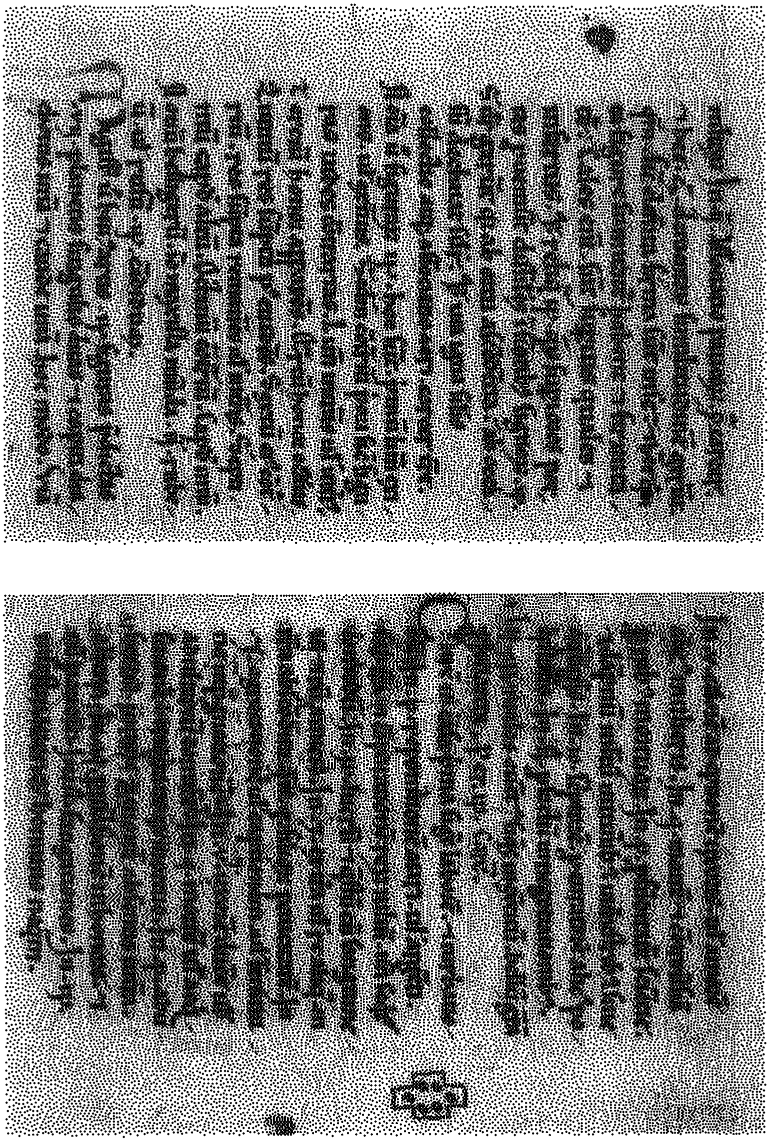


**Plate 11.** From the supplement to Hugo of Santalla's *Ars geomancie* in MS Oxford, Bodleian Library, Digby 50, fols. 94v (note the number 4 in the second line) and 100r (note the sequence of numbers from 1 to 12).

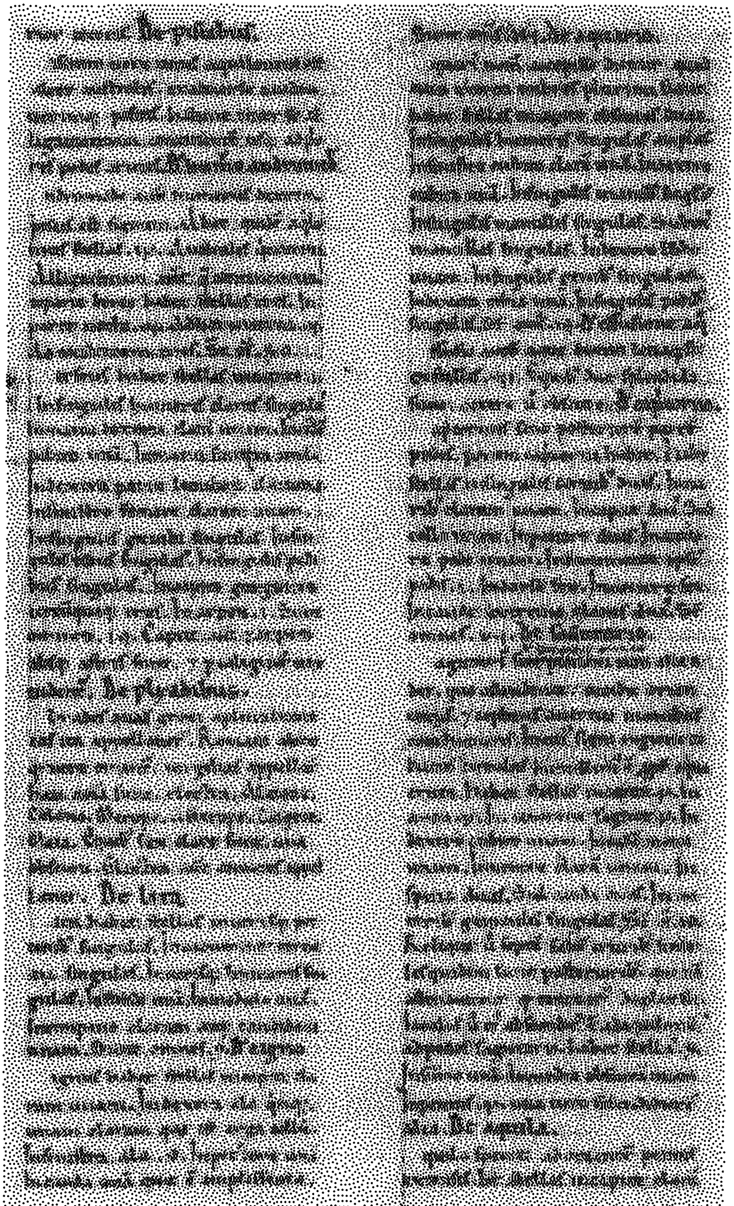


**Plate 12.** From the *Aratea* in British Library, Arundel 268, fol. 87r. Note the numbers 12, 3 and 40 within the description of Andromeda.

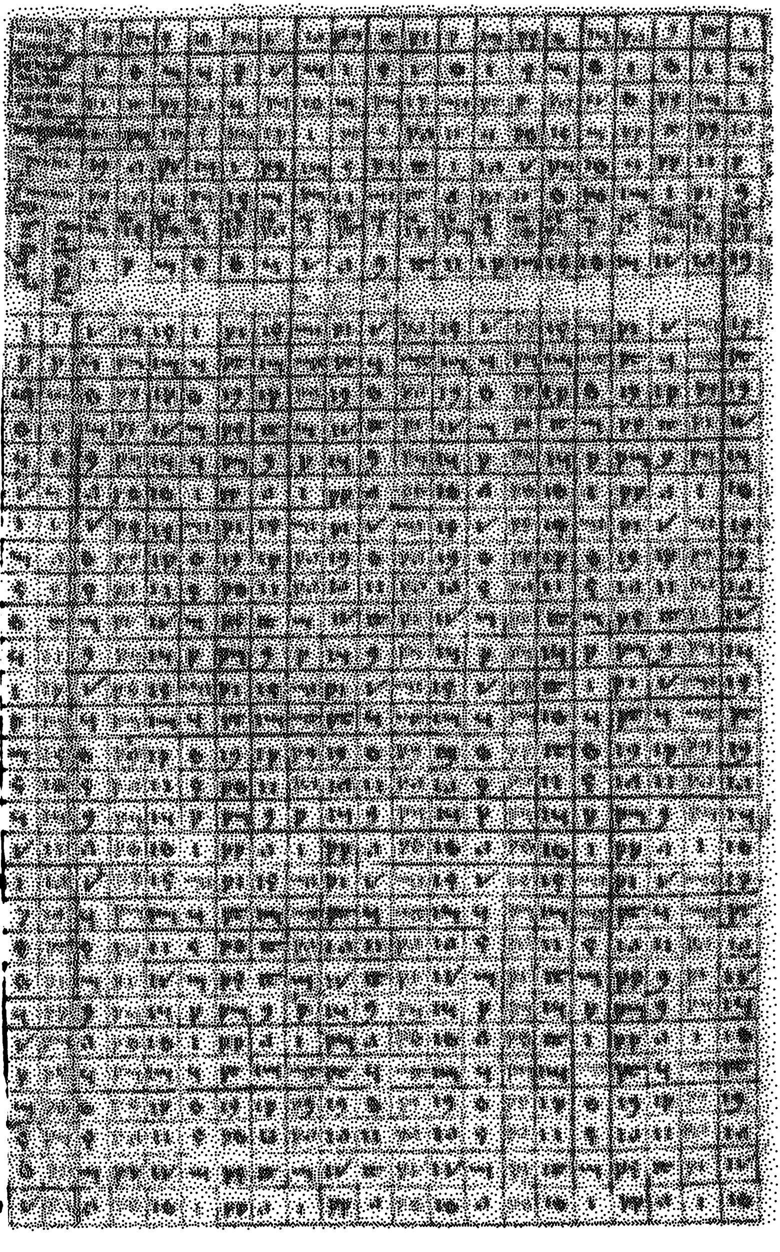


**Plate 13.** Part of the computus table in MS Paris, Bibliothèque nationale de France, lat. 7373, fol. 176v.

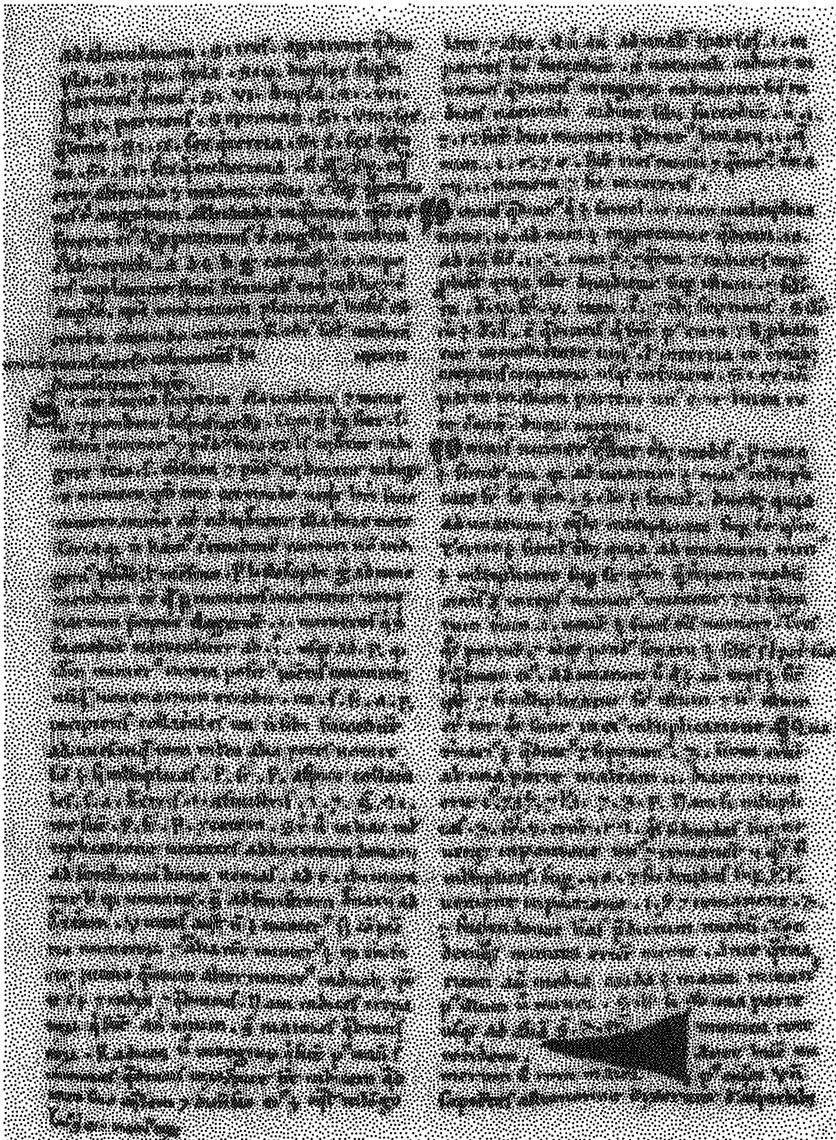


**Plate 14a.** Abraham ibn Ezra's *Arithmetic* and *Geometry* in MS Oxford, Bodleian Library, Digby 51, fol. 38v.

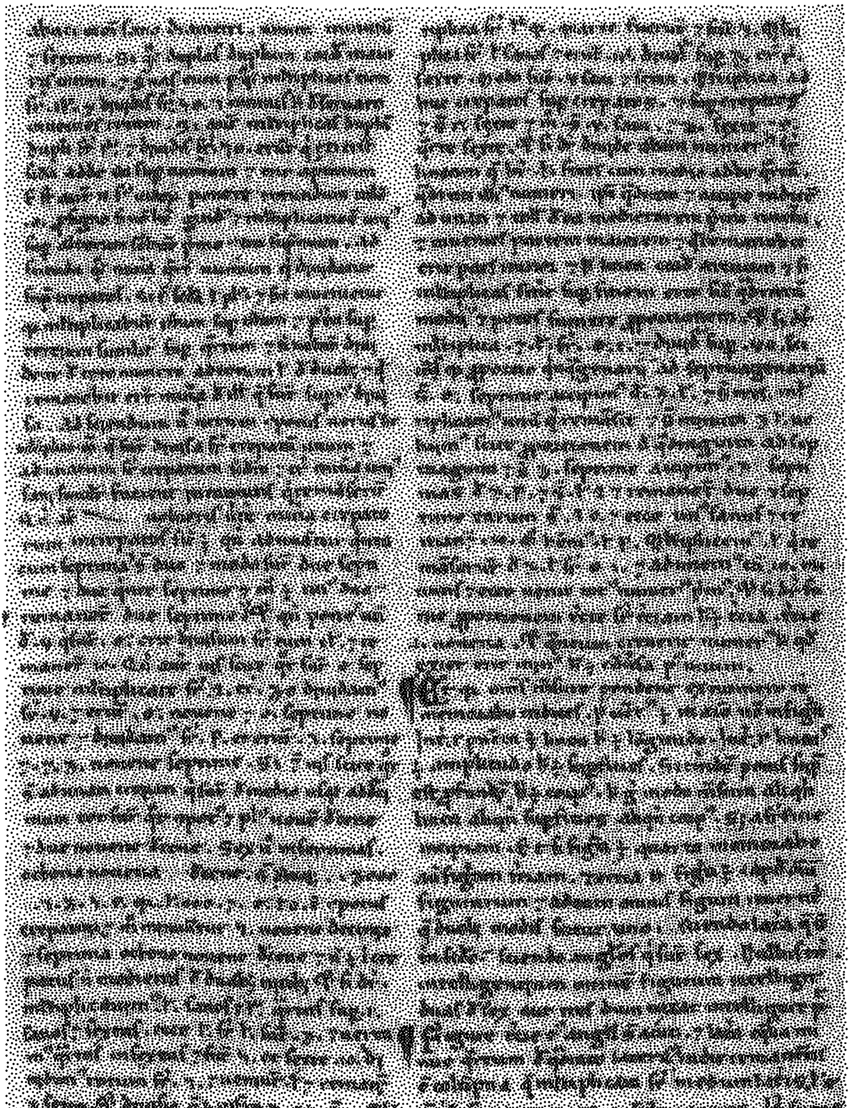


**Plate 14b.** Abraham ibn Ezra's *Arithmetic* and *Geometry* in MS Oxford, Bodleian Library, Digby 51, fol. 39v.

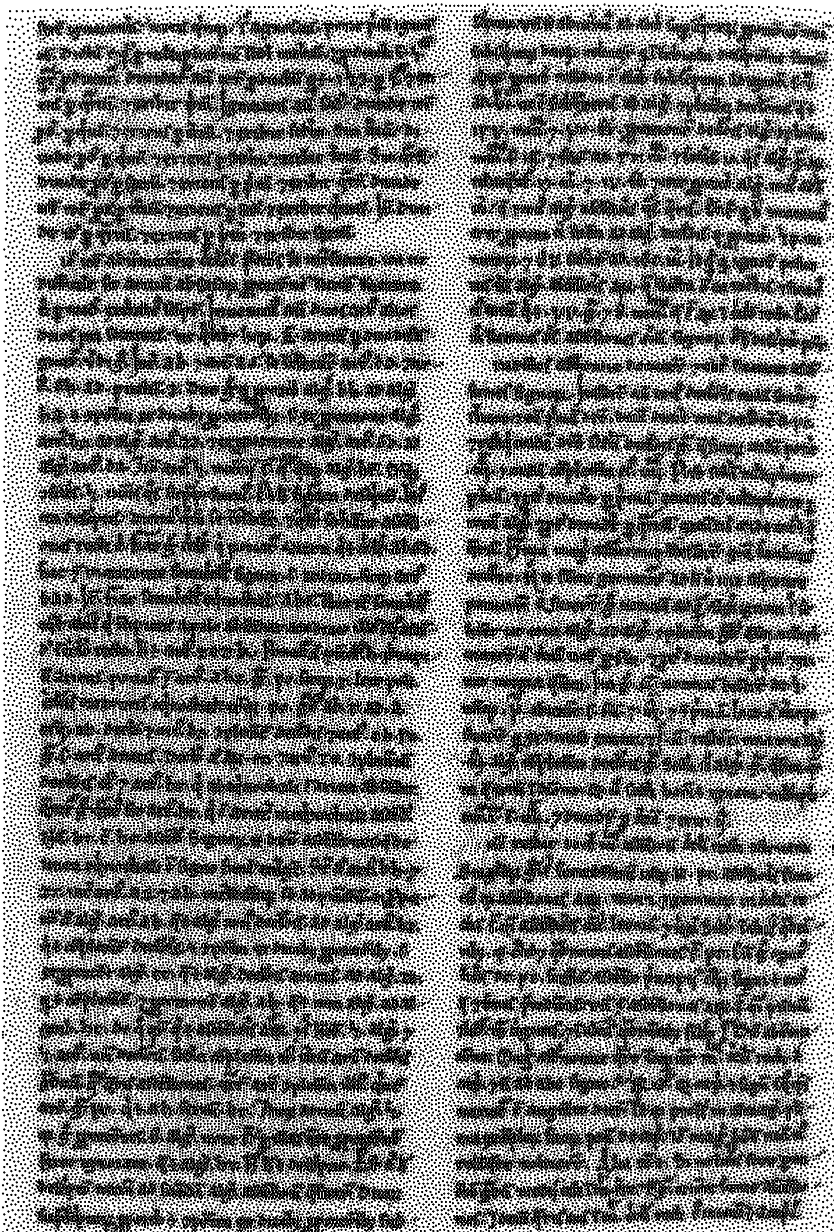


**Plate 15.** Abraham ibn Ezra's *Book on the Foundations of the Astronomical Tables* in MS British Library, Cotton, Vespasian A II, fol. 29v.

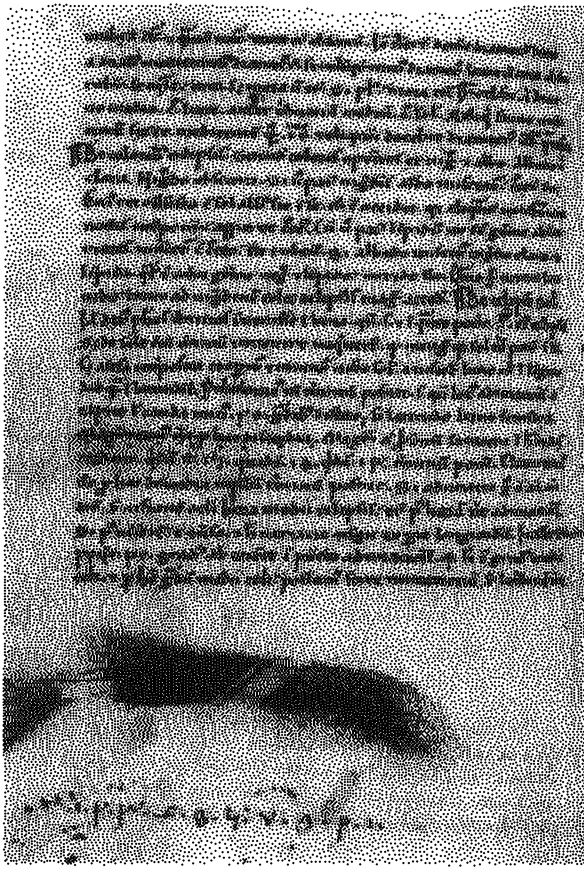


Plate 16. The same in MS Oxford, Bodleian Library, Digby 40, fol. 88v.

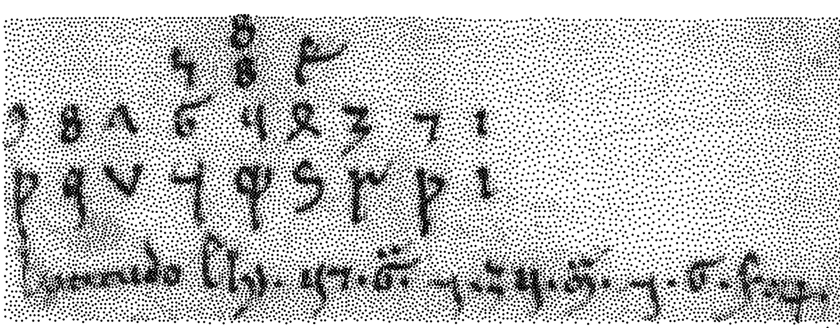


Plate 17. The varieties of numerals in MS British Library, Arundel 377, fol. 56r. Note that, in the lower row, the scribe has written two forms of 4 and omitted 5.

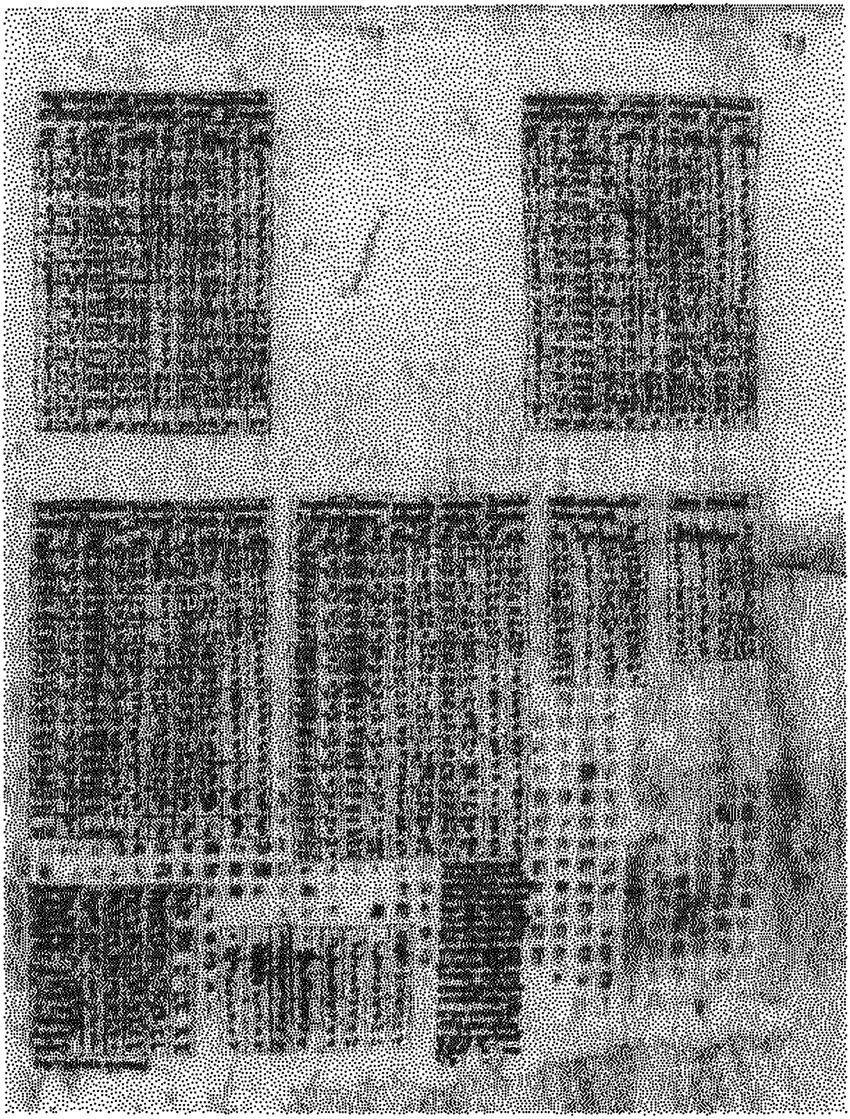


**Plate 18.** The tables of Pisa in MS Berlin, Staatsbibliothek Preußischer Kulturbesitz, lat. fol. 307, fol. 28r.

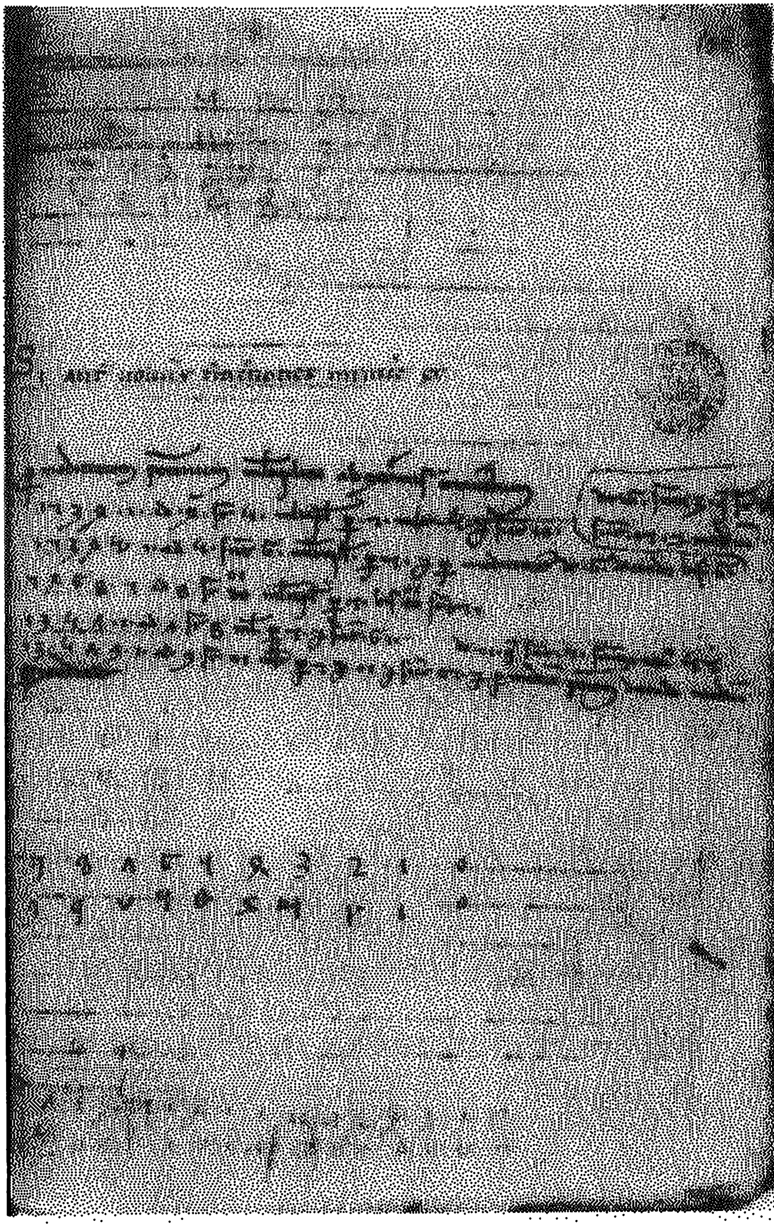


**Plate 19a.** The *Liber Alchorismi / Liber pulveris* in MS Oxford, Bodleian Library, Selden supra 26, the key on fol. 106r (with some later additions).

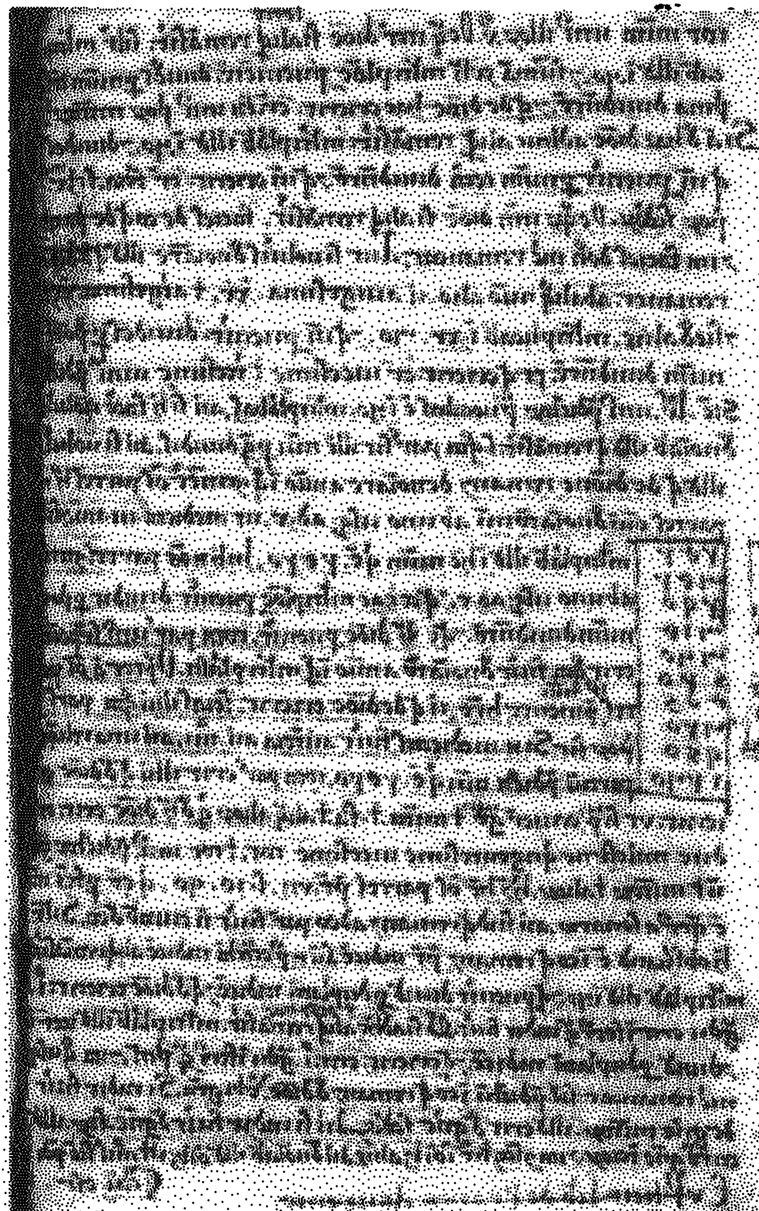


**Plate 19b.** The *Liber Alchorismi / Liber pulveris* in MS Oxford, Bodleian Library, Selden supra 26, fol. 121v.

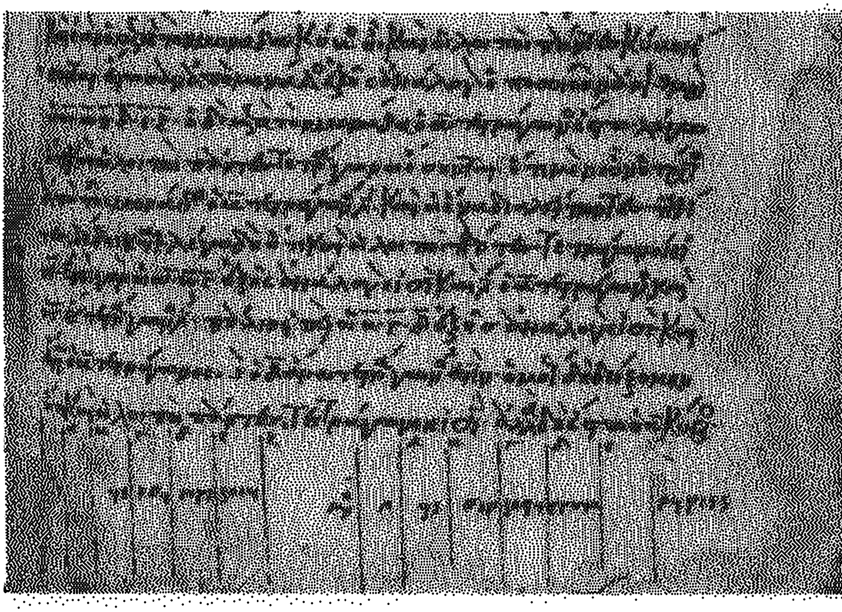


Plate 20. Indian numerals in annotations in MS Oxford, Bodleian Library, Auct. F.6.23, fol. 127r.

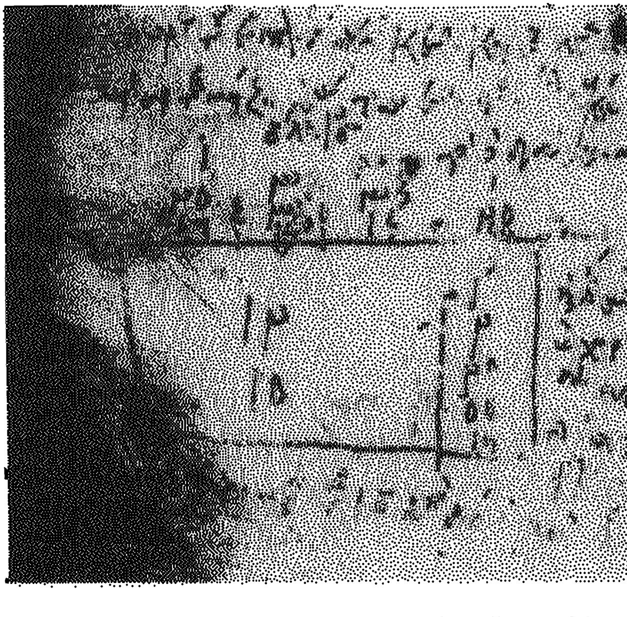


Plate 21. The same in MS Oxford, Bodleian Library, D'Orville 301, fol. 1v.

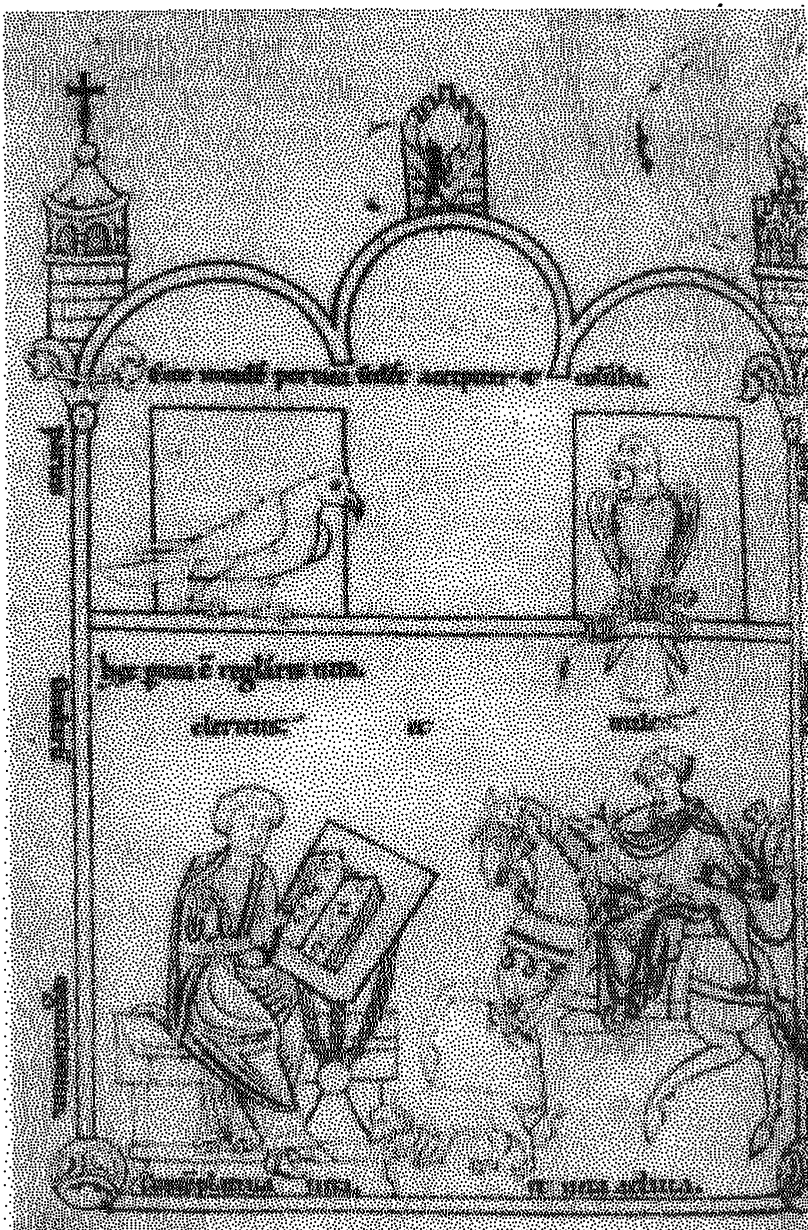


**Plate 22.** The frontispiece to Hugh of Fouilloy's *Aviarium* in MS Heiligenkreuz, Stiftsbibliothek 226, fol. 129r.

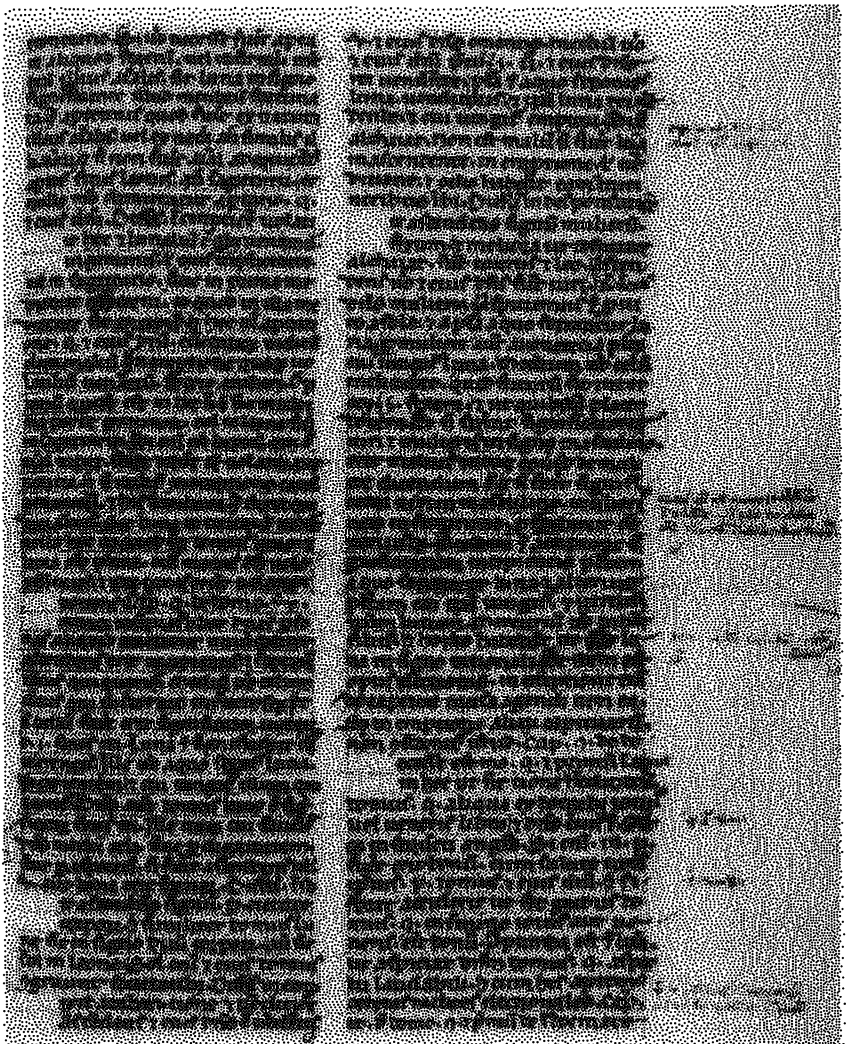


**Plate 23.** The Eastern forms in the chapter-headings of 'Alī ibn Aḥmad al-Imrānī's *Elections* in MS Madrid, Biblioteca nacional, 10009, fol. 37r. Note "Cap. 8 de domo 7" and "Cap. 9 de domo 8."

# Jealous Husbands Crossing the River: A Problem from Alcuin to Tartaglia

by RAFFAELLA FRANCI

The problem of the *jealous husbands crossing the river* involves three couples who must cross a river; the boat is capable of holding just two people and no wife can be in the presence of a man unless her husband is also present. This puzzle first appears in Alcuin's *Propositiones ad acuendos juvenes* (9th century), spreading afterwards through all Europe. This paper investigates its diffusion through an examination of 13th- and 14th-century collections of Latin problems and manuscripts and of printed abacus treatises from the 14th to the 16th centuries. Also noted is the first mention of a generalisation of the problem to more than three couples, in particular in two manuscripts of the beginning of the 16th century.

## Introduction

The classic problem of the *jealous husbands crossing the river* involves three couples who must cross a river using a boat that holds just two people. Because the husbands are so jealous, no wife can be in the presence of another man without her husband being present. This puzzle first appears in the *Propositiones ad acuendos juvenes*, a collection of mathematical problems attributed to Alcuin and written around the year 800.

Almost all ancient collections of mathematical problems contain recreational ones,[1] but none of the actually known Indian, Arabic, Chinese and Greek collections contains crossing puzzles. Thus although it seems highly probable they have an European origin, it is nevertheless likely they were already known before Alcuin. It is possible he had heard them at Charlemagne's court, where they could have had an oral tradition. This kind of puzzle is in fact very suitable for court entertainments.

In this paper I will investigate the spreading of the jealous husbands problem in Northern Europe and in Italy up to the 16th century, analysing numerous manuscripts so far unexplored.[2]

1 In concurrence with [Smith 1925], we define recreational problems or puzzles as those "questions that have no application to daily life, the chief purpose being to provide intellectual pleasure."

2 [Smith 1925; Tropfke 1980] provide a classification of mathematical recreational problems. In particular, Tropfke sketches a history of many of them including the crossing problems, and the analysis of new sources greatly adds to this reference.

## Alcuin's Version of the Problem

The *Propositiones ad acuendos juvenes* is the oldest collection of mathematical problems in Latin and the greatest part of them pertains to recreational mathematics.[3] It presents numerous new problems, the most famous of them being three types of river crossing puzzles. In various forms, they have been popular ever since.

The first of the three problems is that of the jealous husbands, the second is the popular problem of the wolf, goat and cabbage, the third concerns a man and a woman who each weigh a cartload, with two children who together weigh a cartload, who must cross a river with a boat that can only take one cartload. In the present paper I focus my attention only on the first of these problems. Its Latin version is as follows:

> *Propositio de tribus fratribus singulas habentibus sorores*
>
> *Tres fratres erant, qui singulas sorores habebant et fluvium transire debebant. Erat enim unicuique illorum concupiscentia in sorore proximi sui. Qui venientes ad fluvium non invenerunt nisi parvam naviculam, in qua non poterant amplius nisi duo ex illis transire. Dicat, qui potest, qualiter fluvium transierunt, ut ne una quidem earum ex ipsis maculata sit.*
>
> *Solutio*
>
> *Primum omnium ego et soror mea introissemus in navem et transfretassemus ultra, transfretatoque fluvio dimisissem sororem meam de navi et reduxissem navem ad ripam. Tunc vero introissent sorores duorum virorum, illorum videlicet, qui ad litus remanserant. Illis itaque feminis navi egressis soror mea, quae prima transierat, intraret ad me navemque reduceret. Illa egrediente foras duo in navem fratres intrassent ultraque venissent. Tunc unus ex illis una cum sorore sua navem ingressus ad nos transfretasset. Ego autem et ille, qui navigaverat, sorore mea remanente foris ultra venissemus. Nobisque ad litora vectis una ex illis duabus quaelibet mulieribus ultra navem reduceret, sororeque mea secum recepta pariter ad nos ultra venissent. Et ille, cuius soror ultra remanserat, navem ingressus eam secundum ultra reduceret. Et fieret expleta transvectio nullo maculante contagio* [Folkerts & Gericke 1993: 314].

The English translation is as follows:

> Three friends and their sisters.
>
> Three friends each with a sister needed to cross a river. Each one of them coveted the sister of another. At the river they found only a small boat, in which

3 The *Propositiones* has long been attributed to Alcuin though we have no definite proof. Folkerts [1993] has found 13 manuscripts of the text, the earliest being from the late ninth century. In [1978] Folkerts published the first critical edition of the *Propositiones*, which introduction we refer to for all the matters of attribution, extant manuscripts, description of contents, etc. [Folkerts & Gericke 1993] contains an improved edition together with a German translation, whereas [Hadley & Singmaster 1992] provides an English translation. [Singmaster 1998] investigates the history of some of Alcuin's problems.

only two of them could cross at a time. How did they cross the river without any of the women being defiled by the men?

Solution.

First of all my sister and I would get into the boat and travel across; then I'd send my sister out of the boat and I would cross the river again. Then the sisters who had stayed on the bank would get into the boat. These having reached the other bank and disembarked, my sister would get into the boat and bring it back to us. She having got out of the boat, the other two men would board and go across. Then one of them with his sister would cross back to us in the boat. Then the man who had just crossed and I would go over again, leaving our sisters behind. Having reached the other side, one of the two women would take the boat across, and having picked up my sister could take the boat across to us. Then he, whose sister remained on the other side, would cross in the boat and bring her back with him. And that would complete the crossing without anything untoward happening [Hadley & Singmaster 1992: 111].

The exact rules of behavior are not made explicit by Alcuin, but the traditional interpretation is that no man is willing to allow another man to be with his sister unless he is present.

Alcuin's solution in eleven one-way trips is minimal, though the crossing may be somewhat varied. In fact one could instead begin with the crossing of two women, one of which brings back the boat. Afterwards the solution works as in Alcuin. Another variation is possible after the ninth trip, when on the first bank there is only a woman. Here, in fact, a sister instead of a brother may go back.

## The Northern European Latin Tradition

The jealous husbands problem had a large diffusion in North Europe from the 13th to the 15th century. In fact, the collections of problems in Latin that include it are numerous. Perhaps the most ancient version actually known is that contained in *Annales Stadenses*, which date back to the middle of the 13th century. The problem is the sixth of a series that two young scholars, Tirri and Firri, propose one to the other:

*O Firri, ego ad rem similem te provocabo.*[4] *Tres viri cum uxoribus suis Renum transire volebant. Modica erat navicula, nec amplius, quam duos homines, capiebat. Sicque transeumdum erat, ut in transvectione quilibet vir uxorem suam custodiret ab altero. Vocabantur ita: Bertoldus et Berta, Gherardus et Greta, Rolandus et Rosa. Quomodo transibunt? Firri modicum dubitante, Tirri ait: Berta et Greta primo transierunt. Greta rediit, et Rosam attulit. Ecce tres mulieres transierunt. Berta ad Bertoldum rediit. Gerardus et Rolandus transierant ad suas uxores. Gerardus et Greta redierunt. Bertoldus et Gerardus transierunt. Rosa sola rediit, tribus viris in ulteriore littore manentibus. Berta et*

4 Before, in fact, Tirri had proposed the puzzle of the wolf, goat and cabbage.

*Greta transierunt ad suos viros. Rolandus ad Rosam rediit, et eam secum transvexit. Sic omnes bene transierunt.*

*Quo audito, Firri gavisus ait: Scis aliqua ad retinendum ea? Cui Tirri: Scio ecce duo versus:*

*Binae, sola, duae, mulier, duo, vir mulierque,*
*Bini, sola, duae, solus, vir cum muliere.*

*Ad haec Firri: Expone. Et Tirri: Binae, Berta et Greta; sola, Greta; duae, Greta et Rosa; mulier, Berta; duo, Gherardus et Rolandus; vir mulierque, Gherardus et Greta; bini, Bertoldus et Gherardus; sola, Rosa; duae, Berta et Greta; solus Rolandus; vir cum muliere, Rolandus et Rosa* [Lappenberg 1859: 334].[5]

It should be noted that here, as in all the other texts examined in this paper, the protagonists are changed from brothers and sisters into husbands and wives. This version is particularly interesting for it is the only one I know of in which the wives and the husbands are given names, an occurrence that makes the solution easier to understand. Another point of interest is the presence of two hexameters that summarize the solution. The same verses conclude the presentation of the problem in the manuscript Cod. lat. Monac. 14684 (sec. XIV) edited by M. Curtze.[6]

It is evident that a poem is easier to memorise, so it is not surprising if other authors have rendered the whole problem in verse. This is the case of the manuscript Cambridge Trinity College 1149 where at folio 32 we read:

1 *Tres uia forte uiros totidem uirum mulieres*
2 *Associat pariter modicum trans flumen ituros.*
3 *Ergo coniugii iuratur soluere nexus*
4 *Si foret absque suo coniunx inuenta marito.*
5 *Inveniunt cimbam sine remige fune retentam*
6 *Que tres non caperet: vix sufficit illa duobus.*

5 English translation: "Firri I will challenge you to a similar thing. Three husbands with their wives wished to cross the Rhine. There was a small boat, which held no more than two people. They had to cross in such a way that during the crossing each husband guarded his wife. Their names were Bertoldus and Berta, Gherardus and Greta, Rolandus and Rosa. How should they cross? To Firri, a little uncertain, Tirri said: at first Berta and Greta crossed, Greta went back and embarked Rosa. Thus the three wives crossed. Berta went back to Bertoldus: Gerardus and Rolandus crossed to their wives. Gerardus and Greta went back. Bertoldus and Gerardus crossed. Rosa went back alone, the three husbands stayed in the other bank. Berta and Greta crossed to their husbands. Rolandus went back to Rosa and crossed with her. So they all well crossed. Having heard this, Firri, satisfied, asked: Do you know anything to memorize it? To which Tirri: I know these two verses: Women, woman, women, wife, men, man and wife, men, woman, women, man, man and wife. To this, Firri: Explain. And Tirri: Women, Berta and Greta; woman, Greta; women Greta and Rosa; wife, Berta; men, Gerardus and Rolandus; man and wife, Gerardus and Greta; men, Bertoldus and Gerardus; woman, Rosa; women, Berta and Greta; man, Rolandus; man and wife, Rolandus and Rosa." The translation of the two verses is that of [Singmaster 1998: 20].

6 See [Curtze 1895: 83]. At folios 30–33 the manuscript contains a collection of 34 recreational problems entitled *Subtilitas enigmatum*. The 24[th] of the series is the jealous husbands problem. Curtze claims to have found the same collection in another manuscript, Cod. Amplonianus Qu. 345, of the first half of the 14[th] century entitled *Cautele Algorismorum*.

*Sic igitur binos transfert ne federa soluant.*
*Transuehit una duas remanetque reuersa marito.*
*Inde duos fert cimba uiros, redeunt uir et uxor.*
*Sunt duo suntque due citra, uir uterque recedit.*
*Tercia iam rediens prius hanc post transuehit illam.*[7]

The manuscripts mentioned above are only a few of a large number containing collections of recreational problems that were written from the $13^{th}$ to the $15^{th}$ century. They were composed in the circle of the monastic schools in Northern Europe (France, England and Germany). These collections, which are often entitled *Cautele, Enigmata,* and *Subtilitates*, although of different length, show a great number of similarities in content and form. Almost always only the solutions are given without an explanation of how they are obtained; furthermore, final verses often summarize the solutions. Menso Folkerts has analysed 33 of these manuscripts located in the Bodleian, British Museum, Berlin, Munich, Vienna and Erfurt libraries.[8]

The collections contain anywhere from one to 83 problems, and twelve of them present the jealous husband problem, which is sometimes presented together with that of the wolf, goat and cabbage.

Many of these manuscripts summarize the solution with the following hexameters:[9]

*Uxor abit duplex redit una meat quoque duplex*
*Una redit geminus vir abit redit unus et una*
*Vir geminus vehitur redit una duoque vehuntur*
*Vir redit cimbam post transeat unus et una.*[10]

If we consider the aforementioned Latin manuscripts together with the copies of Alcuin's *Propositiones* we can easily conclude that the problem was widely circulated in Latin Northern Europe.

## The Jealous Husbands Puzzle in the Italian Abacus Tradition

A considerable part of the Italian mathematical treatises from the $14^{th}$ to the $16^{th}$ centuries is represented by abacus treatises (*Trattati d'abaco*), the composition of which is linked to the teaching of *Liber abaci* (1202) of Leonardo of Pisa.[11]

The principal subject of these treatises is the application of practical arithmetic to the solution of business problems. However, many of them contain numer-

7 This version has been found by André Allard, who has kindly sent me the transcription and authorised me to publish it.

8 A complete list of the manuscripts with their analysis is contained in [Folkerts 1971].

9 See for example Oxford, Bodleian Library, Digby 193, ff. 25v–26v; Munich BSB Clm 534. M. Folkerts has kindly sent me the transcription of some of these problems.

10 Oxford, Bodleian Library, Digby 75, f. 132r.

11 A catalogue of these treatises together with a general presentation of the abacus mathematics can be found in [Van Egmond 1980].

ous recreational problems, sometimes collected into a single section, sometimes scattered throughout the text.

The remarkable number of abacus treatises actually known, about 250, has prevented us from examining all of them. However, the inspection of about 40 manuscripts[12] that seem to be the most representative has made it clear that the jealous husbands problem was seldom included. To explain this it should be noted that this puzzle does not appear in *Liber abaci*.

Only one of the twelve 14th-century texts examined contains our problem. It is the *Rascioni di Algorismo*, written about 1350 by an anonymous author who uses the vernacular of Cortona, a town in central Italy:[13]

> *Fammi questa rascione: 3 conpangni che ànno 3 loro mogliere e uogliono passare uno fiume chon una barcha che non porta se non è 2 persone. Adomando, chomo passarà dall'altra parte, che ll'uno di chonpangni non possa auere a ffare chola molgliere dell'altro; e llo de' passare in prima la molglie cholo suo marito e torna lo marito e passono le duo molgliere e torna l'una donna e passo li 2 mariti e torna lo marito chola sua dona e passo li dui mariti e la donna torna chol alt(ra) donna e poj lo marito passa chola sua donna e portala dall'altra partte e ssono tuctti passati oltra.*[14]

It should be noted that the solution follows closely Alcuin's.

The only Italian text of the 15th century in which I have found the problem belongs to the abacus tradition although it is much more than an abacus treatise. It is the *Pratica d'arismetrica* written by Maestro Benedetto da Firenze in 1463. Three copies of this treatise are known, the most complete being L.IV.21 of the Biblioteca di Siena. Its 500 folios, which measure 28 × 40 cm, comprise a large encyclopaedia of abacus mathematics.[15] The tenth of the sixteen books in which it is divided is devoted to "*Chasi dilettevoli*" (recreational problems). This part occupies folios 233–300 and contains a very large collection of recreational problems, including the jealous husbands problem. Benedetto's version of the problem is as follows:

> *Tre mariti gelosi chon 3 loro moglie vogliono passare uno fiume e ànno una navicella che chon 2 di loro è charica. Adimandasi in che modo passeranno a mo' che ciaschuno sia libero dell'altro. Chosì farai. E pon bene l'ontelletto che chon breve parlerò. Passesi prima 2 donne, e la nave rimeni una donna, et passino l'altre 2 donne. E aremo dall'uno lato 3 mariti e dall'altro 3 donne et*

12 Appendix 3 gives a list of these manuscripts.

13 The manuscript of this treatise is X511.A13 of Columbia University Library, edited by Vogel [1977].

14 English translation: "Solve this problem for me: 3 friends who have 3 wives wish to cross a river by a boat that holds only two people. I ask how they could cross so that none of the friends can stay with the wife of another. First a wife must cross with her husband, then the husband goes back and two wives cross; a woman goes back and the 2 husbands cross; a husband goes back with his wife; the two husbands cross and the woman goes to the other woman; after the husband crosses with his wife, and taking her on the other side, all of them have crossed." Cf. [Vogel 1977: 132].

15 An accurate description of this treatise is given in [Arrighi 1965].

*la nave è dal lato delle donne. E di poi una donna torni cholla nave e starà chol marito, et li due mariti passino alle loro moglie. E di poi uno marito cholla moglie passi de qua. E di là poi passino e 2 mariti. E rimarrà di qua 2 moglie et di là la nave, la quale chondurrà di qua una moglie, et virranno 2 moglie a loro mariti, all'altra moglie menerà la nave lo suo marito. Et aranno passati ciaschuno salvo sança sospetto.*[16]

Benedetto presents the problem at the end of the chapter after that of the wolf, goat and cabbage and before the Josephus problem,[17] referring to them as *favole* (fairy tales), perhaps because their solution does not involve calculations.

At the beginning of the 16th century in Italy two large anthologies of recreational problems were composed by authors belonging to the abacus tradition. They are *Libro dicto giuochi mathematici* by Piero di Nicolao, and *De viribus quantitatis* by Luca Pacioli. These treatises are very similar and seem to have been drawn from the same sources; both include the jealous husbands puzzle.

Piero di Nicolao lived in the first part of the 16th century in Tuscany. His *Libro dicto giochi mathematici*, dedicated to Giuliano de' Medici, son of the more famous Lorenzo, consists of 183 folios. It is divided into four parts and contains 91 problems.[18] In the introduction the author claims to have devoted himself to the study of mathematics since his youth. Concerning his sources he claims only to have collected them in many places.[19]

The jealous husbands problem is the second problem of the third part of the treatise and is contained in the folios 124r–126r. Piero's exposition is very long and filled with embellishments suitable to its presentation as a court entertainment; it is contained in its entirety in Appendix 1. The most interesting feature of this version is the final generalisation of the problem:

> Afterwards you can, with your imagination, make [this problem] with 4 husbands and 4 wives, with 5 and with 6, etc. But note that the boat has to be capable of holding one less than the number of the couples, so that if there are four husbands and as many wives, the boat must be capable of [holding] three

16 English translation: "Three jealous husbands with their 3 wives wish to cross a river and they have a little boat that holds only 2 of them. We ask how shall they cross that each is free of the other. You shall do so and be careful, for I will speak shortly. At first 2 women cross, and a woman takes the boat back and the other 2 women cross. And we have, on one side, 3 husbands and on the other, 3 women, and the boat is on the side of the women. After, a woman comes back with the boat and stays with her husband, while the two husbands go to their wives. Then a husband and his wife go back. Then 2 husbands cross to the other side. And on this side will remain 2 wives, and the boat is on the other side, which a wife shall take on this side, and 2 wives will come to their husbands, her husband will take the boat to the other wife. And each one will have crossed freely and without suspicion." Cf. f. 293v.

17 This is a well-known puzzle that relates that fifteen Christians and fifteen Pagans were on a ship in danger, and thus half had to be sacrificed. The question is how they could be arranged in a circle so that, in counting round, every ninth one should be a Pagan.

18 [Arrighi 1971] gives biographical notes on Piero and a transcription of the introduction.

19 Piero di Nicolao, *Libro dicto giochi mathematici*, Biblioteca Nazionale di Firenze, Magl. XI, 15: f. 4v. "Io ho raccolti di più luoghi questi mia secreti et forze, et forse parte rubbati et a questo et a quello."

> people, and if they are 5 [couples], it is necessary that the boat be capable of [holding] 4 people and so on infinitely, otherwise you will tire yourself in vain.
>
> However, here you could propose that [the couples] be 4 and the boat be capable of [holding] 2 people and no more; it would be impossible and you would make the crowd fantasize and you could put up a prize. And so it would be if there were 5 and the boat capable of 2 or 3 at the most; it would be similarly impossible.

Therefore Piero says the problem of crossing $n$ couples can be solved if the boat holds $n - 1$ people and ends reaffirming the problem is impossible if the people are five and the boat holds two or three people. However, he does not give any explanation for his assertions. The latter remark is in part wrong. In fact five couples can cross with a boat holding only three people in eleven trips.

Finally I should like to remark that even if it is true that $n$ couples can cross with a boat holding $n - 1$ people, one can nevertheless prove that for each $n > 5$ it is sufficient to have a boat holding four people.[20]

Luca Pacioli (1440/45–1517) is perhaps the most well-known Italian writer of abacus mathematics; very famous indeed is his *Summa de arithmetica, geometria, proportioni et proportionalità* (1494). Less well-known is his *De viribus quantitatis*, the only hand-written copy of which is kept in the Library of the University of Bologna. This manuscript consists of 309 folios containing a large collection of recreational problems and resembles that of Piero di Nicolao; it too was composed at the beginning of the 16th century.[21]

Pacioli presents the problem of the jealous husbands in folios 103v–105r. His treatment is similar to Piero's including the remarks about the generalisation to more couples, but more correctly he says:

> But for 4 couples the boat must hold 3 people and for 5 couples it must hold 4 or 3 otherwise it is impossible.

Neither does Pacioli provide any explanation for his assertions. However, he gives only the solution for three couples and begins with the crossing of two women, but he wrongly remarks there is another solution that begins with the crossing of two men, and requires the same number of trips. I wish to point out further that while Pacioli indicates the order of the trips, he only assigns a number to the trips from the first to the second bank. Another interesting characteristic of this version is the use of letters to denote the elements of the couples, namely red capital letters A, B, C for the husbands, and black small letters a, b, c for their respective wives. Pacioli's version of the problem seems to me to be the most complete and correct I have seen, therefore it is given in Appendix 2.

I would also like to remark that although neither Piero's nor Pacioli's text presents the wolf, goat and cabbage puzzle, Pacioli appears to have intended to include it. In fact, he lists it in the index just before the jealous husbands problem, but the problem does not actually appear in the text.

20 A close examination of the problem can be found in [Lucas 1992: 6–14].

21 An accurate description of the content of this treatise is given in [Agostini 1924].

A concise and correct version of our problem is also found in a 16th-century manuscript of an abacus treatise entitled *Fiori di aritmetica e geometria* (Flowers of Arithmetic and Geometry), written by Bertholamio Verdello in 1550 in Verona. It is interesting that Bertholamio claims to be the grandson of Francesco Feliciano da Lasize, the author of a printed abacus book of which fourteen editions were published from 1517 to 1696 and which does not include the jealous husbands problem.

## Nicolas Chuquet's *Triparty*

Nicolas Chuquet (ca. 1440–1488), Parisian, bachelor of medicine, algorist, wrote in 1484 in Lyon a large mathematical treatise entitled *Triparty en la science des nombres*, which has a great affinity with Italian abacus treatises even if it also contains elements of a different tradition.[22]

The first part of *Triparty is* concerned with arithmetic; the second with roots; the third presents algebra. The treatise ends with a collection of 166 problems illustrating applications of the previously expounded principles. The last problems are recreational mathematics, and in particular Problem 164 presents the jealous husbands problem. Its French version is as follows:

> *Le jeu des troys maryz et de lers femmes*
>
> *CLXIV. Ilz sont trys hommes auec chascun sa femme qui veulent passer vne riuier et nont que ung petit bateau ou quel ne peuent passer plus de deux personnes a la foiz. Or est il ainsi ordonne entre eulx que nulle de lers femmes ne se doit trouuer auec homme nul que son mary ne soit present ne deca la riuiere ne dela. Et si aultrement elle fait elle est reputee deshonneste et desloyale a son mary. Lon demande maintenant commant ces six personnes pourront passer la riuiere lonnoeur des femmes saulue. Response. Deux femmes passent et lune ramene le bateau pius deux femmes passent encores et lune ramene le bateau et demeure avec son mary. Et les deulx aultres marys passent. Puis lung diceulx auec sa femme retornent le bateau. Les deux hommes passent. Et la femme repasse. Puis deux femmes passente. Ne reste plus fors que lung des maryz voyse querir sa femme et sta fait.*[23]

22 *Triparty* was ignored by historians of mathematics until Aristide Marre [1880–81] drew attention to it publishing large excerpts. On the occasion of the fifth centenary of *Triparty* an ample study on Chuquet was edited in [Flegg & Moss 1985].

23 English translation: "The puzzle of the three husbands and their wives. There are three men each with his wife, who wish to cross a river and they have a small boat in which only two people can cross at a time. It happens they have agreed that none of their wives must be with a man without her husband being present on either this side of the river or the other. Otherwise she will be believed dishonest and disloyal towards her husband. One asks how these six people could cross the river, preserving the honour of the women. *Answer.* Two women cross and one brings back the boat, then two women cross again and one brings back the boat and remains with her husband. And the two other husbands cross. Then one of them with his wife brings back the boat. The two men cross. And a woman goes back. Then two women cross. It remains that one of the husbands goes back to take his wife and it is made." Cf. [Marre 1881: 459–460].

Chuquet's version is worthy of note, for it states explicitly the rules of behavior. This is very important; in fact, as we shall see in the next section, the misunderstanding of these rules gave rise to erroneous solutions.

## The Printed Versions in the Sixteenth Century

The importance and the diffusion of abacus mathematics in Italy are shown also by the fact that one of the first printed books was just an abacus treatise: the *Aritmetica di Treviso* (1478). A few years later it was followed by the *Aritmetica* of Piero Borghi (Venice, 1484), *Pitagoras aritmetice introductor* of Filippo Calandri (Florence, 1491), *Compendion de lo abaco* of Pellos (Turin, 1492) and *Summa de aritmetica* of Luca Pacioli (Venice, 1494).

The printed abacus treatises are similar enough to the hand-written ones, even if they tend towards a more standard format. The most part of them includes recreational problems, but none of the books mentioned above presents the jealous husband problem.

*Le pratiche delle due prime matematiche* (1546) of Pietro Cataneo, one of the best Italian arithmetic treatises, contains the first printed version of the problem I have found.

> *Et dicendosi che tre mariti gelosi con le lor mogli volessero passare un fiume, et havessero una barchetta che non ne passasse per volta più che 2 di loro, et volessi sapere in che modo convenisse loro passare, acciò che in fra loro non havesse a cadere nessuna gelosia. Dico che si passi prima due donne, et passate che elle sieno, una ne resti, et l'altra ritorni con la barca per l'altra donna, et così haverai da un lato le tre donne, et dall'altro li tre mariti, et la barca serà dal lato delle donne, hora torni indietro una di quelle, et restisi col suo marito, et li due altri mariti passino alle donne loro. Di poi un marito con la sua moglie passi di qua dal fiume, et di là passino li due mariti, et rimarrà di qua dal fiume due mogli, et di là tre mariti, et un di quelli haverà accanto la sua moglie, et faralla passare di qua, acciò che l'altre due mogli passino ai loro mariti, et passate ch'elle sieno uno di quelli se ne ritorni per la sua, et in tal modo seranno tutti passati senza sospetto di gelosia.*[24]

24 English translation: "And if one says that three jealous husbands with their wives want to cross a river, and they have a boat in which only 2 of them could cross at a time. One wishes to know how it comes tha' they cross, so that no jealousy occurs among them. I say that first two women must cross, after they have crossed, one remains and the other goes back with the boat for the other woman. So you will have on one side three women and on the other the three husbands, and the boat will be on the side of the women, now one of them goes back and remains with her husband, and the other two husbands go to their wives. Then a husband with his wife crosses to this side of the river and the two husbands cross beyond, and on this side of the river will remain two wives and beyond three husbands, and one of them will have with him his wife, and make her cross on this side, so the other two wives cross to their husbands, and afterwards one of them goes back for his wife. Thus all people have crossed without suspicion of jealousy." Cf. [Cataneo 1546: f. 54v].

It should be noted how careful Cataneo is in checking each time that the conditions of jealousy are fulfilled. However, he does not mention any generalisation.

The *General trattato* (1556) of Niccolò Tartaglia (ca. 1505–1557) is the most important treatise of practical arithmetic of the 16th century. It includes many recreational problems, among which we find the jealous husbands puzzle. Tartaglia also includes the wolf, goat and cabbage problem.

Tartaglia's version, like Cataneo's, is concise. He further gives a solution for the crossing of four couples with a boat holding two people. It is obvious that Tartaglia did not know the works of Piero di Nicolao and Luca Pacioli, which had asserted that the problem was impossible. The solution is the following:

> *Et se fossero stati 4 huomini et quattro donne prima manda fuora 2 donne, & fa che una di quelle ne venga a torre un'altra, & la conduce de la, poi una di quelle vien di qua con il navetto, & tolse la quarta donna, & se la condusse de la, condutte che siano de la tutte quattro, ne vien di qua una con il navetto, & si accosta appresso a suo marito, poi si levano duoi huomini, & entrano nel navetto, & si ne vanno de la appresso alle sue donne, poi quella donna che è de la discompagnata la intra nello navetto, & vien a torre suo marito. & lo mena de la, & di qua sono rimasti solamente marito e mogliere, poi se ne venne di qua uno di quelli huomini, & mena da la quell'huomo chi era di qua con la donna sua, menato ch'el fu de la quello di prima andette appresso a sua moglie, & quest'altro ritornò di qua a tor sua mogliere, & la condusse de la, & cosi furno condotti fuora tutti sani, & salvi.*[25]

In Tartaglia's solution, by the seventh trip there are two couples and a woman on the second bank, situation not allowed by the strict jealousy condition. The mistake of Tartaglia may be caused by a misunderstanding of this condition; in fact the woman without a husband on the second bank goes immediately back with the boat. But authors with a more pessimistic attitude towards human nature do not permit such a situation and so this solution was considered wrong.

A mistaken interpretation of the jealousy situation like that of Tartaglia is at the origin of the following wrong solution of the problem for three couples given by Dionigi Gori in his *Libro di arimetricha* (1571).[26]

25 English translation: "And if they should have been 4 men and four women, first you send to the other side 2 women, and make one of them come to take another and bring her to the other side, then one of them comes to this side with the boat and takes the fourth woman and brings her to the other side, so that on the other side all the four women, one of them comes to this side with the boat and goes over next to her husband, then two men enter into the boat and go to their women, then that woman that is not paired goes into the boat and comes to take her husband, and brings him to the other side, so that on this side only one husband with his wife remains, then one of them came to this side and takes to the other side that man who was on this side with his wife, when he is on the other side, one went to his wife and the other went back to take his wife, and brought her on the other side. Thus all people were crossed safe and sound." Cf. [Tartaglia 1556: 256].

26 The manuscript of this treatise is kept in the Biblioteca Comunale of Siena. Some parts of it including the problem in question are published in [Franci & Toti Rigatelli 1982].

> *Tre belle done com 3 loro mariti molto gelosi, volendo passare el fiume in una barcha che non passa si no 2 per volta, e nisuno di questi mariti fida la sua dona all'atro maritto, essendo di bissognio passare el fiume, si dimanda senza fidarsi l'uno maritto dell'atro, come fecieno al passarlo: prima passò una moglie e uno maritto, el maritto rimena la barcha e lassa la moglie, poi passorno le 2 mogli e resta el maritto, come sonno passatte una di quelle done rimena la barcha e resta dal suo maritto, e lli 2 mariti che ànno già le mogli passatte passano dalle loro mogli, e rimena in qua la barcha una moglie e ritorna coll'atra moglie, e quella che rimenò la barcha resta e l'atra ritornò per suo maritto et cossì passorno tutti e 3 le mogli e mariti.*[27]

The above solution in nine trips rather than eleven is to be considered incorrect since the jealousy condition is not always fulfilled. In the sixth and seventh trip, in fact, there is a woman on a bank in presence of men without her husband being present. However, note that exactly as in Tartaglia's solution for four couples, the woman goes back immediately. So it seems that someone gave a wider interpretation to the jealousy condition.

Claude-Gaspar Bachet (1581–1638), author of the first printed book entirely devoted to recreational mathematics, *Problèmes plaisant & délectable* (1621), is not one of them. In fact in his treatise he includes the three jealous husbands problem and points out that the solution of Tartaglia for four couples is wrong. He claims the problem impossible, but he does not give a proof. A proof can be found in a later edition of the *Problems* by A. LaBosne [Bachet 1993: 150]. The latter gives also a solution to the problem of crossing for $n$ couples with a boat holding $n - 1$ people in $2n - 1$ trips.

Finally, Edouard Lucas [1881–94] in his *Récreations mathématiques* remarks that the correct generalisation of the problem is to ask for the least number $x$ of people the boat must hold so that the crossing of $n$ couples is possible. He proves that for $n > 5$, $x$ must be 4 and for $n = 4$ or 5, $x$ must be 3. Thus with Lucas the problem of the jealous husbands is clarified definitively.[28]

## Final Remarks

The relatively numerous Latin sources testify to the wide circulation of the problem of the three jealous husbands in Northern Europe from the 13th to the 15th century.

27 English translation: "Three beautiful women with their 3 very jealous husbands, wishing to cross a river with a boat which can cross only two of them at a time, and any husband does not entrust his wife to another husband, for it is necessary to cross the river, one asks how they made the crossing without trusting each of the other. At first a wife and one husband cross, the husband brings back the boat and leaves his wife, then two wives cross and the husbands stays, when they are crossed one of the women bring back the boat and she remains with her husband, and the 2 husbands whose wives have already crossed, cross to their wives, and one wife brings back the boat and goes back with the other wife, and she who brought back the boat stays and the other went back for her husband and so all 3 wives and husbands crossed." Cf. [Franci & Toti Rigatelli 1982: 105].

28 [Hadley & Singmaster 1989] reports on modern variations and generalisations of the puzzle.

More problematic is the analysis of its diffusion in Italy. In spite of its infrequent inclusion in abacus treatises, it is included in the large collections of recreational problems of Piero di Nicolao and Luca Pacioli, which even refer to generalisations. The lack of previous Italian written sources leads us to conjecture that the problem had a widespread oral tradition. To support this hypothesis I would like to remind the reader that many abacus writers who present recreational problems remark that they are very suitable for the brightening of social gatherings.

## Acknowledgements

During the Bellagio meeting, André Allard and Menso Folkerts, pointed out the existence of the Latin tradition I did not know. The latter in particular has kindly sent me the materials that have allowed me to write the second section of the paper.

## Appendix 1

Piero di Nicolao d'Antonio da Filicaia, *Libro dicto Giuochi mathematici*, early sec. XVI, Florence, Biblioteca Nazionale, Magl. XI, 15.

//f. 124v// Secondo caso di 3 mariti et 3 mogli gelosi et quattro et cinque
Voglio mostrare modo a ffare quella favola di tre mariti et tre mogli gelosi per chi e non //f. 125r// lo sapessi et anchora pochi la saprà, dirò qualche cosa in questa di piacere. Capitando adunque 3 mariti et loro donne seco ad uno fiume grandissimo el quale senza la barcha era impossibile potersi dall'altra banda trasferirsi et erano in paese costoro che non senza qualche suspecto si caminava et maxime nelli (...) et era luogho silvestre dove era charistia di habitazioni et perchè avevano lor giornate ordinatamente compartite non bisognava che troppo badassino per non essere alla campagna scoperta dalla obscura nocte sopragiunti. Onde giuncti a questo fiume et disidirosi passar via con velocità non ritrovorno el barcharulo che subito li haveria passati, ma solo trovorno una piccola barchetta alla ripa ligata et quella presa et dalla ripa disciolta et volendo in ipsa tucti ad uno tracto passar viddono per experientia che dicta barcha non era capace di più che due persone per volta et così quasi confusi rimasono, et desiderosi passar via pensavano el modo passare senza gelosia et suspecto l'uno dell'altro. Se domanda in questo caso se è possibile farlo, a che si risponde che si, et fia in questo modo. Prima dirai che passi in là due donne et poi una rimangha di là et l'altra rimeni la barcha in qua et meni delà l'altra donna, et sono in due volte passate tucte tre le donne di là, et li homini sono di qua et le donne hanno seco la barca. Dunque le donne si potivan andar con Dio et lassare abaiare li loro mariti de là del fiume, ma perchè omniuna era gelosa del suo marito non harieno per niencte facto maj tale errore. Onde per tornare al mio proposito et volendo fornire di passarlj, faraj ora passar una delle 3 donne allo in qua et così el suo maritto et li due maritti di quelle che

sono di là dal fiume montino in barcha et vadino a trovare le lor donne che hanno la barcha seco li duj mariti con le loro donne et resta a passare uno marito e una moglie. Hora passi allorquando uno marito con la sua donna et hanno di qua due mariti con le donne loro et di là sarà uno marito et la donna soli et la barcha è de qua. Hora passino di là //f. 126v// li dui mariti lassando le loro 2 donne di qua e poi mandino con la barcha la donna ch'è già passata per una delle due che restano a passare, et poi mandino o al suo marito oppure una altra donna per la 3ª. Ma è più conveniente mandare el suo marito et così li haveraj passati tucti senza gelosia o suspecto l'uno dello altro non che habbino ad esser traditori uno all'altro, ma per levar via ruggine fra loro intrinseca amicizia, perchè el suspecto che ha uno della sua sposa o donna è immenso et grandissimo et non si può armare el suspecto si che questo modo suproadicto leva via omnj fantasia. Potrai tu poi col tuo ingenio fare di 4 mariti et 4 mogli di 5 et di 6 etc.. Ma nota che la barcha bisogna che sia capace di uno mancho de il numero delle coppie che sono, perchè se sono quattro mariti tante mogli, bisogna che la barcha sia capace di tre persone et se fussino 5 bisogna che la barcha sia capace di 4 persone et così in infinito, et altrimenti indarno ti affaticheresti et però qui tu proponi che sieno 4 et la barcha //f. 127r// sia capace di 2 persone et non più; fia impossibile et farai fantasticare le brighate et potrai metter pegno. Et così se fussino 5 et la barcha fusse capace di 2 o 3 al più similmente sarà impossibile.

## Appendix 2

Luca Pacioli, *De viribus quantitatis*, early sec. XVI, Library of the University of Bologna, ms 250.

//f. 103v// LXI C. de 3 mariti et 3 mogli gelosi
Capitaronno a un fiume per passare 3 huomini //f. 104r// ognuno con la sua moglie o voglia dir foemina et trovando la barca legata alla rippa con li remi el barcaruolo non n'era, costoro per la fretta de passare deliberaro o vogarse per esser usate a vinegia fra loro stessi o chiogia et credendo poter passare tutti a un tratto, la barcha none potia regere più de dua al suo carco, onde costoro se vegano inpaciati si per pericolo delo inducio che dubitavano non havere le nimico ale spalle si et perchè dubitavano non potere rehaver la barcha de qua et di non rompere la compagnia giurata alo viagio et ancho che fra loro non havesse a nascere rugine de suspecto de lor donne capitando ale mani de l'uno senza el suo marito, con ciò sia commo dici la scriptura vinum et mulieres faciunt hominem apostatare. Et così stando in tal pensiero et agonia l'uno di loro, de numero pratico experto, disse a tutti servate l'ordine, il modo ch'io ve dirò et siremo di là tutti senz'altri fra noi passati senza l'amico suspecto et così fò observato. Se domanda si sia possibile et commo passaro senza suspecto de gelosia et nonne portando la barca più de doi alla volta. Dirai el caso esser possi //f. 104v// bile et fecero in questo modo. Et poniamo per non sbagliar et ancho equivocare nel trascrivere che gli huomini sienno letre del alphabeto A.B.C rosse maiuscole et le donne a.b.c picole nere et la

barcha sia la letra D. Prima passaro a.b foemine et restaro delà gli 3 huomini con una donna appresso el suo marito et la barca sia di là. Poi una di queste tornò la barca in qua et acostose al suo marito lasciando la compagna di là et poi presse la 3ª donna sua compagna .c. et menola di là. Ora di qua restaro gli 3 huomini et di là sonno 3 donne con la barcha et sonno facti doi viaggi. Poi una di quelle qual vuoli retornando la barcha in qua s'acosta al suo marito havendo lasciate le doi altre sue compagne di là. Ora montano in barcha gli doi mariti di quelle che sonno di là et vano al loro lasciando di qua l'una con lo suo marito et sonno facti 3 viagi et la barca è di là con li 2 mariti et loro mogli senza suspecto. Poi l'uno di questi con la sua mogli montano in barca et tornano di qua lasciando el marito con sua mogli di là. Et poi gli 2 huomini montano in barcha et vanno di là lasciando //f. 105r// di qua le doi lor donne et ànno facto 4 viaggi et la barcha est di là. Poi la donna di là torna la barca in qua sola, lasciando tutti li 3 huomini di là et al 5° viaggio passa la compagna et consegnala al suo marito. Et poi torna per l'altra di qua et portala di là, medesimamente consegnandola al suo marito, al 6° viagio ponendo tutto al fine. Et la barca che prima era di qua se ritrovarà di là et sel barcarolo virrà o lui notarà over per l'altro ingegno passarà di là se lui vorrà. Et così sia fatta tutta senza suspecto de gelosia fra loro però che de donne con donne l'uomo non ha suspecto, neanche quando vi sonno seco li lor mariti, perché non vale a far violenza in simil casi, ma solo se atende a levar gli suspecti, jux illud suspectos cavem nesis misce omnibus hominis dammnatis (...). Et sapi si comme tu comenzasti con le donne tu medisimamente potevi comenzare dali huomini usando el passaggio per lor via commo festi per le donne et in tanti viaggi similmente gli passarai franchi senza alcun suspecto, como per te porrai experimentando provare, ponendo da te le disposizioni del caso como //f. 105v// vedi di qui figurate in margine adove la rippa di qua del fiume metiamo f.g et di là la metiamo h.z.[29] Porrai anchora da te, per tuo ingenio negoziando, disponere di 4 mariti et 4 mogli et così di 5 mariti et 5 mogli et sic de quotlibet. Ma per li 4 bisogna la barca possi portar 3 e per li 5 ne port 4 o 3 aliter impossibile laboratur. Et però fu bello a proporre de 4 cola barca de 2 quod non potest et ancho de 5.

## Appendix 3

*Abbreviations for the Libraries*

| | |
|---|---|
| BCS | Biblioteca Comunale degl'Intronati, Siena |
| BMLF | Biblioteca Mediceo-Laurenziana, Florence |
| BNF | Biblioteca Nazionale, Florence |
| BPP | Biblioteca Palatina, Parma |
| BRF | Biblioteca Riccardiana, Florence |
| CUL | Columbia University Library, New York |

29 In the actual manuscript the figures are missing.

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# De l'arabe à l'hébreu: la constitution de la littérature mathématique hébraïque (XIIe–XVIe siècle)

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Les premiers écrits mathématiques rédigés en hébreu et datables apparaissent au XIIe siècle. Entre le XIIIe et le XIVe siècle se constitue, essentiellement en Provence, un véritable corpus textuel, riche et diversifié, à partir de sources arabes, traduites ou adaptées. Cet essor se prolonge aux XVe et XVIe siècles en s'ouvrant aux sources latines et vernaculaires. Après avoir présenté ce mouvement, nous en dégageons quelques traits caractéristiques.

The first dated mathematical writings composed in Hebrew appear in the XII$^{th}$ century. Between the XIII$^{th}$ and the XIV$^{th}$ centuries, mainly in Provence, a fairly coherent corpus of texts is made up of translations and adaptations from Arabic sources. This movement goes on during the XV$^{th}$ and XVI$^{th}$ centuries, exploiting Latin and vernacular sources. We try to sketch the main features of this mathematical activity.

Dans cet essai, nous proposons:

I. une périodisation des mathématiques médiévales d'expression hébraïque, telle qu'elle se dégage des sources identifiées à ce jour, et qui n'ont pas encore toutes été systématiquement étudiées,

II. un tableau du premier grand mouvement de traductions et d'adaptations, réalisées pour l'essentiel en Provence, entre 1230 et 1350,

III. une caractérisation de ce corpus, qui le rapporte à ses sources, sa langue, son contexte culturel. La comparaison avec le développement initial des mathématiques d'expression latine nous paraît éclairer utilement les problèmes historiques et méthodologiques qui sont en jeu.

## I. Mathématiques d'expression hébraïque (XIIe–XVIe siècle): une périodisation

La littérature mathématique hébraïque prend véritablement son essor en Europe, vers la fin du XIe siècle. La carte géographique de son développement est, au début, celle des communautés juives intéressées par la culture judéo-arabe d'Espagne et n'ayant pas (ou plus) d'accès directs aux ouvrages arabes: il s'agit du nord-est de l'Espagne et du sud de la France entre XIIe et XIVe siècle; au cours de cette étape initiale, on voit se constituer à la fois une culture mathématique de base (ouvrages généraux d'arithmétique et de géométrie, encyclopédies scientifiques) et un fonds de textes destinés plus proprement à un public savant;

cette production a pour source quasi-exclusive les écrits mathématiques arabes, dont elle représente un témoin et un relais. Dans l'Italie de la Renaissance, aux XVe et XVIe siècles, on constate une influence grandissante des sources latines ou vernaculaires. A la même époque, le regain d'intérêt pour les sciences manifesté dans les communautés judéo-byzantines se nourrit aussi bien de sources arabes et grecques.

*1. Le* Mishnat ha-Middot (*Livre des mesures*): *témoin d'une tradition orientale ancienne?* Sans doute le plus ancien ouvrage de mathématique rédigé en hébreu, cet opuscule relève du genre de "la géométrie pratique"; sa dernière partie est faite de commentaires numériques relatifs aux dimensions du Tabernacle, tel qu'il est décrit dans le récit biblique.[1] Nous sommes, pour l'heure, incapables de préciser la date, le lieu, et l'auteur de cette composition, malgré les suggestions de Gandz tendant à en situer la rédaction au IIe siècle.[2] Peut-être entre IXe et XIIe siècle; peut-être à Bagdad; peut-être un savant juif arabophone soucieux d'ouvrir le monde des études traditionnelles aux savoirs scientifiques. Quoi qu'il en soit, il paraît difficile d'isoler cet opuscule d'une possible tradition *scientifique hébraïque* ancienne (médecine, astronomie-astrologie, mathématiques pratiques) dont les contours et les sources sont encore flous.

*2. La naissance d'une langue et d'une littérature scientifiques dans l'Espagne du XIIe siècle.* Deux personnages dominent cette époque: Abraham bar Ḥiyya (ca. 1065–1145) et Abraham ibn 'Ezra (m. 1067).

Abraham bar Ḥiyya (connu en latin sous le nom de Savasorda) a été éduqué et formé scientifiquement dans l'une des principautés arabes issues de la chute du califat de Cordoue, sans doute à Saragosse sous la dynastie hudide (1039–1118), dont plusieurs souverains furent des savants renommés. C'est toutefois à Barcelone, en pays chrétien, qu'il composa ses œuvres originales en hébreu, à la demande pressante des dignitaires juifs de Provence; il y collabora aussi avec Platon de Tivoli pour traduire en latin plusieurs ouvrages arabes.[3] Abraham bar Ḥiyya a traité de philosophie, d'astronomie et d'astrologie; nous lui connaissons deux compositions mathématiques: une encyclopédie scientifique[4] et un ouvrage de "géométrie pratique"[5] traduit en latin par Platon de Tivoli en 1145.[6] Dans ses divers ouvrages, l'auteur rappelle son recours aux sources arabes, insiste sur le respect scrupuleux desdites sources — lorsqu'elles existent —, et prétend simplement exploiter en hébreu la science arabe. A y regarder de près, on se rend compte que l'auteur ne s'est pas contenté de forger une langue mathématique hébraïque, il a aussi adapté les objectifs des ouvrages qu'il a exploités, pour les mettre à la portée des lecteurs hébréophones de Provence.

1 Pour l'édition la plus récente du texte: [Sarfatti 1993].
2 [Gandz 1932].
3 [Lévy 2001b].
4 [Millás Vallicrosa 1952].
5 [Guttmann 1912; Millàs i Vallicrosa 1931].
6 [Curtze 1902].

Abraham ibn 'Ezra est né en Espagne chrétienne, à Tudèle (Navarre). De culture arabe et connaissant sans doute le latin, il a passé une grande partie de sa vie en voyages à travers l'Europe, après avoir quitté l'Espagne à la fin des années trente. C'est un poète, un grammairien, un astronome-astrologue (l'astrologie tient un rôle majeur dans sa pensée) et un commentateur biblique très réputé. Son ouvrage d'arithmétique *Le livre du nombre* (*Sefer ha-Mispar*) a constitué d'emblée une référence et exercé une influence durable.[7] Abraham ibn 'Ezra a, lui aussi, façonné une nouvelle langue scientifique, différente de celle de son aîné Bar Ḥiyya, en ce qu'elle est plus proche de l'hébreu biblique (la langue de Bar Ḥiyya exploite plutôt les ressources de l'hébreu rabbinique) et plus éloignée des tournures et de la terminologie arabes.

*3. Le premier mouvement de traduction de textes classiques (XIIIe–XIVe siècles).* La dynastie des Tibbon, originaire de Grenade, a créé en Provence les conditions d'émergence d'une véritable culture scientifique hébraïque. C'est à la troisième et quatrième générations qu'apparaissent les traducteurs d'ouvrages mathématiques (et astronomiques). Sur une période qui couvre un peu plus d'un siècle (1230–1350), on verra se constituer ce que nous avons appelé "la bibliothèque mathématique du savant juif médiéval":[8] un corpus de textes, dont la base est constituée par des œuvres majeures de la mathématique hellénistique, complétée par des commentaires ou des œuvres originales arabes. Cet ensemble peut s'ordonner selon la classification suivante:

- Euclide et la tradition euclidienne
- La géométrie des sphères
- L'*Almageste* et la tradition théorique ptoléméenne
- Arithmétique et théorie des nombres
- Archimède et la tradition archimédienne
- Apollonius et la tradition apollonienne

Les principaux traducteurs sont Jacob Anatoli (ca. 1194–1256), lequel a travaillé à Naples à la cour de Frédéric II; Moïse ibn Tibbon de Montpellier, qui fut actif entre 1240 et 1283; Jacob ben Makhir (Don Profeit Tibbon), astronome à Montpellier (ca. 1236–1305); Qalonymos ben Qalonymos (Maestro Calo) d'Arles (1287–après 1329); Samuel ben Juda (Miles Bongodas) de Marseille (1294–après 1340). On peut qualifier certains de ces savants de "traducteurs professionnels" (par exemple, Moïse ibn Tibbon); Jacob ben Makhir est l'auteur d'écrits astronomiques originaux; Qalonymos ben Qalonymos a travaillé au service du roi Robert II d'Anjou.

*4. Deux astronomes-mathématiciens provençaux au XIVe siècle.* L'activité traductrice ne s'interrompt pas brutalement au milieu du XIVe siècle, pas plus qu'elle ne se limite géographiquement aux centres provençaux que nous avons mentionnés: des textes scientifiques arabes continuent de circuler en Espagne chrétienne, par-

7 [Silberberg 1895; Lévy 2000b, 2001a; Langermann et Simonson 2000].
8 [Lévy 1997c].

mi les savants juifs, à Tolède, à Saragosse, à Burgos, à Salamanque; ils y sont étudiés ou traduits. Même si les compositions originales qui voient le jour en Provence au XIVe siècle sont largement tributaires des textes "classiques" désormais disponibles en hébreu, elles exploitent manifestement des sources que nous n'avons que partiellement identifiées: bien des indices nous font soupçonner que l'héritage d'al-Andalus est encore mal connu.

Les deux auteurs qui dominent incontestablement cette période sont des astronomes réputés.

Lévi ben Gershom, dit Gersonide (1288–1344) est l'auteur d'une œuvre astronomique considérable,[9] traduite en latin;[10] ses écrits mathématiques sur les nombres harmoniques,[11] sur l'arithmétique et la combinatoire,[12] sur la géométrie d'Euclide,[13] manifestent une connaissance (indirecte?) de sources arabes et latines, dont il n'existe pas — dans l'état actuel de la recherche — de versions ou d'adaptations en hébreu.

Immanuel ben Jacob Bonfils de Tarascon, actif entre 1340 et 1365, est aussi un astronome, dont les Tables astronomiques seront connues en latin et en grec byzantin. Sa maîtrise du latin semble avoir été suffisante pour lui permettre de rendre en hébreu le *Roman d'Alexandre*.[14] Nous lui connaissons plusieurs recueils d'écrits arithmétiques[15] et géométriques,[16] dispersés dans les collections manuscrites, dont il conviendrait de faire un inventaire et une analyse minutieux, avant d'en évaluer la portée: développements originaux? notes de lecture d'un astronome? exploitation de sources arabes et/ou latines non attestées en hébreu?

5. *Traductions et commentaires dans l'Italie du XVe siècle.* Le déplacement vers l'Italie n'est pas seulement géographique: les textes dont nous disposons témoignent d'un recours de plus en plus fréquent au latin et aux langues vernaculaires.

Mordekhay (Angelo) Finzi de Mantoue (m. 1475) est un des personnages les plus représentatifs de cette période. Astronome, traducteur, commentateur, Finzi a associé son nom à plusieurs écrits mathématiques.[17] Il est sans doute le traducteur (peut-être l'auteur) d'un ambitieux résumé des savoirs géométriques en 11 parties, mentionnant les noms de Campanus et de Jordanus de Némore;[18] il est l'auteur d'un ouvrage de stéréométrie, le *Traité de mesures des cuves et des tonneaux*, mentionnant aussi bien Abraham bar Ḥiyya que des "abacistes" chrétiens.[19]

9 [Goldstein 1992].
10 [Mancha 1992, 1997].
11 [Chemla et Pahaut 1992].
12 [Lange 1909; Simonson 2000].
13 [Langermann 1996: 45–46; Lévy 1992a, 1992b].
14 [Lévi 1881].
15 [Gandz 1936].
16 [Rabinovitch 1974].
17 [Langermann 1988].
18 [Steinschneider 1893–1901: 194].
19 [Langermann 1988: 32–33].

Dans le domaine de l'algèbre, nous connaissons une composition due à un ami de Finzi et dédiée à ce dernier. Simon ben Moïse ben Simon Motot, l'auteur de ce texte, y traite des équations canoniques du second degré ainsi que des équations qui s'y ramènent; dans cet ouvrage rédigé dans les années 1460, on peut relever des traces évidentes du *Liber abaci* de Fibonacci, auquel il convient d'ajouter les œuvres d'"abacistes" italiens du XIVe ou XVe siècle.[20] Mordekhay Finzi, encore lui, traduisit l'ensemble de l'*Algèbre* d'Abū Kāmil,[21] y compris la deuxième partie, consacrée au pentagone et décagone,[22] ainsi que la troisième partie, sur les équations indéterminées.[23] Finzi est aussi le traducteur (1473) d'un important traité de Maestro Dardi de Pise (1344) composé en italien et comportant des équations du troisième ou du quatrième degré irréductibles.[24]

Il n'est pas exclu que les contacts des milieux juifs savants d'Italie avec ceux de Constantinople, surtout après l'arrivée des Ottomans en 1453, aient constitué des canaux de diffusion des idées ou des textes originaires de l'Orient arabe.

*6. Les mathématiques dans le monde savant judéo-byzantin (XVe–XVIe siècle).* A partir du XVe siècle, et jusque vers le milieu du XVIe siècle, on relève un regain d'intérêt pour l'apprentissage et l'enseignement des mathématiques et de l'astronomie au sein des communautés judéo-byzantines. On copie beaucoup: bien des précieux recueils manuscrits que nous connaissons sont de la plume de copistes travaillant à cette époque à Constantinople, répondant ainsi nécessairement à une demande.[25] On compile: nous connaissons ainsi un volumineux commentaire anonyme des *Eléments* d'Euclide, mentionnant al-Kindī, al-Fārābī, Ibn al-Haytham, al-Anṭākī, à côté d'auteurs grecs. On traduit encore des livres arabes: c'est vers 1460 que Shalom ben Joseph 'Anavi traduit et commente en hébreu le texte arabe de Kūshyār ibn Labbān consacré au calcul indien.[26] Et on compose. Les compositions que nous connaissons manifestent le recours à un triple héritage: celui des textes déjà disponibles en hébreu (Abraham bar Ḥiyya, Abraham ibn 'Ezra, classiques grecs ou arabes traduits en hébreu, Gersonide), celui de textes arabes non traduits en hébreu, et enfin celui de textes grecs ou gréco-byzantins.

Mordekhay Komtino (1402–1482) est l'auteur du *Livre du calcul et des mesures (Sefer ha-Ḥeshbon we-ha-Middot)*,[27] où sont repérables des sources grecques

20 [Sacerdote 1892–94].
21 [Levey 1966].
22 [Sacerdote 1896].
23 Cette troisième partie ne nous est parvenue en latin qu'à travers un court fragment; voir: [Sesiano 1993: 447–452].
24 [Van Egmond 1983; Hughes 1987].
25 L'expulsion des juifs d'Espagne, en 1492, entraîne un afflux de familles anciennement espagnoles dans l'Empire ottoman, et tout particulièrement à Constantinople. Un savant de Constantinople, de surcroît copiste prolifique, Kalev Afendopulo, signale en 1499 que plusieurs manuscrits mathématiques lui ont été accessibles grâce à l'arrivée des exilés d'Espagne; voir [Steinschneider 1896: 93].
26 [Levey et Petruck 1965].
27 [Silberberg 1905–06].

anciennes telles que Héron, ainsi que des procédés exposés par des auteurs byzantins des XIIIe et XIVe siècles.[28]

Elie Mizraḥi (1455?–1526), élève de Komtino, auquel il succéda à la charge de grand rabbin de Constantinople, témoigne d'une vaste culture scientifique et d'un grand souci de pédagogie dans *Le livre du nombre (Sefer ha-Mispar)*.[29]

Kalev Afendopulo (1460?–1525?), responsable spirituel de la communauté qaraïte (opposée au judaïsme rabbinique et ne reconnaissant que l'autorité de la Loi écrite) de Constantinople, fort prospère à cette époque, fut aussi un élève de Komtino. Copiste prolifique et exégète, Afendopulo a traité d'astronomie et de mathématiques. Nous lui connaissons un copieux commentaire de l'*Introduction arithmétique* de Nicomaque, qui n'est pas le seul dans son genre, à Constantinople, à la même époque.[30]

Les sources mentionnées dans les sections 5 et 6 n'ont pas reçu encore l'attention qu'elles méritent. En particulier, le développement des mathématiques hébraïques en Italie aux XVe et XVIe siècles est encore mal évalué: les sources n'y sont plus seulement arabes, mais latines et vernaculaires. De surcroît, le contexte culturel environnant ainsi que les transformations des communautés juives elles-mêmes ont exercé des effets d'impulsion qu'on ne mesurera que lorsque les textes seront correctement édités et analysés. En revanche, nous sommes mieux instruits aujourd'hui sur la portée du grand mouvement qui se développe en Provence aux XIIIe et XIVe siècles.

## II. L'essor de la littérature mathématique hébraïque et les premières traditions textuelles

Dans cette partie, nous mettons l'accent sur les problèmes de transmission et d'acculturation des savoirs mathématiques arabes dans le monde hébréophone. Pour ne pas nous en tenir au seul inventaire de la transmission, nous pointons les traditions arabes dont les textes mis à jour et étudiés se font l'écho, avant de nous demander si lesdits textes se constituent à leur tour en traditions proprement hébraïques. On s'en tiendra ici à la classification des traductions et adaptations réalisées, pour l'essentiel en Provence, entre 1230 et 1350.

*Euclide et la tradition euclidienne.*

• Les *Eléments*. L'examen systématique du corpus nous a permis d'identifier 31 manuscrits de versions proprement dites du texte euclidien.[31] De l'analyse de ce riche matériau, nous avons dégagé l'existence de quatre traductions issues de sources arabes (au moins deux manuscrits offrent des textes issus de sources

28 [Schub 1932].
29 [Mizraḥi 1534; Wertheim 1896].
30 [Steinschneider 1894: 76–77; 1896].
31 [Lévy 1997a]. Par comparaison, on connaît à ce jour une vingtaine de manuscrits du texte arabe; voir: [Folkerts 1989: 27–29].

latines), réalisées au XIIIe siècle, dans des milieux proches les uns des autres. L'une de ces versions, inédite, est indiscutablement liée aux plus anciennes versions arabes (les versions "ḥajjājiennes"), pour lesquelles elle apporte des éclairages nouveaux.[32] Ajoutons que l'ouvrage euclidien n'est pas seulement un outil pour le savant; il fait aussi partie des livres recherchés par "l'homme cultivé", comme en témoigne l'inventaire de certaines bibliothèques de médecins juifs.[33]

A ces recensions (versions au sens strict), il convient d'ajouter deux adaptations arabes du texte euclidien qui ont fait l'objet d'une traduction en hébreu. C'est ainsi que nous avons identifié une version hébraïque anonyme de l'adaptation-résumé des *Eléments* composée par Avicenne, soit la partie géométrique de la grande encyclopédie avicennienne *al-Shifā'*, intitulée *Les fondements de la géométrie*.[34] D'autre part, nous avons étudié la partie géométrique d'une autre encyclopédie, qui a sans doute exploité la précédente, et qui est l'œuvre d'un auteur juif du XIIIe siècle, Juda ben Salomon ha-Kohen de Tolède; cette encyclopédie, rédigée initialement en arabe, ne nous est parvenue que dans la version hébraïque réalisée par l'auteur lui-même. L'ouvrage intitulé *Midrash ha-Hokhma*, L'enseignement de la sagesse (ou de la science), comprend, dans sa deuxième partie, une version abrégée des *Eléments*, conçue comme une introduction à l'étude de l'*Almageste* de Ptolémée.[35]

Plusieurs commentaires arabes des *Eléments* furent traduits au cours de cette même période: celui d'al-Fārābī,[36] celui d'Ibn al-Haytham, celui de Jābir ibn Aflaḥ.[37]

L'exceptionnelle importance accordée au texte des *Eléments* se mesure aussi par l'existence de commentaires directement composés en hébreu: certains sont des commentaires mathématiques, comme celui de Gersonide ou d'anonymes;[38] d'autres ont un caractère philosophique, comme celui d'Abraham ben Salomon ha-Yarḥi (= de Lunel), au XIVe siècle.[39]

• *Les Données*, *L'Optique*, ainsi qu'un ouvrage pseudo-euclidien, *Le livre des miroirs*, furent aussi traduits.

*Astronomie théorique et géométrie des sphériques.* L'astronomie mathématique constitue une part importante des premières traductions: à côté des ouvrages de Ptolémée, nous disposons de versions hébraïques des écrits de Geminus, al-Biṭrūjī,[40]

32 [Lévy 1997b].
33 [Vajda 1958; Iancou-Agou 1975].
34 [Lévy 1997a: 80].
35 [Lévy 2000].
36 [Freudenthal 1988a].
37 L'original arabe ne nous est pas connu; nous en avons identifié un fragment en hébreu.
38 [Lévy 1992c]. Parmi les commentaires anonymes importants, signalons un substantiel "Livre d'Euclide", en fait une vaste encyclopédie mathématique en douze parties, incluant, entre autres, une présentation des *Eléments*; voir [Langermann 1984b].
39 [Freudenthal 1998].
40 [Goldstein 1971].

Jābir ibn Aflaḥ, al-Farghānī, Ibn al-Haytham,[41] Averroès,[42] et d'auteurs juifs rédigeant en arabe tels que Joseph ibn Naḥmias et Joseph ibn Israël.

Parallèlement, on traduit des écrits consacrés aux sphériques, relevant de "la petite astronomie" ou, comme les désigne parfois la tradition arabe, des "[livres] intermédiaires": ceux d'Autolycus, Théodosius, Ménélaus;[43] ainsi que des ouvrages arabes qui prolongent cette tradition, comme ceux de Thābit ibn Qurra ou Jābir ibn Aflaḥ sur *La figure sécante* de Ménélaus. Cette tradition textuelle a été suffisamment vivante pour inclure, à côté de la traduction de commentaires arabes (non encore étudiés), des écrits directement composés en hébreu, comme ceux que Gersonide a consacré à Ménélaus.[44]

*Arithmétique et théorie des nombres.*

- Nicomaque de Gérase: *Introduction à l'arithmétique*. La version hébraïque ne correspond exactement ni au texte grec, ni à la version arabe donnée par Thābit ibn Qurra: elle est issue d'un texte arabe (apparemment perdu), qui fut lui-même traduit d'une version syriaque (perdue aussi); on y relève d'importantes gloses attribuées à al-Kindī. L'étude continue du texte hébreu se manifeste par l'existence de commentaires jusqu'au XVIe siècle, à Constantinople.[45]
- Abū al-Ṣalt: ce savant andalou (1068–1134) fut très probablement l'auteur d'une encyclopédie des sciences, que nous ne connaissons qu'à travers deux parties qui nous sont parvenues en version hébraïque, l'une sur la musique[46] et l'autre sur l'arithmétique (traduite en 1395, à Saragosse). Nous avons montré que cette partie arithmétique est constituée, pour l'essentiel, par une traduction littérale de la section arithmétique de l'encyclopédie d'Avicenne, le *Shifā'*. Nous avons d'autre part relevé diverses références à Abū al-Ṣalt en relation avec l'optique, la géométrie et l'astronomie;[47] dans ce dernier cas, nous disposons peut-être d'une partie de son ouvrage.[48]
- al-Ḥaṣṣār: le "petit" traité d'arithmétique, rédigé sans doute au XIIe siècle par cet auteur actif en al-Andalus ou au Maghreb,[49] fut traduit par Moïse ibn Tibbon en 1271. Il importe de souligner ceci: même s'il ne comporte pas un chapitre spécial concernant l'algèbre (pourtant prévu par l'auteur, mais absent du texte arabe tel qu'il nous est parvenu, ainsi que de la version hébraïque), l'ouvrage se réfère plusieurs fois, à l'occasion de problèmes arithmétiques (problèmes "pratiques" ou problèmes associés à certaines suites d'entiers) à "la méthode de l'algèbre" (dans l'hébreu de Moïse ibn Tibbon, "la méthode de ce qui est caché"), ainsi qu'à sa

41 [Langermann 1990].
42 [Lay 1996].
43 [Ginsburg 1943].
44 Nous ne les connaissons que par la mention que l'auteur en donne dans l'inventaire de sa bibliothèque: voir: [Weil-Guény 1992: 361].
45 [Steinschneider 1894; Langermann 2001].
46 [Avenary 1974].
47 [Lévy 1996a: 65, 83].
48 [Lévy 1997c: n. 11].
49 [Aballagh et Djebbar 1987].

terminologie élémentaire ("la chose", "le bien"). Cet aspect de l'ouvrage, dont on connaît de nombreux exemples dans l'histoire de l'arithmétique arabe, présente un intérêt particulier pour l'historiographie des mathématiques hébraïques: il témoigne d'une certaine connaissance de l'algèbre, qui a été sous-évaluée jusque là.

A l'appui de cette réévaluation, signalons la découverte récente d'un important texte d'arithmétique, traduit de l'arabe en Sicile à la fin du XIVe siècle;[50] l'ouvrage comporte une section sur l'algèbre. L'auteur de la version hébraïque, Isaac ben Salomon al-Aḥdab, est un astronome, originaire de Castille. Cet ouvrage d'Isaac ben Salomon constitue un témoignage important quant à la connaissance, dans les milieux savants hébréophones, de traditions arithmétiques et algébriques nées et développées en pays d'Islam. Même si nous ne pouvons pas mesurer, pour l'heure, l'importance de la diffusion d'un tel ouvrage et, partant, l'ampleur des connaissances acquises dans le domaine de l'arithmétique et de l'algèbre arabes, on ne peut esquiver la question du rapport possible entre de telles sources et l'intérêt manifesté pour l'algèbre dans la deuxième moitié du XVe siècle par certains savants juifs italiens (Simon Motot, Mordekhay Finzi).

- Thābit ibn Qurra: L'opuscule *Sur les nombres amiables* a été traduit, sans mention de l'auteur. On retrouve en effet l'ensemble des théorèmes du texte arabe, sans les démonstrations, dans une composition hébraïque arithmétique originale, rédigée très probablement par Qalonymos ben Qalonymos, à la demande du roi Robert d'Anjou: *Le livre des rois*. Nous avons établi que le théorème de Thābit était connu dans des cercles de savants juifs, entre XIVe et XVIe siècles en Espagne, en Provence et en Italie.[51]

Il convient de relever l'imbrication des traditions que mettent en évidence ces résultats. Les recherches récentes des historiens des mathématiques arabes ont établi l'existence d'une tradition textuelle et conceptuelle continue dans ce domaine de l'arithmétique, du IXe jusqu'au XVIIe siècle, comportant d'importants développements théoriques au XIIIe siècle; c'est ainsi que les résultats établis par Fermat et Descartes entre 1636 et 1638 sont avérés dans le monde scientifique arabe bien avant cette époque.[52] La circulation de textes hébraïques attestant la diffusion de la tradition arabe dans l'Europe médiévale réduit la possibilité que ladite tradition soit restée inconnue des mathématiciens d'Europe; pour autant, il n'est pas établi que les développements formulés par Descartes et Fermat soient directement liés à une connaissance de la tradition arabe.

*Archimède et la tradition archimédienne.*

- *La mesure du cercle*: à la version hébraïque anonyme, issue d'une source arabe et déjà signalée par les bibliographes,[53] il convient d'ajouter une deuxième version, que nous avons identifiée dans un recueil de textes traitant du problème

50 Nous présentons ce texte dans une étude en cours de rédaction, consacrée aux traces de l'algèbre arabe dans les textes mathématiques hébraïques.

51 [Lévy 1996a: 76–82].

52 [Rashed 1983; Hogendijk 1985].

53 [Steinschneider 1893: 502; Sarfatti 1968: § 267].

des isopérimètres. Ce texte, issu d'une autre source arabe, distincte de la précédente, porte la marque — style, terminologie — des traducteurs connus du XIIIe ou XIVe siècle, tandis que le premier porte plutôt la marque stylistique d'écrits plus anciens tels que la géométrie de Bar Ḥiyya.[54]

• *La sphère et le cylindre*, ainsi que le Commentaire d'Eutocius qui l'accompagne généralement dans les sources arabes, furent rendus en hébreu.

Comme indice supplémentaire d'une tradition *textuelle* archimédienne en hébreu, il nous faut citer l'ouvrage d'un auteur espagnol du XIVe siècle, Abner de Burgos, converti au christianisme sous le nom d'Alfonso de Valladolid: *Celui qui rend droit le courbe* (*Meyasher ʿAqov*) composition philosophico-mathématique, contenant de nombreuses références (explicites ou non) à des écrits "archimédiens".[55]

*Apollonius et la tradition apollonienne*. L'œuvre majeure d'Apollonius, *Les sections coniques*, était disponible en arabe dès le IXe siècle. Nous n'avons aucune trace d'une version hébraïque, fût-elle partielle, de ce traité. Néanmoins plusieurs textes "apolloniens" nous sont parvenus, qu'ils soient liés aux *Sections coniques*, à d'autres ouvrages du géomètre grec, ou bien qu'ils relèvent d'un développement novateur dans la géométrie des courbes et des surfaces.

• Un opuscule anonyme sur la propriété asymptotique de l'hyperbole (qui fait l'objet de la proposition II,14 des *Sections coniques*), traduit deux fois de l'arabe, a fait l'objet d'une large diffusion, encouragée sans doute par l'intérêt philosophique qu'avait accordé Maïmonide à cette propriété.[56]

Nous connaissons aussi:

• Un opuscule anonyme, traduit de l'arabe et intitulé *Résolution des doutes relatifs à la dernière prémisse du Livre des sections coniques*.

• Un *Opuscule sur le triangle* d'Abū Saʿdān, faisant usage des propriétés de la parabole.

• Un recueil de *Problèmes géométriques*, dont certains renvoient aux ouvrages "perdus" d'Apollonius.

• Un ouvrage consacré aux propriétés des cinq polyèdres réguliers, élargissant les analyses d'Hypsiclès consignées dans le Livre XIV des *Eléments* d'Euclide. Le nom d'Apollonius y est mentionné en relation avec son traité (perdu) sur le dodécaèdre et l'icosaèdre.[57]

Parler d'une tradition apollonienne en hébreu ne nous paraît toutefois pas fondé, en dépit des indices que constituent ces diverses traductions, relais évident d'une tradition apollonienne arabe vivante. La comparaison avec la situation des mathématiques d'expression latine à la même époque nous permettra de préciser cette remarque. Avant les grandes traductions latines de la Renaissance, la con-

54 Nous avons édité, traduit et commenté ces deux versions dans une étude en cours de publication: La mesure du cercle *d'Archimède en hébreu: deux versions distinctes*.

55 [Gluskina 1983; Langermann 1988: 38–39; Lévy 1992a: 44–58; Langermann 1996: 33–40].

56 [Freudenthal 1988b; Lévy 1989a, 1989b].

57 [Langermann et Hogendijk 1984].

naissance directe du texte des *Sections coniques* n'était guère plus importante que celle que nous avons relevée dans le domaine hébreu: les toutes premières définitions ont été traduites de l'arabe par Gérard de Crémone en introduction à sa version latine du traité d'Ibn al-Haytham sur les miroirs ardents; ajoutons-y l'opuscule arabe sur la propriété asymptotique de l'hyperbole (mentionné ci-dessus) traduit en latin vers 1230 par Jean de Palerme. En revanche, on relève un intérêt soutenu pour l'étude des miroirs ardents, et plus généralement pour l'optique, dont témoignent la version latine de l'*Optique* d'Ibn al-Haytham (Alhazen), la *Perspectiva* de Witelo, et plusieurs écrits consacrés aux coniques en relation avec les problèmes de miroirs paraboliques.[58] C'est dans cet intérêt pour l'étude des miroirs ardents que gît la principale différence — pensons-nous — avec les mathématiques hébraïques: on peut parler, à la suite de M. Clagett, d'une tradition latine des coniques, mais pas d'une tradition hébraïque.

Un indice, toutefois, nous permet de penser qu'une tradition apollonienne d'un autre type s'est, peut-être, constituée en hébreu. Il s'agit d'un important texte consacré à l'ellipse, traduit de l'arabe en 1312 par Qalonymos ben Qalonymos sous le titre *Traité sur les cylindres et les cônes*.[59] Son auteur, le géomètre andalou Ibn al-Samḥ (m. 1035), qui fut l'élève de Maslama al-Majrīṭī, a composé un ambitieux traité de géométrie des courbes (qui ne nous est pas parvenu en arabe), prolongeant les recherches novatrices entreprises au IXe siècle par les Banū Mūsā, en particulier dans le domaine des coniques. Le texte hébreu, après une introduction traçant un vaste programme portant sur les cylindres, les cônes et les sphères, traite principalement de l'ellipse, caractérisée par sa propriété bifocale et considérée comme section plane d'un cylindre. Nous avons là un précieux témoignage des recherches théoriques les plus avancées du XIe siècle andalou. Pour qui, et à quelle fin, Qalonymos a-t-il traduit ce texte difficile et dense? Nous n'avons pas de réponse à l'heure actuelle: l'unique manuscrit connu est l'œuvre d'un copiste de Constantinople, en 1506, qui en souligne la difficulté et la rareté!

## III. Mathématiques hébraïques: quelle définition?

Comment *désigner* ces textes que nous avons évoqués? Littérature mathématique hébraïque? mathématiques d'expression hébraïque? mathématiques hébraïques?

L'expression de "mathématiques hébraïques" offre un parallèle commode avec celle de "mathématiques arabes" ou de "mathématiques latines", mais elle n'est pas sans poser problème si l'on veut restituer une intelligibilité historique à la genèse et au développement du corpus ainsi désigné, à savoir *la littérature mathématique rédigée en hébreu*, dont les premiers écrits datables apparaissent au XIIe siècle.

Sur le plan de la méthode, nous ne devons pas esquiver les difficultés que suscite l'expression "mathématiques hébraïques"; ces difficultés sont de deux

58 [Clagett 1980].
59 [Lévy 1996b].

ordres: historique et épistémologique. Historiquement, les études — voire les textes — nous manquent encore, concernant la constitution d'une littérature mathématique hébraïque qui se serait développée dans le monde juif arabophone: l'énigmatique *Mishnat ha-Middot* ("énigmatique" du point de vue de ses sources, de sa composition, de son public), par son caractère "exceptionnel", en témoigne. A cet égard, la comparaison avec la littérature mathématique persane, rédigée dans les milieux savants arabophones, pourrait se révéler féconde. D'autre part, quel statut accorder aux "savoirs" mathématiques, si parcellaires et rares qu'ils soient, dispersés dans la littérature hébraïque biblique ou post-biblique? Epistémologiquement, un autre difficulté surgit: on pourrait, à bon droit, nous objecter que l'expression "mathématiques hébraïques" confère d'emblée aux textes mathématiques rédigés en hébreu une unité, une cohérence, voire une "logique" de développement, dont nous ne sommes pas en mesure de rendre compte.

Les remarques qui suivent ne peuvent donc pas et n'entendent pas trancher ce débat: on s'y propose seulement de caractériser l'essor de la littérature mathématique hébraïque, qui se traduit entre XIIe et XIVe siècle par la constitution d'un corpus à la fois riche et diversifié, et qui se prolonge aux XVe et XVIe siècles.

En circonscrivant ainsi notre domaine, nous en excluons une partie importante de textes témoignant de l'activité mathématique dans le monde juif médiéval, ce que Steinschneider désignait par "Mathematik bei den Juden".[60] Précisons cela.

Il existe des textes ou commentaires mathématiques rédigés en arabe par des savants juifs arabophones: Sa'adiah Gaon al-Fayyūmī[61] ou Dunas ibn Tamīm al-Qarawī au Xe siècle, Maïmonide[62] ou Ibn 'Aknīn[63] au XIIe siècle.[64] Ces ouvrages relèvent de l'historiographie des mathématiques arabes; pour autant, on ne peut ignorer les effets que cette activité scientifique a pu induire dans la constitution du domaine qui nous concerne, qu'il s'agisse d'effets directs (traductions ou adaptations en hébreu) ou indirects.

Les passages de type arithmétique ou géométrique que l'on relève dans la littérature hébraïque biblique ou post-biblique constituent certes des "savoirs mathématiques"; toutefois, ils sont relativement peu nombreux, n'ont aucun caractère systématique et ne prennent sens que rapportés aux problèmes (pratiques, juridiques, liturgiques) qui les convoquent. On pourrait sans doute tenter d'inscrire ces savoirs dans la tradition des "mathématiques babyloniennes", "égyptiennes", voire certaines traditions grecques spécifiques (géométrie "pratique"). Soulignons cependant ceci: les premières compositions mathématiques rédigées en hébreu au XIIe siècle ne sont pas un déploiement de ces savoirs, formulés eux aussi en hébreu (quelquefois en araméen). Il nous faut insister sur cette discontinuité de *nature* et non pas seulement de *forme* qui sépare les savoirs mathématiques "rabbiniques" des écrits que nous considérons ici. Pour autant, il

60 [Steinschneider 1893–1901].
61 [Gandz 1943].
62 [Langermann 1984a].
63 [Güdemann 1873: 1–62].
64 Ajoutons que des savants juifs, ou désignés comme tels, sont signalés par les bibliographes arabes pour leurs compétences et leur activité mathématiques. Voir: [Poznanski 1905].

nous faut réfléchir sur un phénomène qui frappe l'historien engagé dans l'étude de ce *corpus*: l'insistance des acteurs de la vie scientifique médiévale à pointer la continuité entre ces deux ordres de savoir afin de légitimer l'activité scientifique et ses exigences. Ce phénomène relève d'une analyse historique, difficile mais nécessaire: comment penser l'effort des savants juifs médiévaux pour faire place aux sciences "profanes" dans le cadre de la culture communautaire, ciment essentiel et garant de la pérennité du monde juif? De surcroît, sur le plan de la terminologie, le lexique mathématique hébraïque (on songe à Abraham bar Ḥiyya ou Abraham ibn 'Ezra, les premiers artisans de la langue mathématique hébraïque) s'est élaboré en exploitant les ressources langagières offertes par les textes anciens.[65]

Qu'en est-il enfin des rapports entre mathématiques et astronomie dans ce travail historique qui est le nôtre? Il est banal de rappeler que pour les savants médiévaux l'astronomie est la première des "sciences mathématiques"; de plus, ces savants sont souvent aussi des astronomes. S'agissant de "l'astronomie hébraïque", il nous faut donc au minimum faire référence aux écrits d'astronomie théorique, pour saisir le rôle moteur que les exigences de la "sciences des astres" (qu'on songe aux recherches liées aux problèmes de calendrier) ont exercé, par exemple, dans la constitution d'une partie du corpus géométrique hébraïque, voire, tout simplement, dans la légitimation des études et recherches scientifiques. Le cas de Maïmonide est exemplaire à cet égard: la maîtrise de l'astronomie de son temps, manifestée avec éclat dans son ouvrage rabbinique sur le calendrier (*Sanctification de la nouvelle lune*), et sur un plan essentiellement critique dans le *Guide des égarés*, est légitimée avec vigueur au nom de l'éclairage ainsi apportée aux exigences de la Loi.[66]

Les écrits composant le domaine ainsi circonscrit sont caractérisables par trois traits: l'unité de la langue, le caractère de leurs sources constitutives — les traditions mathématiques arabes —, et leurs rapports à la culture traditionnelle. Les deux premiers éléments sont strictement homologues à ceux qu'on pourrait citer pour définir les "mathématiques latines"; il n'est donc pas sans intérêt de détailler les différences spécifiques de chacune de ces traditions; elles nous permettront d'évaluer plus finement les caractères du domaine qui nous concerne.

La proximité linguistique de l'arabe et de l'hébreu est une donnée dont on ne peut minorer la portée, même s'il convient de l'intégrer à une donnée d'ordre plus général: l'activité de générations de savants juifs arabophones en pays d'Islam du VIIIe au XIIe siècle, phénomène dont les effets se font ressentir bien au-delà de cet espace géographique et de cette période historique.

Le développement de la langue mathématique hébraïque s'effectue autour de deux pôles: la fidélité à la langue des sources traditionnelles d'une part, et d'autre part la volonté de s'ouvrir, par le biais de l'emprunt linguistique, aux nouvelles

65 [Sarfatti 1968].

66 Nous avons étudié cet aspect des analyses maïmonidiennes dans "Maïmonide et les mathématiques", dans: Tony Lévy et Roshdi Rashed, éd., *Maïmonide et les traditions philosophiques et scientifiques médiévales* (en cours de publication).

idées véhiculées par l'arabe. L'intensité de cette familiarité avec la langue, les belles lettres et les sciences arabes, et l'impérieuse nécessité d'en entretenir et d'en transmettre la connaissance sont illustrées, par exemple, par le "testament spirituel" adressé par Juda ibn Tibbon, le "père" des traducteurs, lui-même exilé de Grenade en Provence au milieu du XIIe siècle, à son fils Samuel.[67] La maîtrise de l'arabe et de sa culture manifestée par ce dernier suscitera les éloges de Maïmonide, qui lui confiera avec confiance la traduction de son *Guide des égarés*. Le gendre de Samuel, Jacob Anatoli, nous indique qu'il étudiait les mathématiques avec ce dernier (sans doute à Marseille) "dans les livres arabes".[68] Jacob Anatoli est le traducteur de l'*Almageste* (1235); Moïse, le fils de Samuel, est l'auteur d'une version des *Eléments* d'Euclide (1270). Il n'est pas rare, dans les colophons, de voir ce traducteur rappeler fièrement son ascendance espagnole ("... Moïse, fils de Samuel, fils de Juda ben Tibbon, de Grenade en Espagne").

S'il est vrai que le grand mouvement de traductions, en Provence, répond à une "demande", explicitement formulée par des responsables de communautés juives (Narbonne, Lunel, Marseille, Béziers), l'essor de ce mouvement n'a été possible que par l'immersion de ses premiers artisans dans le monde de la culture et de la science arabes. En pays d'Islam, en Orient comme en Occident, les juifs composent en arabe non seulement dans le domaine de la philosophie et des sciences, mais aussi dans celui de la grammaire, de la "morale", de la jurisprudence rabbinique. Seules la liturgie et la poésie (au départ, synagogale) restent, en général, fidèles à l'hébreu.[69] Et l'on sait que la pratique de la langue et de la culture arabes se perpétue dans l'Espagne chrétienne au moins jusqu'au XVe siècle (Tolède, Saragosse, Barcelone, Grenade), même si elle concerne des cercles de plus en plus restreints.

S'agissant plus précisément des sciences, il importe de mesurer le poids des "anciens" (les savants juifs arabophones) dans l'orientation des activités scientifiques et éducatives qu'impulsent les "modernes" (les savants juifs hébréophones). A cet égard, on distinguera deux types "d'héritage": celui des traités scientifiques rédigés par des savants juifs, que rien ne distingue, *a priori*, d'autres traités arabes; et celui des ouvrages, scientifiques ou non, qui abordent les questions scientifiques en relation avec la philosophie, l'enseignement et l'éducation, la théologie. Bien que nous ne disposions d'aucun inventaire systématique, il semble bien que les ouvrages du deuxième type ont été traduits ou adaptés en hébreu bien plus fréquemment que ceux du premier type.

Deux exemples illustrent notre propos. Dunas ibn Tamīm de Kairouan est, au Xe siècle, l'auteur d'ouvrages d'astronomie et de mathématiques arabes (par exemple, sur le calcul indien), dont au moins un volumineux traité d'astrolabe nous est parvenu.[70] Toutefois seul son commentaire philosophico-scientifique sur

67 [Abrahams 1926: 51–93].

68 [Anatoli 1866: 11–12].

69 Pour autant, il convient de souligner que la poésie médiévale hébraïque fut profondément marquée par la poésie arabe.

70 [Stern 1956].

le traité mystique (hébraïque) *Le livre de la Création* a été traduit de l'arabe en hébreu.[71] Maïmonide est l'auteur de *Notes sur les Sections coniques d'Apollonius*;[72] or non seulement nous ne connaissons aucune version hébraïque de ce texte mathématique, mais les sources bio-bibliographiques hébraïques sont muettes à son sujet et les auteurs médiévaux qui se sont passionnément intéressés à la propriété asymptotique de l'hyperbole semblent totalement ignorer ce travail du grand maître. En revanche, la "philosophie mathématique" de Maïmonide, sa conception du statut des mathématiques, telles qu'elles ont été développées dans divers écrits arabes (traduits en hébreu) ou hébraïques, ont pesé d'un poids considérable dans la représentation des savants hébréophones des XIIIe et XIVe siècles.[73]

On voit par là ce qui sépare la genèse de la tradition scientifique arabo-hébraïque au XIIIe siècle de celle de la tradition scientifique arabo-latine au XIIe siècle. Les itinéraires de grands traducteurs comme Gérard de Crémone, Adélard de Bath ou Robert de Chester sont singuliers (même s'ils aboutissent à la constitution de cercles, voire d'écoles scientifiques): ces personnages ont quitté leur terre natale pour aller recueillir la science là où ils ont entendu dire qu'elle se trouvait; et il leur a fallu apprendre l'arabe. De plus, ces démarches s'inscrivent dans un contexte culturel où les savoirs scientifiques grecs ne sont pas inconnus en latin: la tradition géométrique des agrimenseurs, les textes boéciens, les traductions gréco-arabes d'Euclide et de Ptolémée n'ont pas d'équivalent dans le monde hébréophone; pour ne pas parler de l'œuvre de Guillaume de Moerbeke, traduisant au XIIIe siècle l'ensemble du *corpus* archimédien à partir de sources grecques, quand bien même ce travail semble avoir été peu exploité.

On sera d'autant plus intéressé par toute analyse instruite du développement comparé du domaine latin et du domaine hébraïque. Ainsi, connaissant le rôle des universités dans le monde latin d'Europe au XIIIe siècle par rapport au mouvement massif de traductions mathématiques arabo-latines du XIIe siècle (comment ce mouvement a-t-il été connu? a-t-il été relayé ou ignoré? amplifié ou simplement enregistré?), on pourrait, par comparaison, mieux analyser les effets de la situation sociale et institutionnelle — comment l'ignorer? — propre aux communautés juives de Provence après le mouvement massif de traductions du XIIIe siècle (quel usage a-t-il été fait de ce tout nouveau *corpus* scientifique? a-t-il été réinvesti dans un domaine particulier? a-t-il été relayé par de nouvelles recherches?).

Interroger le développement des mathématiques hébraïques sous l'angle de ses rapports à la culture traditionnelle est complexe. Divers ordres d'analyse y sont impliqués: la conception et la pratique de l'étude (dont le modèle dominant reste celui des textes de la Tradition: Bible et littérature rabbinique), la prégnance d'un système d'éducation très structuré dès le plus jeune âge, les représentations de la science chez les divers acteurs de la vie intellectuelle, la nature du travail mathématique, les domaines d'application susceptibles de le convoquer et de le

71 [Vajda 1947–54].
72 [Langermann 1984a].
73 [Freudenthal 1993: 113–117; 1995: 36–38].

relancer. L'examen des discours de légitimation (ils ne sont pas rares) élaborés par les acteurs de la vie scientifique devrait nous aider à mieux baliser un tel questionnement.[74]

Résumons-nous. Les mathématiques hébraïques, c'est d'abord un *corpus* de textes, qu'il convient de rechercher, d'identifier, d'éditer, d'analyser, selon les règles les plus exigeantes de la méthode historique; cet objet textuel, découpé par le geste du chercheur dans l'immense littérature médiévale d'expression hébraïque, ne prend toutefois sens (et intérêt) que rapporté aux traditions textuelles *et* conceptuelles qui trament le domaine, autrement plus vaste, des "sciences médiévales" d'expression arabe, latine ou vernaculaire.

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74 Nous avons abordé cet examen, dans le cas de la géométrie d'Abraham bar Ḥiyya, dans: [Lévy 2001b].

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75 A l'origine, ce travail fut publié en plusieurs livraisons dans deux revues d'histoire de mathématiques: *Bibliotheca Mathematica*, Neue Folge 7 (1893) 65–72 et 105–112; **8** (1894) 37–45, 79–83 et 99–105; 9 (1895) 19–28, 43–50 et 97–104; **10** (1896) 33–42, 77–83 et 109–114; **11** (1897) 13–18, 35–42, 73–82 et 103-112; **12** (1898) 5–12, 33–40 et 79–89; **13** (1899) 1–9, 37–45 et 97–104; *Bibliotheca Mathematica*, 3. Folge **2** (1901) 58–76; *Abhandlungen zur Geschichte der Mathematik* 9 (1899) 473–483. Les tirés-à-part, rassemblés et précédés d'un index rédigé par les soins d'une collaboratrice de Steinschneider, Adeline Goldberg, furent publiés en 1901 en un petit nombre d'exemplaires.

# Islamic and Chinese Astronomy under the Mongols: a Little-Known Case of Transmission

by BENNO VAN DALEN

Thanks to the Mongol conquests in the 13th century, a scientific exchange between the Iranian part of the Islamic world on the one hand and the Yuan Dynasty in China on the other became possible. This exchange resulted in the use of a Chinese type of lunisolar calendar in Iran and the construction of instruments and compilation of handbooks with tables by Muslim astronomers in China. In this article, we describe the exchange of astronomical knowledge between Muslims and Chinese in the Mongol period in some detail. In order to obtain more insight into the process of transmission, we first sketch the historical background of the exchange. In an Appendix, methods for investigating the relationships between astronomical tables are described.

## 1. Introduction: Investigating Transmission

This volume contains many examples in which the transmission of mathematical theorems, algorithms, problems, and their solutions is plausible, but cannot be established with certainty. This is a difficulty often encountered in the study of transmission of early mathematics. On the one hand, we find hardly any explicit historical information on the origin of the contents of mathematical works. On the other hand, a comparison of mathematical works from different cultures usually does not allow us to decide with certainty whether similar theorems, problems and solutions were transmitted, or independently invented or constructed.

The present article deals with a little-known case of transmission of mathematical ideas, namely that of mathematical-astronomical knowledge between the Islamic world and China in the Mongol period (13th century). In judging to what extent Chinese astronomers took over Islamic astronomical methods in their own calendars and instruments, and vice versa, we encounter the problem sketched above: we find hardly any explicit attributions, and a direct comparison of astronomical handbooks and instruments does not lead to unambiguous conclusions concerning possible connections.

Four specific cases of scientific contact between Muslim and Chinese astronomers in the 13th century are discussed below. In these cases, attention will be paid to three aspects in particular that may provide information important for judging the possibility of transmission:

1. The historical background of the transmission and the types of contacts that were possible between Muslims and Chinese;

2. Detailed technical descriptions of the objects to be compared, namely astronomical handbooks with tables and various types of astronomical instruments, and of the way in which they were used;
3. Mathematical methods for testing the similarity of astronomical tables and the possibility of connections between them.

The historical information (see Section 2 below) makes clear that there were ample opportunities for contact between Muslims and Chinese. In fact, Muslims in Mongol China were so common that it need not surprise us that the scientific level at the Islamic Astronomical Bureau in Beijing was very high. A detailed technical description of the properties of the so-called "Chinese-Uighur calendar" (Section 3.2) allows us to make well-founded statements about its origin. Finally, the mathematical methods described in the Appendix for testing the relationships between astronomical tables turn out to be exceptionally useful tools. Since the tabular values are highly idiosyncratic, it is often possible to decide whether two tables for the same function stem from the same source (in spite of incidental differences), and, with the aid of statistical or numerical methods, whether a table for a given function was calculated from another table for a different function.

## 2. Historical Background: A Short History of the Mongol Expansion in the 13th Century

In the first decade of the 13th century, Chinggis Khan united the Mongol tribes living on the steppes to the north of the Great Wall and started the expansion of what would soon become the Mongol world empire. His first raids against China resulted in the conquest of the capital of the Jin dynasty, present-day Beijing, in 1215. Chinggis then turned his attention to the west, occupied the province of Khwarezm (south of the Aral Sea) and, in 1220, took the important cities of Bukhara and Samarkand (in present-day Uzbekistan) without resistance. However, neither in Northern China, nor in Transoxiana was the Mongol rule solidly established at this stage.

After Chinggis's death in 1227, the first division of the Mongol empire took place between his four sons. The third son, Ögödei, became Chinggis's successor as Great Khan. Together with the youngest son Tolui, he achieved the final submission of the Jin dynasty in 1234. The oldest son, Jochi, received the lands north of the Caspian Sea, which stayed under Mongol reign for several centuries under the name Golden Horde. The spectacular campaign against Russia and Eastern Europe, which only came to a halt due to the death of Ögödei in 1241, was led by Jochi's son Batu. At that time, the Mongol empire stretched from Hungary in the west to Korea in the east.

From 1251 onwards the most important roles in the Mongol empire were played by the sons of Tolui. His oldest son Möngke was Great Khan from 1251 to 1259 and extended the capital Karakorum in Central Mongolia with a number of palaces designed by Chinese, Iranian, and Russian architects. Möngke's brother and successor Khubilai Khan, Tolui's second son, continued the conquest of China. He made Beijing the new capital of the Mongol empire and named himself the first emperor of the Yuan dynasty. In 1279 he finally defeated the Southern Song dynasty and reunified China under one rule.

Meanwhile, in the 1250s, the third son of Tolui, Hülegü, had led extensive military campaigns in the Middle East. By taking Baghdad in 1258 he made an end to the Abbasid dynasty and as the first Ilkhan ("Submissive" Khan) he reigned over Iran and Iraq from 1256 to 1265. After the death of Khubilai in 1294 and the final conversion of the Ilkhans to Islam in the following year, the Ilkhans became practically independent from the Mongol empire. Their power gradually decreased by the middle of the 14th century, when they were replaced by the Timurids. In China the Mongolian Yuan dynasty was defeated and replaced by the Ming in 1368.[1]

*2.1 Foreigners in the Service of the Mongols.* The Mongols were good warriors, but no rulers. Therefore they entrusted the administration of their empire to loyal subjects of the peoples they subjugated. Already in the first decade of the thirteenth century the Uighurs, a highly civilized Turkic people, became subordinates of the Mongols. The Uighurs had lived in present-day Xinjiang (China's westernmost province) for more than 300 years and had converted to Islam in the tenth century. In their submission to Chinggis Khan they saw the possibility of escaping the suppression exerted on them by the Western Liao dynasty. They soon became to play an important role in the Mongol administration and, since the Uighur alphabet was adopted for the Mongol language, also in education.

In 1218, after his first campaign in China, Chinggis Khan made Yelü Chucai (1189–1243), a Chinese statesman of Mongol descent, his personal advisor. Yelü was the most influential Chinese in the Mongol administration and accompanied Chinggis on all of his expeditions. When the Mongols planned the definite conquest of northern China in the early 1230s, it was Yelü who convinced Ögödei Khan that it would be more profitable to raise taxes from the Chinese population than to lay waste the whole country and reduce it to pastures for the horses.

In particular during the campaigns against the Southern Song dynasty in the 1260s and 1270s, tens of thousands of Muslims arrived in China. Many came voluntarily to pursue a career in the Mongol administration where they served, for instance, as tax collectors. Craftsmen, artists and scholars were forced to settle in China, where they left traces in such areas as pottery, poetry, music, architecture, medicine, astronomy, and military technology. Muslim merchants had frequented the Chinese trade centres on the Silk Road and on the coast before the Mongol conquest, but now benefited particularly from the fast and safe connections within the Mongol empire.

Among the Muslims in China were Uighurs and other people from territories bordering on China, but also, thanks to the good relations between the Yuan dynasty and the Ilkhanate, very large numbers of Iranians. As a result, Persian became one of the main languages of the Mongol administration in China. The Muslims mostly lived in separate quarters of the towns, where they were allowed to build mosques

1 More information on the Mongols and their conquests in the 13th century can be found, e.g., in [Spuler 1972]. The Persian history of the Mongols written in the late 13th century by the famous historian Rashīd al-Dīn was partially translated into English in [Boyle 1971].

and to practice their religion with only very few limitations. Tibetans also played an important role under Khubilai Khan and exerted a strong spiritual, Lamaistic influence on the Mongols.[2]

*2.2 Exchange of Astronomical Knowledge under the Mongols.* Within the historical context sketched above, the following contacts between Muslim and Chinese astronomers are known to have taken place:

- During the "western expedition" of Chinggis Khan (ca. 1220), Yelü Chucai learned of Islamic astronomical handbooks with tables, so-called *zījes*, and used some techniques from them to adjust the official Chinese calendar.[3]
- The Great Khan Möngke (1251–1259) had plans to build an observatory in the Mongol capital Karakorum. For this purpose he intended to make use of the service of a Muslim astronomer from Bukhara named Jamāl al-Dīn Muḥammad ibn Ṭāhir ibn Muḥammad al-Zaydī. It is very probable that this is the same person referred to as Zhamaluding in Chinese annals from the year 1267 onwards. Due to various problems Möngke's plans were never realized.[4]
- At the observatory of Maragha, founded in 1258 by Hülegü, a Chinese astronomer is known to have been active. It is plausible that he was largely responsible for the descriptions of the so-called Chinese-Uighur calendar found in many Persian *zījes* from the Mongol period onwards.
- In 1271 Khubilai Khan founded in his capital Beijing an Islamic Astronomical Bureau with an observatory that operated parallel to the official Chinese Astronomical Bureau, and whose first director was the above-mentioned Zhamaluding. Based on newly made observations, the Muslims active at the Bureau compiled a *zīj*, which is extant in a Chinese translation of 1383.

2 The role of the Muslims in Yuan China is discussed in various contributions to [Langlois 1981], in particular in M. Rossabi, *The Muslims in the Early Yüan Dynasty*, pp. 257–295. [Chen Yuan 1989] deals more in general with the foreigners in Mongol China and their heritage, whereas [Allsen 1983] treats specifically of the Uighurs. More information on Yelü Chucai can be found in [de Rachewiltz 1993: 136–175]. His report of the western expedition of Chinggis Khan with a polemic against Taoism was translated into English in [de Rachewiltz 1962].

3 More than 200 different *zījes* in Arabic, Persian and some other languages were written by Muslim astronomers from the 8th to the 19th centuries. The typical topics treated in these works include chronology, trigonometry and spherical astronomy, planetary longitudes and latitudes, eclipses, and mathematical astrology; in many cases we also find tables of geographical coordinates and stellar positions. Nearly all Islamic *zījes* were based on the geocentric, geometrical models for planetary motion as expounded by Ptolemy in his *Almagest*. The tables in *zījes* allow the calculation of planetary positions and the prediction of the times and magnitudes of solar and lunar eclipses by means of only very few simple arithmetical operations. The text in *zījes* consists of instructions for using the tables and, less often, explanations and proofs for the underlying models. [Kennedy 1956] offers a survey of Islamic *zījes*, an update of which is currently being prepared by the present author. See also [King & Samsó 2001] and the article "Zīḏj" in the *Encyclopaedia of Islam, new edition*.

4 Since we have no further information about the Muslim astronomers in the service of Möngke, this item is not treated in more detail in the following section. The full name of Jamāl al-Dīn is mentioned by the historian Rashīd al-Dīn; see also Chapter II, Section 10 of [Yamada 1980, in Japanese].

- In 1280 Guo Shoujing, one of the most famous Chinese scholars from the Yuan and Ming periods, completed the *Shoushili*, the new official astronomical system of the Mongol dynasty. Thanks to the presence of the Islamic Astronomical Bureau, it can be assumed that Guo Shoujing had access to Arabic and Persian sources, and Islamic influence has been suspected in the *Shoushili* as well as in the instruments built by him.

In the following section, the above-mentioned cases of scientific contact between Iranians and Chinese in the Mongol period will be discussed in more detail. Some methods for investigating the relationships between astronomical tables are described in the Appendix.

## 3. Contacts Between Iranian and Chinese Astronomers

In this section details will be given of the exchange of astronomical knowledge between Iranians and Chinese during the Mongol period. Explicit information on the relationships between the Islamic and Chinese astronomical tables and instruments that may have been involved in these exchanges can only rarely be found in astronomical or historical sources. Therefore a comparison of the technical characteristics of tables and instruments is indispensible. In particular for the analysis of the Chinese translation of an Islamic *zīj* called *Huihuili* (see Section 3.3 below), the methods described in the Appendix for investigating the relationships between astronomical tables were extensively used. References to publications giving more details of such analyses are also provided. In each case the history of the Mongol empire in the 13th century provides essential background information.

*3.1 Yelü Chucai's Adjustment of the Calendar of the Jin Dynasty.* When the Mongols conquered Beijing, the capital of Jin, in 1215 they also took over the official astronomical system of that dynasty, the *Revised Damingli* (lit. "Great Enlightenment System"). A Chinese astronomical system or "calendar," as the character *li* 曆 is also often translated, is a set of algorithms for calculating the positions of the sun, moon and planets, the times and magnitudes of eclipses, and various other astronomical quantities. Every Chinese dynasty since the Han (206 BC–AD 221) and, especially in later times, also many individual emperors promulgated their own official astronomical system which was compiled by the astronomers at the imperial Astronomical Bureau.

During Chinggis Khan's long expedition to Transoxiana around the year 1220, his advisor and astrologer / astronomer Yelü Chucai introduced a novelty for Chinese calendars by adjusting the calculations in the *Revised Damingli* for the difference in geographical longitude between mainland China and Samarkand. Sun Xiaochun [1998, in Chinese] has recently argued that he did this under the influence of Islamic *zījes*. It is in fact known from Chinese sources that Yelü Chucai became familiar with Islamic astronomy during his stay in Samarkand and that he highly appreciated the accuracy of the predictions of eclipses that were possible with tables from *zījes*.

According to one source he even wrote an astronomical work based on Ptolemaic methods himself, which was referred to as *Madabali* 麻答把曆, but is not extant.

Most Islamic as well as Chinese astronomical tables are designed for a specific locality, producing planetary positions and times and magnitudes of solar and lunar eclipses that are correct only for that locality. For instance, in the case of China the results of the calculations were valid for the capital only, since all official astronomical activity took place at the imperial court. The adjustment to a locality with a different geographical longitude is nothing more than the correction for the difference in local time. (Local time is defined in such a way that the so-called "mean sun," which moves with the same average velocity as the true sun but on the equator instead of on the ecliptic and with a uniform angular velocity instead of a variable one, always culminates precisely at noon.) Thus the planetary position calculated for noon at the base locality of a *zīj* is the position at 11 am at a locality 15° further west and at 1:30 pm at a locality 22°30′ further east. Similarly, if a lunar eclipse is predicted to begin at 3:30 am at the base locality of a *zīj*, it will begin at 2:30 am at a locality 15° further west and at 5 am at a locality 22°30′ further east.

The simplest way to adjust the planetary positions obtained from a *zīj* to a different geographical longitude, is to modify the mean positions. These are the linear functions of time from which the actual positions are found by applying a non-linear correction consisting of one or two so-called "equations." For each 15° west of the base locality, the mean motion in one hour should be added to the mean position at the base locality (since local noon occurs one hour later); for each 15° east of the base locality, the mean motion in one hour should be subtracted. After the adjusted mean position has been determined, the non-linear correction is applied in precisely the same way as before.

Many Islamic *zījes* provide with each mean motion table a small table for "the difference between the two longitudes" (*mā bayn al-ṭūlayn*) indicating for each geographical longitude the correction to be applied. If $\mu$ is the daily mean motion of a given planet and $\lambda_0$ the longitude of the base locality, then for each longitude $\lambda$ the tabulated value $\Delta$ is given by

$$\Delta(\lambda) = \frac{|\lambda - \lambda_0|}{360^\circ}\mu,$$

to be added to the mean position found from the table if $\lambda < \lambda_0$, otherwise subtracted (only incidentally were longitudes measured from a zero meridian in the east, in which case the two conditions should be interchanged).

Instead of mean motions, Chinese calendars make use of period relations and calculate the numbers of days between the beginnings of these periods and certain phenomena such as the winter or summer solstice and new or full moon. Since all quantities involved are expressed in days, the correction for a difference in geographical longitude can also be expressed in days and hence is the same for each calculation.

The reworking of the *Revised Damingli* prepared by Yelü Chucai is entitled *Western Expedition Calendar for the Epoch Year* Gengwu, and is extant as Chapters 56–57 of the *Yuanshi*, the official annals of the Yuan dynasty. The epoch mentioned in the title, the starting point of all calculations of planetary motions in this calendar, corresponds to the year AD 1210. Besides taking Samarkand as its base locality, Yelü Chucai's reworking also introduces a concept called *licha* 里差 (lit. "*li* difference"; during the Song and Yuan dynasties a *li* was equal to 441 m). The *licha* is a correction to be applied for localities different from Samarkand and is calculated as $0.04359 \cdot \Delta L$, where $\Delta L$ is the (east-west) distance from Samarkand in *li*s [cf. Sun Xiaochun 1998: 4]. The resulting number is the difference in local time expressed in units of which 5230 equal one day.[5] Thus in order to obtain the time difference in hours, the resulting number must still be divided by 5230/24. Finally, it is added to the local time at Samarkand for localities further east, and subtracted for localities to the west, to obtain the local time of the desired astronomical phenomenon.

Using information from an explanatory work by Yelü Chucai, it is possible to deduce the geographical data used by him to arrive at the algorithm for the *licha* given above. He writes that at Samarkand he observed a partial lunar eclipse approximately 2.6 hours ahead of the time predicted by the *Revised Damingli* for the former Song capital and important astronomical centre Kaifeng (this time was converted; the actual report uses so-called "watches" of the night). By calculating backwards with the algorithm for the *licha* we find that this corresponds almost exactly to a distance of 13,000 *li*s between Kaifeng and Samarkand. Sun Xiaochun notes that this distance is too large by a factor of roughly 1.4 and suggests a possible relationship with the data in Ptolemy's *Geography* (and hence in Islamic geographical tables), in which the longitude difference between Samarkand and cities in China is also around 1.4 times too large. However, there are various uncertainties in his analysis, such as the precise length of the *li* during the Yuan dynasty and the identification of localities in China in Ptolemy's *Geography*. It is also unclear how the *licha* should be calculated for a locality with a completely different latitude from Samarkand, since it is not explicitly specified in Yelü Chucai's reworking of the *Revised Damingli* that the east-west distance in *li*s must be used. Nevertheless, it is safe to conclude that Yelü Chucai was inspired by Islamic examples when he implemented the *licha*; however, in doing so he stayed completely within the traditional Chinese framework.

*3.2 The Chinese-Uighur Calendar.* Hülegü Khan made Maragha in northwestern Iran the capital of his newly founded Ilkhanate. In 1259, on the instigation of the famous polymath Naṣīr al-Dīn al-Ṭūsī, he had an astronomical observatory built on a hill just outside of the city. A comprehensive observational program was planned to last at least 12 years (the time that Jupiter takes to complete one revolution around the sun), but the *Īlkhānī Zīj* completed by al-Ṭūsī shortly after 1270 did not yet contain the results of these observations, which were only incorporated in a later work by Muḥyī al-Dīn al-Maghribī.[6]

5 The use of such units is a characteristic of Chinese calendars. The base periods, such as the solar year and the lunar month, are all expressed as fractions with a common denominator, which is called *rifa* 日法, i.e., "day divisor," in this case 5230. The day divisor is found by solving a set of linear congruence relations and is typical for each calendar.

6 Information on many aspects of the observatory in Maragha (construction, instruments, astronomers and their works, financial administration, instruction) can be found in [Sayılı 1960,

From references by a contemporary astronomer and a somewhat later historian we know that a Chinese scholar Fu Mengchi or Fu Muzhai was active at the observatory in Maragha.[7] It seems probable that he was the main source for the information on the so-called "Chinese-Uighur calendar" that was included in many Persian *zījes* from the Mongol period. This calendar was used by the Ilkhans for almost a century and has left traces on modern Iranian almanacs in the form of the use of the Chinese duodecimal animal cycle. The Chinese-Uighur calendar was a lunisolar calendar of standard Chinese type with some elements that were more commonly found in unofficial Chinese calendars. Below follows a brief description of its characteristics with an attempt to trace these back to particular Chinese calendars. A complete description of the technical treatment of the Chinese-Uighur calendar in the *Īlkhānī Zīj* can be found in [van Dalen, Kennedy & Saiyid 1997]; its use in Iranian historical sources from the Mongol period is discussed in [Melville 1994].

The Chinese-Uighur calendar has its basic characteristics in common with the official Chinese calendars of the Song and Jin dynasties (11th to 13th centuries). As in every Chinese calendar, each month starts with the day of new moon and hence lasts 29 or 30 days. In determining the day of new moon, first the time of the *mean* new moon is calculated on the basis of the average length of the lunar month; in the Chinese-Uighur calendar this length is taken to be 29.5306 days. To obtain the time of the *true* new moon, a correction has to be applied to the time of the mean new moon; in the Chinese-Uighur calendar this correction consists of two periodic components, the solar equation and the lunar equation, with maximum values of 0.1840 days and 0.3844 days respectively. The period of the solar equation is equal to the solar year (see below), but that of the lunar equation, called the anomalistic month, is somewhat smaller than a lunar month, namely 27.5546 days. The names of the months are given by means of Turkish numerals as well as in transliterations of the Chinese.

The solar year in the Chinese-Uighur calendar starts with the passage of the sun through the midpoint of the zodiacal sign Aquarius and is of length 365.2436 days. It is divided into 24 equal parts, the so-called *qi* 氣, whose transliterated Chinese names are given in the *Īlkhānī Zīj* and other Persian works. The beginning of a Chinese-Uighur year (i.e., the actual lunisolar year) is the day of the (true) new moon immediately preceding the entrance of the sun into the sign Pisces. (This implies that the beginning of the lunisolar year precedes the beginning of the solar year in approximately half of the cases.) An ordinary year consists of twelve lunar months (354 or 355 days), but to stay in pace with the solar year a leap month is inserted every second or third year (close to seven times in each period of nineteen

Chapter 6]. The description by Mu'ayyad al-Dīn al-ʿUrḍī of the instruments available at the observatory was translated into German by Seemann [1928]. An extant notebook by al-Maghribī lists some of the observations made at Maragha and shows how new planetary parameters were derived from them; see [Saliba 1983].

7 The Chinese name in Persian transliteration is mentioned by al-ʿUrḍī in a Tehran manuscript of the work indicated in footnote 6 which was not used by Seemann, and the historian Banākatī, who adds the honorific *Sīng sīng* (for Chinese *xiansheng* 先生, "professor," translated as *ʿārif* "sage"); see [Boyle 1963: 253, n. 4].

years). The place of the leap month within a Chinese-Uighur leap year is not fixed; it is the month that contains the initial point of only one of the 24 divisions of the solar year, whereas all other months contain the beginnings of two such divisions.

Chinese calendars traditionally make use of a Grand Conjunction epoch (Chinese: *shangyuan* 上元) in the far past. In the case of the Chinese-Uighur calendar this epoch is said to fall 88,639,679 years before the accession to the throne by Chinggis Khan in AD 1203. The years since the epoch are reckoned in *wan* 萬, i.e., tenthousands. However, different from many official Chinese calendars, all practical calculations in the Chinese-Uighur calendar are carried out on the basis of a sexagesimal cycle of years (the cycle that is mostly used in the *Īlkhānī Zīj* starts in the year 1264). Note that the Chinese sexagesimal cycle was obtained by combining the duodecimal cycle of earthly branches or animals with the decimal cycle of heavenly stems. It was also used for counting the days in a way similar to the use of the days of the week.

Finally, the descriptions of the Chinese-Uighur calendar in Persian sources state that a day was divided into 12 double-hours, the first of which started at 11 pm, and that each double-hour was further divided into 8 quarters (Chinese *ke* 刻; traditionally a *ke* was defined as a hundredth of a day and hence slightly shorter than the *ke* of 15 minutes used in the Chinese-Uighur calendar).

On the basis of the characteristics of the Chinese-Uighur calendar listed above we may now try to draw conclusions concerning its origin. The general characteristics, such as the beginning of the month and year, the use of a correction to determine the true new moon from the mean new moon, and the insertion of the leap month, are common to most contemporary Chinese calendars. The length of the lunar month is correct to tenthousandths; most Chinese and Islamic astronomical works give this parameter with a higher precision. The lengths of the anomalistic month and the solar year agree with values found in various official Chinese calendars from the Song and Jin dynasties (11th to 13th centuries). The only characteristic that can with certainty be associated with one particular Chinese calendar is the Grand Conjunction epoch, which agrees precisely with the epoch of the *Revised Damingli* of the Jin dynasty, but not with any other known calendar. It thus seems probable that the Chinese-Uighur calendar as described in the *Īlkhānī Zīj* was dependent on the *Revised Damingli*.

However, the Chinese-Uighur calendar also has a number of characteristics that are rather atypical for official Chinese calendars. These are: the use of tenthousandths of a day instead of values to a larger number of decimal places or fractions with a common denominator (cf. footnote 5); the use of parabolas for the solar and lunar equations instead of linear or quadratic interpolation between observed values; the use of the Babylonian approximate value $\frac{248}{9} \approx 27.5556$ days for the length of the anomalistic month in the calculation of the lunar equation; the use of a recent epoch for all practical computations instead of the Grand Conjunction epoch. Such characteristics are more typical for the unofficial Chinese calendars that were made by various private scholars not connected to the imperial Astronomical Bureau.

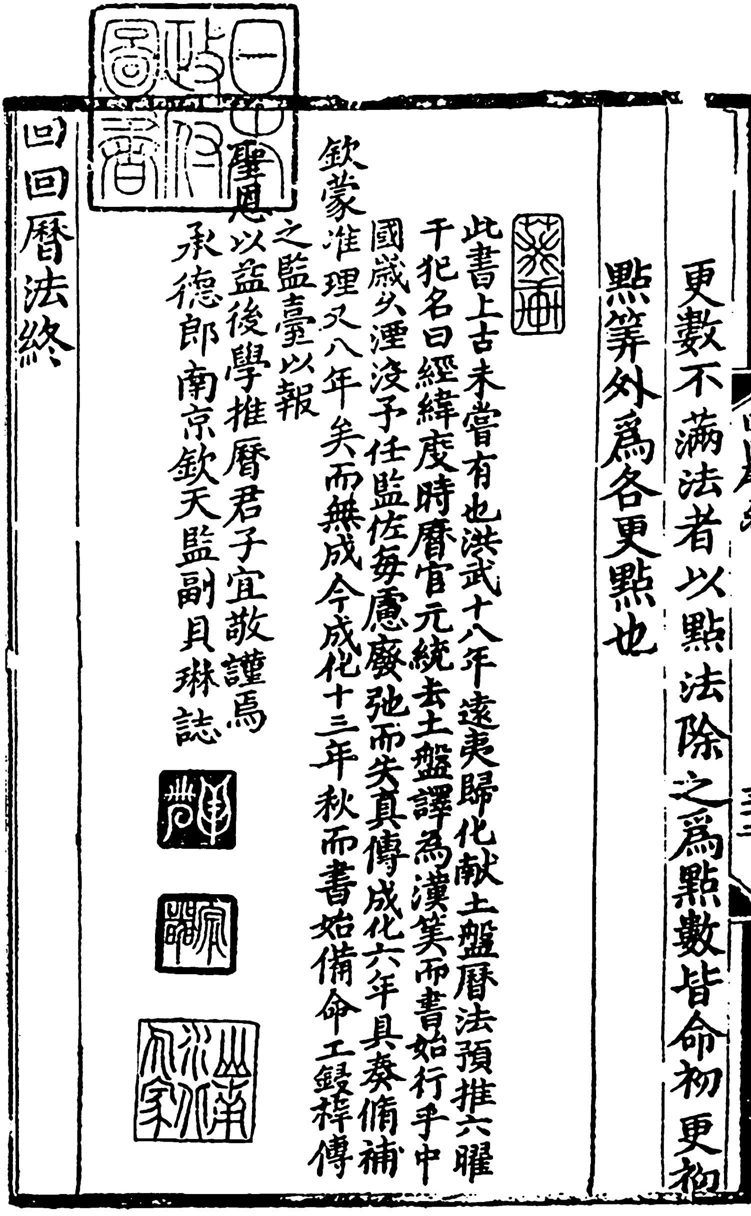
Most unofficial Chinese calendars are lost, but there is one calendar in particular that is known to have had all four characteristics mentioned above. This was the

*Futianli* 符天曆 ("Heavenly Agreement System") compiled by Cao Zhiwei 曹士蔿 around the year 780 (see [Yabuuti 1982], in Japanese). It was used for the astronomy examinations at the Imperial Academy for several centuries and is extant in a small fragment in Nara (Japan). Some Chinese sources indicate that Cao Zhiwei lived in the western part of China, i.e., in or near the Uighur empire that flourished in the same period. It is therefore tempting to conjecture that the above-mentioned characteristics of the Chinese-Uighur calendar that stem from unofficial Chinese calendars derive from the original calendar of the Uighurs. In that case the Chinese-Uighur calendar as described in Persian *zījes* from the Mongol period would be a mixture of the *Revised Damingli* of the Jin dynasty, adopted by the Mongols when they captured Beijing in 1215, and the calendar of the Uighurs, which undoubtedly came to play a role in the administration of the Mongols at the time when the Uighurs entered their service around the year 1208.

*3.3 The Huihuili ("Islamic Astronomical System").* In 1271 the Mongol Great Khan Khubilai founded an Islamic Astronomical Bureau in the Yuan capital of Beijing and appointed the Muslim scholar Zhamaluding 札馬魯丁 as its first director. Presumably this Zhamaluding was the same person as the Jamāl al-Dīn Muḥammad ibn Ṭāhir ibn Muḥammad al-Zaydī from Bukhara who, according to the famous historian Rashīd al-Dīn, served Khubilai's brother Möngke in the Mongol capital Karakorum in the 1250s. In 1267 Zhamaluding had presented to Khubilai Khan a *zīj* as well as models or diagrams of seven astronomical instruments of Islamic type (cf. Section 3.4 below). The Islamic Astronomical Bureau operated parallel to the Chinese Bureau and had a staff of around 40 people including scholars, teachers and administrative personnel.[8] It is known that the Bureau remained in existence until the early Qing dynasty (17th century), but there are very few direct records of its activities. However, the extent of the observational program that was carried out at the Bureau in the early Yuan dynasty can be judged indirectly from two sources dating from the last third of the 14th century, namely:

- The *Huihuili* 回回曆 (lit. "Islamic Astronomical System"), a Chinese translation prepared in Nanjing in AD 1383 of a Persian *zīj* that was available at the Islamic Astronomical Bureau in Beijing at the time when the Yuan dynasty was defeated by its successor, the Ming (1368–1644). The original version of the *Huihuili* does not seem to be extant. A restoration made in the year 1477 by Bei Lin 貝琳, vice-director of the Astronomical Bureau of the Ming dynasty in Nanjing, is available in the National Library of China in Beijing (without tables) and in the National Archives of Japan in Tokyo (complete); it is nowadays easily accessible in the facsimile edition of the *Sikuquanshu*, the enormous collection of literary works produced under the Qing emperor Qian Long in the late 18th century. The copy of the *Huihuili* in the *Mingshi*, the official historical annals of the Ming dynasty, is an abridged version of Bei Lin's restoration which lacks

8 More information on the Islamic Astronomical Bureau of the Yuan dynasty can be found in [Yabuuti 1954, 1987, 1997; Yamada 1980 (in Japanese)]. The primary sources availabe for an investigation of the achievements of the Muslim astronomers in Yuan China are discussed in more detail in [van Dalen 2002].



更數不滿法者以點法除之爲點數皆命初更初
點筭外爲各更點也

此書上古未嘗有也洪武十八年遠夷歸化獻土盤曆法預推六曜
干犯名曰經緯度時曆官元統去土盤譯爲漢筭而書始行乎中
國歲久湮沒予任監佐每慮廢弛而失眞傳成化六年具奏脩補
欽蒙准理又八年矣而無成今成化十三年秋而書始備命工鋟梓傳
之監臺以報
恩以益後學推曆君子宜敬謹焉
承德郎南京欽天監副貝琳誌

回回曆法終

**Plate 1.** Colophon of the copy of the *Huihuili* in the National Archives of Japan, Cabinet Library (Naikaku Bunko) in Tokyo.

various important tables. More useful is a Korean reworking of the original translation that was prepared on the order of King Sejong in 1442 and contains accurate copies of the tables, whereas the text was adjusted for use in Seoul.[9]

- The *Sanjufīnī Zīj*, the Arabic astronomical handbook by a certain al-Sanjufīnī, written in 1366 for the Mongol viceroy of Tibet. This work is extant in the unique manuscript arabe 6040 of the Bibliothèque Nationale de France in Paris. It contains librarian's notes in Chinese, Tibetan transliterations of month-names, and Mongolian translations of the titles of tables.[10]

I have applied various of the methods described in the Appendix to investigate the relationship between the *Huihuili* and the *Sanjufīnī Zīj*. In the first place, it turned out that almost twenty tables, mostly intended for the determination of planetary positions, are identical in the two works. Of some other tables that have a completely different structure in the two sources, it can be shown that they are based on the same planetary parameters: in the case of the planetary mean motions, the tables in the *Sanjufīnī Zīj* were derived from those in the *Huihuili*, whereas for the planetary latitudes it is the other way around.[11] We can thus conclude that the *Huihuili* and the *Sanjufīnī Zīj* had a common ancestor.

A comparison of the planetary parameters underlying the tables in the *Huihuili* and the *Sanjufīnī Zīj* with those from Arabic and Persian astronomical works included in a parameter file started by E.S. Kennedy (cf. the Appendix), showed that the former do not occur in any other known *zīj*, and in particular not in the *zīj*es resulting from the contemporary activities at the observatory in Maragha, namely the *Īlkhānī Zīj* by Naṣīr al-Dīn al-Ṭūsī and the *Adwār al-anwār* by Muḥyī 'l-Dīn al-Maghribī. Also the setup of various tables in the *Huihuili* and the *Sanjufīnī Zīj* is different from what we find in other *zīj*es. We may therefore conclude that the common ancestor of the two works constitutes a highly original compilation based on an extensive observational program with determination of new planetary parameters and various innovations in the presentation of the tabular material.

That this compilation was produced at the Islamic Astronomical Bureau of the Yuan dynasty is plausible for a number of reasons. Firstly, both the *Huihuili* and the *Sanjufīnī Zīj* were written in China and, as was remarked above, we know of no Islamic astronomical work from outside of China that is based on the same parameters. Secondly, the tables for planetary mean motion in the *Sanjufīnī Zīj* are said to be based on the "observations of Jamāl," which may refer to Zhamaluding.

9 An edition and translation of the text of the *Huihuili* with commentary and transcription of the tables is currently being prepared by the present author. The technical contents of the work were outlined in [Yabuuti 1997] and discussed in some more detail in [Chen Jiujin 1996, in Chinese]. The Korean version of the *Huihuili* was studied by Shi Yunli (to appear in *Archive for History of Exact Sciences*).

10 Philological and historical aspects of the *Sanjufīnī Zīj* were explored in [Franke 1988]; the technical material on eclipses, parallax and lunar visibility was analysed in [Kennedy 1987/88; Kennedy & Hogendijk 1988].

11 The tables for planetary latitude in the *Huihuili* were described and compared with the tables in other *zīj*es in [Yano 1999; van Dalen 1999]. The table for Mercury's equation of centre discussed in the Appendix is investigated in [Yano 2002].

Finally, the Oriental Institute in St. Petersburg is in the possession of a Persian astronomical manuscript with tables numerically identical to those in the *Huihuili*. This manuscript had been obtained in China and was described by A. Wagner [1882] when it was still in the library of the Pulkovo Observatory near St. Petersburg. On paleographical grounds it has been variously dated between the 12th and 13th centuries, and a preliminary investigation indicates that it was a working copy for the Chinese translation prepared in 1383. It contains tables for Beijing that must have stemmed from the original work and were not included in the translation, as well as tables for Nanjing that must have been additions by the translators.

Although the *Huihuili* and the *Sanjufīnī Zīj* are standard Islamic *zīj*es based on Ptolemy's geometrical models for planetary motion, they both show distinguishable Chinese influences. In the case of the *Huihuili* these influences probably stem from the translators in the early Ming dynasty and can be recognized in the following topics:

- The chronological material at the beginning of the work was reformulated with the use of Chinese terminology and methods. Thus the number of years since epoch is called "accumulated years" (*jinian* 積年) and all mean positions are first determined for the vernal equinox rather than directly for the desired date in the Islamic Hijra calendar. Also a rule for the determination of the leap month in the Chinese calendar and the confusion of lunar and solar years that led to the use of a Hijra epoch in AD 599 instead of the correct AD 622 must have originated with the Ming translators.
- The star catalogue in the *Huihuili* lists ecliptical longitudes and latitudes as well as magnitudes for 277 fixed stars. The catalogue is of particular interest, since it is one of only two larger Islamic star tables that were based on new observations rather than having been derived directly from the star catalogue in Ptolemy's *Almagest* (cf. [van Dalen 2000]). Furthermore, it is the earliest table to give a correspondence between Ptolemaic and Chinese star-names. The star table in the *Huihuili* is mainly used for the calculation of so-called "encroachments" (*lingfan* 凌犯), passages of the moon and the planets through stellar constellations, an important topic in Chinese astrology. Whereas traditionally encroachments were observed in order to be interpreted as omens, the *Huihuili* provides the additional possibility of calculating them.

In the case of the *Sanjufīnī Zīj* the Chinese influences must have been part of the original work, since the only extant manuscript is an autograph. They include:

- The use of an epoch named after Chinggis Khan, namely the vernal equinox of the year AD 1207.
- A table of the 24 equal divisions of the solar year, the so-called *qi* (cf. Section 3.2 above), with Persian transliterations of their Chinese names.
- A table of the Arabic lunar mansions with longitudes and latitudes for some of the individual stars and Persian transliterations of the names of the corresponding Chinese mansions. In one or two cases the listed stars are not the standard Arabic ones and may have been influenced by Chinese constellations (cf. [van Dalen 2000: 150–151]).

*3.4 Guo Shoujing's Astronomical Instruments and the* Shoushili. In 1276, the famous scholar Guo Shoujing, who had previously been involved in both canal and irrigation projects, was assigned to the task of devising a new official astronomical system for the Mongolian Yuan dynasty. For this purpose he first designed a large number of astronomical instruments and carried out systematic observations of winter and summer solstices, positions of planets and stars, etc. In 1280 the new calendar was finally completed and starting from 1281 it was distributed under the name *Shoushili* (lit. "Season-Granting System"). It was by far the most accurate calendar in the history of Chinese mathematical astronomy and continued to be used for almost 400 years (the calendar of the Ming dynasty was called *Datongli*, lit. "Great Concordance System," but differed only insignificantly from the *Shoushili*).

There has been a continuing debate about to what extent Guo Shoujing and his colleagues were influenced by Islamic astronomy. Thanks to the presence of the Islamic Astronomical Bureau in the Yuan capital, the official Chinese astronomers must have had ample opportunity to familiarize themselves with Islamic astronomical instruments and tables. It seems possible that Guo Shoujing consulted Zhamaluding, the director of the Islamic Bureau, and other Muslim astronomers in person. Moreover, eleven Islamic instruments and many Arabic and Persian books that were present at the Bureau were described in official Chinese sources. Seven of these instruments were presented by Zhamaluding to Khubilai Khan in 1267; four more were kept by him at his residence. Among them were various typically Islamic instruments, such as a parallactic ruler, a terrestrial globe, a plane and a spherical astrolabe with ecliptic rings, and a compass. Among the books whose transliterated Persian titles are listed in Chinese annals, we find Ptolemy's *Almagest*, Euclid's *Elements*, al-Ṣūfī's *Constellations of the Stars*, Kūshyār's *Introduction to Astrology*, one or more *zījes*, and works on topics such as cosmology (*ha'ya*), the construction of instruments, chronology, geometry, arithmetic, and technical devices (*ḥiyal*).[12]

However, there is hardly any information in Chinese sources about the extent to which Chinese astronomers made actual use of the available Islamic knowledge. It is known that during the Qing dynasty there was a harsh competition between the Islamic and the Chinese Astronomical Bureaus (some articles about this topic were published in Chinese by Huang Yilong). For the Yuan and Ming dynasties we have to resort to investigations of the extant calendar works and descriptions of instruments or, in some cases, the instruments of Guo Shoujing themselves. A comprehensive analysis of these sources is beyond the scope of this article; instead I will limit myself to some interesting points put forward, in particular, by Needham, Yabuuti and Miyajima. More information can be found in the works mentioned in footnote 12 and [Yamada 1980]. Recent researches on the *Shoushili* include those by Jing Bing and Wang Rongbin published in Chinese.

A frequently quoted statement concerning the influence of Islamic astronomy on the work of Guo Shoujing is found in the introduction of [Sédillot 1847], the

12 The transliterations of Persian names of instruments, book titles, month-names and the days of the week that are found in Chinese sources, were discussed extensively in [Tasaka 1957]. [Hartner 1950; Miyajima 1982] dealt specifically with the names of the instruments presented to Khubilai Khan by Zhamaluding.

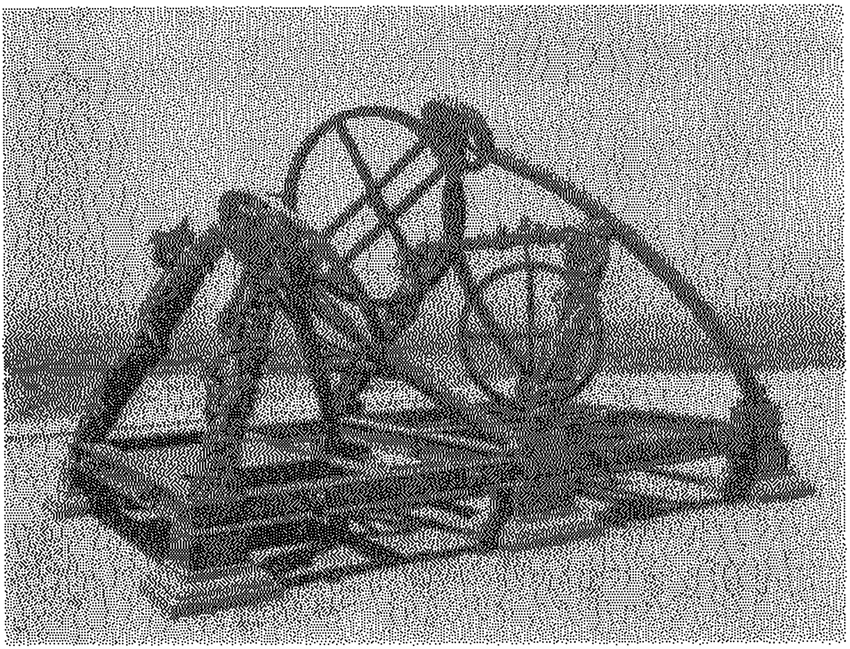


**Plate 2.** The Simplified Instrument of Guo Shoujing at the Purple Mountain Observatory outside of Nanjing (photograph taken from K. Yamada, *Juji-reki no michi*, Tokyo 1980).

edition of the Persian text of the *zīj* of Ulugh Beg. On page *ci*, Sédillot states that "in 1280 Guo Shoujing received the *zīj* of Ibn Yūnus from Zhamaluding and studied it in detail." In [Sédillot 1845–1849, 2: 484, 640, 642], he adds that Guo Shoujing wrote the *Shoushili* together with Zhamaluding, that Arabic scientific treatises were translated into Chinese at the time, and that Guo Shoujing was the first Chinese to study spherical astronomy. As a matter of fact, not a single of these statements can be proved and most of them are simply wrong; they are all distorted versions of translated passages from Chinese historical annals and their interpretations as presented in [Souciet 1729–32].

In the 20th century more reliable studies concerning the interaction between Islamic and Chinese scholars in the Yuan and Ming dynasties were made and the most important results were summarized in [Needham & Wang 1959: 294–302, 367–382; Yabuuti 1997: 14–17]. As far as the instruments of Guo Shoujing are concerned, both authors give room to the possibility that they were to some extent influenced by Islamic instruments. For instance, they assume that the so-called "Tower of the Duke of Zhou," a masonry tower in Gaocheng near Luoyang in the province Henan with a shadow scale and 40 ft. gnomon which is now lost, was influenced by the giant instruments built by, e.g., al-Khujandī (d. AD 1000) near Rayy in Iran. Furthermore, the 6-meter "Simplified Instrument" (*jianyi* 簡儀, see Plate 2) with its equatorial alignment, projections of the heavenly circles onto three different planes and alidades instead of sighting tubes, has similarities to the torquetum, whose invention has been attributed to Jābir ibn Aflaḥ (12th-century Spain, cf. [Lorch 1976]; note, however, that the torquetum was not among the instruments used in Maragha or those brought to China by Zhamaluding). Finally, the "Upward-Looking Instrument" (*yangyi* 仰儀), which consists of a concave hemisphere with a grid and a pointer and was not known in pre-Mongol China, is very similar to the scaphe sundial that had been common in ancient Greece and the Islamic world and is still found among the Islamic-style instruments built by Jai Singh in 18th-century India. Thus in each of these three cases there is a possibility of Islamic influence.

Recently, in his detailed study of the descriptions in the Yuan annals of the instruments brought to China by Zhamaluding, Miyajima [1982, in Japanese] put the Islamic influence on Guo Shoujing into perspective by noting that, in spite of their similar appearance, the usage of the Islamic and Chinese instruments mentioned above is in fact quite different. Thus the Tower of the Duke of Zhou measures the shadow cast by a gnomon (i.e., wooden post) on a horizontal plate, whereas with the giant Islamic instruments like that of al-Khujandī the sunlight falls through an aperture onto a scale on a concave circular arc. Furthermore, except for the equatorial alignment of the Simplified Instrument, its structure and its scales are quite different from that of a torquetum. Finally, the Upward-Looking Instrument of Guo Shoujing was used for measuring positions on the heavenly sphere, whereas the Greek and Islamic scaphe sundials were designed to measure the time of day.

It is also difficult to point out precisely the Islamic influence in Guo Shoujing's astronomical system, the *Shoushili*.[13] The use of an epoch in the recent past instead of a Grand Conjunction epoch millions of years ago, as well as the consistent use of decimal numbers in all parameters and calculations have been related to Muslim practices, but are also found in unofficial Chinese calendars (besides, Islamic astronomical works make use of sexagesimals much more than decimals).

The same holds for the use of cubic corrections for the motions of the planets. As was indicated above for the Chinese-Uighur calendar, the actual position of the sun, moon and planets is calculated, both in Ptolemaic and in traditional Chinese astronomy, by applying a periodic correction to a linear function of time. In Chinese calendars before the *Shoushili* these corrections were parabolic functions calculated by means of first-order differences. However, in the *Shoushili* cubic functions were applied; for example, for the sun the correction $q$ is given by:

$$q = 0.051332d - 0.000246d^2 - 0.00000031d^3$$

for days $d$ reckoned from the the winter solstice, and by

$$q = 0.048706d' - 0.000221d'^2 - 0.00000027d'^3$$

13 The canons of the *Shoushili* are contained in Chapters 54–55 of the *Yuanshi*, an evaluation by a government committee in Chapters 52–53. A summary of the technical contents of the work can be found in [Nakayama 1969: 123–150]. I am grateful to Prof. Nathan Sivin for letting me use his unpublished translation of, and commentary on, the *Shoushili*, as well as for his useful comments on a preliminary version of this section.

for days $d'$ reckoned from the summer solstice. Here $q$ is expressed in Chinese degrees equal to a 1/365.25th of a complete apparent rotation of the sun around the earth (i.e., one Chinese degree corresponds to the average daily solar motion). The actual calculations were performed by means of a table with second-order differences of the corrections. Since in Ptolemaic astronomy all corrections to the linear mean motions of the planets were trigonometric functions (e.g., the correction for the sun, the so-called "solar equation," is given by

$$q = \arcsin\left(\frac{e \cdot \sin \bar{a}}{60 + e \cdot \cos \bar{a}}\right),$$

where $\bar{a}$ is a linear function of time and $e$ is the eccentricity of the circle of uniform motion of the sun with respect to the earth), there is once more no clear indication of transmission. It seems rather that the planetary corrections in the *Shoushili* presented a further development of those in earlier Chinese calendars with at most a very subtle influence of Islamic knowledge.

Because of the equatorial character of Chinese astronomy (most measurements were made with respect to the north pole) and the fact that the planets move on or near the ecliptic, conversion of ecliptical coordinates into equatorial ones and vice versa was essential in Chinese planetary theory. Since the Chinese astronomical systems did not make use of any form of geometry and of only little trigonometry, this conversion had to be approximated rather than carried out exactly. In calendars from the Tang dynasty (AD 618–905) this was done by means of a linear step function. During the Song dynasty we find a more refined numerical approximation of parabolic type. Guo Shoujing, finally, made a first step towards a geometric method, although he made use only of plane triangles, not of spherical ones. The resulting formula, based on the "arc-chord-sagitta relationship" formulated by the Song scholar Shen Gua (see [Nakayama 1969: 137–139]), is only in some parts of the equator and ecliptic more accurate than the earlier quadratic method.

It seems possible that Guo Shoujing's step towards geometry in the conversion between ecliptical and equatorial coordinates was influenced by Islamic astronomy. On the other hand, both the actual method of conversion and the implementation of the necessary calculations by means of a table with first order differences fall clearly within the traditional Chinese framework. In no instance in the *Shoushili* do we find ready Islamic results or tables.

## 4. Conclusion

In this article we have studied various aspects of the transmission of astronomical knowledge between the Islamic world and China under Mongol rule. We have sketched the historical background of the interaction between Muslim and Chinese scholars in the 13th century and have thus made clear that the possibilities for contacts were numerous, and that the transmission took place by means of personal transfer rather than through written works or instruments brought by, for instance, merchants. Whereas it seems that the number of Chinese in Ilkhanid Iran was rather small, in the last third of the 13th century tens of thousands of Iranians, Uighurs and other Muslims were active in Mongol China in administrative functions and also as scholars and artists.

We have investigated four particular cases of transmission in some detail.

- Around the year 1220, Yelü Chucai, advisor of Chinggis Khan, adapted the official Chinese calendar of the Jin dynasty for use in Transoxiana and at arbitrary geographical localities (Section 3.1 above). Yelü is known to have familiarized himself with Islamic astronomy and probably carried out the adaptation under the influence of Islamic *zījes*.
- The Ilkhans introduced into Iran the so-called "Chinese-Uighur calendar" (Section 3.2). This was a typical Chinese calendar whose elements can be traced back to the calendar of the Jin dynasty as well as to unofficial Chinese calendars. Its descriptions in Persian *zījes* from the year 1270 onwards presumably stemmed from a Chinese astronomer active at the famous observatory in Maragha.
- In AD 1383 the first emperor of the Ming dynasty had some Persian astronomical works translated into Chinese that were available at the Islamic observatory of the conquered Yuan capital Beijing. These works included Kūshyār's *Introduction to Astrology* and a *zīj* with tables and parameter values not known from any other Islamic work (Section 3.3). The *zīj* can be assumed to have been an original achievement of the Muslim astronomers active in China around the year 1275 and is based on completely new planetary and stellar observations.
- Guo Shoujing finished the most accurate calendar in the history of Chinese mathematical astronomy, the *Shoushili*, in AD 1280 (Section 3.4). Islamic influence has been suspected in this calendar as well as in Guo's instruments, but closer inspection reveals that this influence was very minor and that calendar and instruments stand clearly in the Chinese tradition.

The transmissions in each of these four cases were of a very different nature. Yelü Chucai visited the Islamic world himself and must have had ample opportunity to learn Persian and to become acquainted with Islamic astronomy through written works as well as personal contacts (even in cities where the Mongols slaughtered the population, they spared the lives of scholars and artisans). The Chinese astronomer active at the observatory in Maragha seems to have been isolated and was probably the only source of information on the Chinese-Uighur calendar that was available to his Arabic and Iranian colleagues. On the other hand, a relatively large number of Muslim scholars worked at the Islamic Astronomical Bureau of the early Yuan dynasty. They must have brought with them from Iran and other regions many Arabic and Persian works on mathematics and astronomy, but in China they compiled a completely new *zīj* based on an extensive program of observations. The same Muslim astronomers and the works they had brought with them may have influenced Guo Shoujing and his colleagues when they compiled the *Shoushili* and built various new instruments; it seems probable, though, that these contacts had to take place through interpreters and hence would have been quite problematic.

Also as far as the assimilation of the transmitted astronomical knowledge is concerned, there are clear differences between the four cases. Both Yelü Chucai in his adaptation of the Jin calendar and Guo Shoujing in the *Shoushili* made use of some Islamic concepts or ideas such as the adjustment of the times of phenomena to the local geographical longitude and the use of geometrical / trigonometrical methods for the conversion between ecliptical and equatorial coordinates. However, neither Chinese astronomer took over actual algorithms or tables from Islamic *zījes* and the resulting calendars fitted in almost every respect within the Chinese tradition. The Chinese translation of the *zīj* compiled by the Muslim astronomers in Yuan China basically left the Islamic (i.e., Ptolemaic) character of the work unchanged, although Chinese concepts were introduced to make certain topics easier to understand for Chinese astronomers. In particular the star table with ecliptical coordinates, displaying Ptolemaic as well as Chinese star names, and its application to the typically Chinese problem of "encroachments," may have been the result of a joint effort of Muslim and Chinese astronomers. Similarly, the descriptions of the Chinese-Uighur calendar in Persian *zījes* contain little Islamic influence. The algorithms are of Chinese type and the technical terminology is given almost exclusively in transliterations of the Chinese. As far as the numerous auxiliary tables are concerned, some of them, such as the solar and lunar equations, may have been present in a Chinese original, whereas most others display simple multiples of the base parameters of the calendar and could have been added by the Muslim authors.

Some mathematical methods for investigating the relationships between astronomical tables are described in an Appendix to this article. Since the numerical data in such tables are highly idiosyncratic, connections between them can usually be reliably established. In cases where tables from different cultures and / or periods can be shown to be connected, we may have found a case of transmission. The situation is much more difficult for computational algorithms and instruments. In various examples discussed briefly in this article (in particular, the calendar and instruments of Guo Shoujing) it is clear that at least complete descriptions of structure as well as usage should be considered in order to be able to make well-founded statements concerning possible connections.

## Appendix: Methods for Investigating Relationships Between Astronomical Tables

In many cases mathematical tables in manuscripts of *zījes* are not clearly attributed to a particular astronomer. Therefore, in my analyses of the connections between astronomical works from different periods and / or geographical regions I have made an extensive use of comparisons of numerical data in tables. Such comparisons have turned out to be very effective for investigating relationships, because the data are generally exact and highly idiosyncratic (two tables for the same function need only have relatively small differences to be able to conclude that they stem from two different sources). Since data in astronomical tables generally show obvious regularities (usually because of the continuity of the tabulated functions), incidental scribal errors can often be reliably corrected. Systematic differences between two tables for the same function can have a number of causes, such as the use of different values for the underlying astronomical parameters or the use of different computational techniques (e.g., approximate algorithms instead of the "exact formula," computation from inaccurate auxiliary tables, use of linear or quadratic interpolation in auxiliary tables, the rounding of intermediate results to a certain number of digits, etc.). Incidental differences between two tables for the same function that cannot be explained as scribal errors may be due to computational mistakes.

Three examples of methods that can be used to investigate the relationships between astronomical tables are presented below. Statistical or numerical tools for performing more sophisticated tasks, such as estimating the parameter values underlying an astronomical table and testing the dependence of two tables for the same or for different functions, are not discussed in detail but are referred to where appropriate.

*Case 1: Comparison of Complete Tables.* The simplest way to investigate the relationship between two tables for a particular astronomical function is to compare them value by value. If all values are identical, we may assume that the tables ultimately derive from the same source. However, an exception to this rule is a pair of tables without errors, since we can hardly ever exclude the possibility that two astronomers independently calculated a correct table for a given function. (In this context the error in a tabular value is defined as the difference between that value and a recomputation based on the modern formula for the tabulated function; a table is correct if it does not have any errors.) Tables without errors can be found among tables of very simple functions such as linear ones, or among tables of more complicated functions with values to only a small number of sexagesimal places (typical examples are sine tables with values to a precision of three sexagesimal places). As was indicated above, scribal errors require a special treatment: if the differences between two tables can all be explained as incidental scribal mistakes, the tables can be considered to be mathematically identical and hence may be related in spite of the differences. If two tables have a very conspicuous set of scribal errors in common, they can be assumed to have been copied from the same erroneous original.

Table 1 shows three fragments of the tables for Mercury's equation of centre in *al-Qānūn al-Masʿūdī* by al-Bīrūnī (Afghanistan, ca. 1030) and the *Huihuili* (China, ca. 1275, see above).[14] The equation of centre is one of the six or seven functions that are tabulated in Ptolemy's *Almagest* and in most Islamic *zīj*es to allow the easy calculation of planetary longitudes. It is a highly complicated trigonometric function that can be derived from Ptolemy's geometrical models for the planetary motions (see, for instance, [Neugebauer 1957, Appendix 1] or, for more details, [Pedersen 1974, Chapters 9 and 10]). It is tabulated as a function of the so-called "mean centrum," a linear function of time.

The third and fifth columns of Table 1 show the errors in the tables in *al-Qānūn al-Masʿūdī* and the *Huihuili*, namely, the differences between the values in these

14 Sexagesimal numbers are given in the standard notation, i.e., sexagesimal digits are separated by commas, whereas the sexagesimal point is represented by a semicolon. For instance, 2;15,29 denotes $2 + \frac{15}{60} + \frac{29}{60^2}$.

| mean centrum | Bīrūnī | Bīrūnī *minus* recomputation | *Huihuili* | *Huihuili minus* recomputation | Bīrūnī *minus* *Huihuili* |
|---|---|---|---|---|---|
| 10 | 0;31 | +5 | 0;31 | +5 | |
| 11 | 0;34 | +5 | 0;34 | +5 | |
| 12 | 0;36 | +5 | 0;37 | +6 | −1 |
| 13 | 0;39 | +5 | 0;40 | +6 | −1 |
| 14 | 0;42 | +6 | 0;43 | +7 | −1 |
| 15 | 0;46 | +7 | 0;46 | +7 | |
| 16 | 0;49 | +7 | 0;49 | +7 | |
| 52 | 2;13 | +4 | 2;13 | +4 | |
| 53 | 2;15 | +3 | 2;15 | +3 | |
| 54 | 2;14 | | 2;16 | +2 | −2 |
| 55 | 2;17 | +1 | 2;18 | +2 | −1 |
| 56 | 2;19 | +1 | 2;19 | +1 | |
| 57 | 2;20 | | 2;21 | +1 | −1 |
| 58 | 2;22 | | 2;22 | | |
| 59 | 2;23 | −1 | 2;24 | | −1 |
| 60 | 2;25 | −1 | 2;25 | −1 | |
| 61 | 2;28 | | 2;27 | −1 | +1 |
| 62 | 2;29 | | 2;29 | | |
| 63 | 2;30 | −1 | 2;30 | −1 | |
| 89 | 2;43 | −18 | 2;43 | −18 | |
| 90 | 2;43 | −18 | 2;43 | −18 | |
| 91 | 2;43 | −18 | 2;42 | −19 | +1 |
| 92 | 2;42 | −20 | 2;42 | −20 | |
| 93 | 2;42 | −20 | 2;42 | −20 | |
| 94 | 2;42 | −20 | 2;42 | −20 | |
| 95 | 2;42 | −20 | 2;42 | −20 | |
| 96 | 2;42 | −20 | 2;41 | −21 | +1 |
| 97 | 2;41 | −21 | 2;41 | −21 | |
| 98 | 2;41 | −20 | 2;41 | −20 | |

**Table 1.** Comparison of the tables for Mercury's equation of centre in al-Bīrūnī's *al-Qānūn al-Masʿūdī* and the *Huihuili.*

two works and a recomputation on the basis of Ptolemy's Mercury model and his parameter value. Since the tables differ systematically from the recomputation by up to 21 minutes of arc, we may conclude that both were calculated according to an algorithm and / or parameter value different from those used by Ptolemy. However, since the two tables differ from each other in only 13 out of 180 values (see the last column), it is highly probable that they were computed on the basis of the same algorithm and parameter value. A statistical test of dependency as described in [Van Brummelen & Butler 1997] may be necessary to determine whether the authors of the *Huihuili* simply copied al-Bīrūnī's table for the first equation of Mercury or calculated it from scratch. Since the differences occur in small groups in which they tend to have the same sign, it seems that the authors of the *Huihuili* carried out at

| *source* | *value* |
|---|---|
| Ptolemy (Alexandria, ca. 140) | 0;59,8,17,13,12,31 |
| Ibn al-Shāṭir (Damascus, ca. 1350) | 0;59,8,19,36, 0 |
| *Alfonsine Tables* (Spain, ca. 1275) | 0;59,8,19,37,19,13,56 |
| Ibn Yūnus (Cairo, ca. 1000) | 0;59,8,19,42,35 |
| Banū Mūsā ibn Shākir (Baghdad, ca. 840) | 0;59,8,19,43,14,18 |
| Naṣīr al-Dīn al-Ṭūsī (Maragha, ca. 1270) | 0;59,8,19,43,47 |
| Shams al-Munajjim al-Wābkanwī (Maragha, ca. 1320) | 0;59,8,19,43,47 |
| Ibn Yūnus (Cairo, ca. 1000) | 0;59,8,19,44,10,30,32, 0 |
| Ghiyāth al-Dīn al-Kāshī (Samarkand, ca. 1420) | 0;59,8,19,44,10,39 |
| *modern value* | 0;59,8,19,48,43,18 |
| *Huihuili* (Beijing, ca. 1275) | 0;59,8,19,49,27,22,58 |
| Muḥyī 'l-Dīn al-Maghribī (Maragha, ca. 1280) | 0;59,8,20, 8, 4,36,38 |
| ʿAbd al-Raḥmān al-Khāzinī (Marv, ca. 1120) | 0;59,8,20,13,18 |
| Yaḥyā ibn Abī Manṣūr (Baghdad, ca. 830) | 0;59,8,20,35,14,38 |
| Ḥabash al-Ḥāsib (Baghdad, ca. 830) | 0;59,8,20,35,25 |
| al-Bīrūnī (Afghanistan, ca. 1030) | 0;59,8,20,38,21,13 |
| Abū 'l-Wafāʾ al-Būzjānī (Baghdad, ca. 975) | 0;59,8,20,43,17,38,41,42 |
| al-Battānī (Syria, ca. 900) | 0;59,8,20,46,49 |

**Table 2.** Values for the daily solar mean motion used by Ptolemy and some important Muslim astronomers.

least some calculations themselves, e.g., they may have corrected some mistakes in al-Bīrūnī's table by carrying out anew the linear interpolation between certain values for multiples of 10°.[15]

*Case 2: Comparison of Parameter Values.* Instead of comparing complete tables, one may compare certain mathematical characteristics of tables, such as computational techniques (see Case 3 below) or the underlying values of the astronomical parameters involved. Methods for extracting parameter values from astronomical tables are described in [van Dalen 1989, 1993, 1996; Mielgo 1996]. Although for certain parameters, such as the obliquity of the ecliptic, only very few different values were in use (the value 23°35′ was particularly common among Muslim astronomers), especially parameters measuring planetary mean motions tend to be highly idiosyncratic. Table 2 shows an excerpt from the file collected by Professor E.S. Kennedy with parameters from about 30 important Islamic *zīj*es and several other astronomical works. The values for the daily solar mean motion listed here are generally very close, but the differences are still large enough to be able to distinguish between the results of different observations or determinations. For example, Shams al-Munajjim al-Wābkanwī apparently used al-Ṭūsī's value, and the value of Ghiyāth al-Dīn al-Kāshī seems to be dependent on that of Ibn Yūnus. On the other hand, the value from the *Huihuili* points to an independent observation.

15 [Yano 2002] contains an extensive analysis of the tables for Mercury's equation of centre in the *Qānūn* and the *Huihuili*. Yano showed that al-Bīrūnī's table is based on Ptolemy's parameter value but a different, apparently erroneous formula for calculating the equation of centre.

*Case 3: Comparison of Mathematical Techniques.* The use of inaccurate auxiliary tables or certain mathematical techniques such as interpolation for the computation of an astronomical table may leave traces in the tabular values in the form of identifiable error patterns. Like the underlying parameter values, the auxiliary tables and computational techniques that were used may be typical for a certain astronomer or school of astronomers and hence may be used to determine the origin of tables or, at least, to establish relationships between them.

In the first example below, it will be shown that the error pattern in a tangent table is tightly connected with the errors in the sine table from which it was calculated. In the case of the sine and tangent tables from the so-called *Baghdādī Zīj* (late 13th century), this helps us to conclude that both tables derive from the important tenth-century astronomer Abū 'l-Wafā' al-Būzjānī, whose *zīj* is non-extant. In the second example, it will be shown how the use of a special type of second-order interpolation can be recognized in the highly accurate sine table of Ulugh Beg (early 15th century). The same type of interpolation also underlies Ulugh Beg's tables of oblique ascensions, but has not been found in the work of any other astronomer.

*Example 1.* The third column of Table 3 shows the errors for arguments between 60° and 90° in the tangent table found on folios 227v–228v of the *zīj* by a certain Jamāl al-Dīn Abī 'l-Qāsim ibn Maḥfūẓ al-munajjim al-Baghdādī, which was written in 1285 and is extant in the unique manuscript Paris BNF arabe 2486. In my dissertation [van Dalen 1993: 164–168] I have conjectured that this tangent table and a number of other tables in the *Baghdādī Zīj* with values to sexagesimal thirds were taken from the astronomical handbook *al-Majisṭī* by the important 10th-century mathematician and astronomer Abū 'l-Wafā' al-Būzjānī. The text of this work is extant as Paris BNF arabe 2494, but the tables are lost except for scattered fragments. Meanwhile I have worked through the text of *al-Majisṭī* and have found more evidence for my conjecture which I hope to publish in due course. In the remainder of this article, when speaking of the "sine and tangent tables of Abū 'l-Wafā'," I mean the tables from the *Baghdādī Zīj*.

The tangent function tabulated by Abū 'l-Wafā' and most other Muslim astronomers was $60 \cdot \tan x$ rather than $\tan x$. As can be seen clearly from the third column of Table 3, which displays in sexagesimal thirds the differences between Abū 'l-Wafā''s table and exact tangent values, the errors in the tabular values increase rapidly as the argument $x$ approaches 90°. This is typical for many historical tangent tables and can be explained as follows. The tangent is calculated according to

$$\tan x = \frac{\sin x}{\cos x}.$$

When the argument $x$ approaches 90°, the denominator in this expression, $\cos x$, nears zero. Therefore, if it is consistently rounded to the same number of sexagesimal places, its relative error (i.e., the rounding error divided by the cosine itself) increases. Since the relative error in the tangent is of the same order of magnitude as that in the cosine, it also increases when the argument approaches 90°. Finally, since the tangent itself becomes very large towards 90°, the absolute error in the calculated

| arc | tangent in the *Baghdādī Zīj* | *Baghdādī Zīj minus* recomputation | *Baghdādī Zīj minus* reconstruction |
|---|---|---|---|
| 60 | 103;55,22,58 | | |
| 61 | 108;14,34,17 | −2 | |
| 62 | 112;50,36,56 | +1 | −1 |
| 63 | 117;45,23,52 | | −1 |
| 64 | 123; 1, 5,38 | | |
| 65 | 128;40,13,33 | +3 | |
| 66 | 134;45,43,54 | −3 | −2 |
| 67 | 141;21, 4,12 | +5 | |
| 68 | 148;30,18,42 | −4 | −1 |
| 69 | 156;18,19,10 | −4 | −1 |
| 70 | 164;50,55, 6 | −1 | −1 |
| 71 | 174;15, 9,27 | −6 | |
| 72 | 184;39,39,40 | +1 | |
| 73 | 196;15, 4,15 | +5 | +1 |
| 74 | 209;14,41,34 | +3 | |
| 75 | 223;55,22,57 | −1 | |
| 76 | 240;38,48,39 | −2 | |
| 77 | 259;53,18,41 | −6 | |
| 78 | 282;16,39,50 | −16 | −1 |
| 79 | 308;40,23,27 | −13 | |
| 80 | 340;16,37, 8 | +16 | |
| 81 | 378;49,30, 6 | −14 | −1 |
| 82 | 426;55,20,10 | +18 | |
| 83 | 488;39,38,28 | −22 | |
| 84 | 570;51,42,35 | −8 | |
| 85 | 685;48,12,11 | +53 | +1 |
| 86 | 858; 2,23,33 | −22 | |
| 87 | 1144;52, 5,34 | +3 | +1 |
| 88 | 1718;10,37,11 | +388 | |
| 89 | 3437;24, 1,51 | +608 | |

**Table 3.** Recomputation of the tangent table of Abū 'l-Wafā' as contained in the *Baghdādī Zīj*, with a reconstruction from his sine table.

tangent, as tabulated in the third column of Table 3, grows even more rapidly than the relative error.

The fourth column of Table 3 displays the differences between Abū 'l-Wafā''s tangent table and values reconstructed by applying the formula

$$\tan x = \frac{\sin x}{\sin(90^\circ - x)}$$

to values from his sine table. This sine table is also included in the *Baghdādī Zīj* (folios 224v–225v) and contains only nine errors of one sexagesimal third. The agreement between Abū 'l-Wafā''s tangent table and our reconstruction is not only nearly perfect, it in fact even allows us to conclude that Abū 'l-Wafā''s sine table was used for the calculation of his tangent table. To make this clear, first note that the nine errors in his sine table occur for arguments 19, 22, 25, 35, 44, 56, 67, 78 and 80°. If a sine table has an error for argument $x$, the tangent values computed from it will have errors for arguments $x$ and $90° - x$. Within the range of arguments displayed in Table 3 the errors in Abū 'l-Wafā''s sine table would thus leave traces in the tangent values for arguments 65 (= 90 − 25), 67, 68 (= 90 − 22), 71 (= 90 − 19), 78, and 80°. Now the use of a correct sine table with values to sexagesimal thirds leads to tangent values for these arguments that differ by +5, +3, −7, −9, −6, and +6 thirds respectively from those of Abū 'l-Wafā' (these differences are not displayed in Table 3), whereas the use of Abū 'l-Wafā''s sine table leads to differences of 0, 0, −1, 0, −1, and 0 thirds (fourth column of Table 3). Thus there can be no doubt that Abū 'l-Wafā''s tangent table was in fact computed from his sine table.

Chapter 4 of [van Dalen 1993] contains various similar results that, in combination with perusal of *al-Majisṭī*, were decisive in the attribution of a total of nine tables from the *Baghdādī Zīj* to Abū 'l-Wafā'. For instance, a table for the equation of daylight for Baghdad with values to sexagesimal thirds [van Dalen 1993: 181–183] was shown to have been derived from another table by means of inverse linear interpolation in an accurate sine table with values to thirds for every 15 minutes of arc, rather than quadratic interpolation, values to seconds, or any other increment of the argument (the tables for the sine and tangent in *al-Majisṭī* in fact had values for every 15 minutes, so that strictly speaking the tables in the *Baghdādī Zīj* are extracts from those of Abū 'l-Wafā').

*Example 2.* Table 4 shows the errors in a small part of the sine table from the *Sulṭānī Zīj* by Ulugh Beg (Samarkand, ca. 1440). This table can be regarded as the culmination of Islamic computational mathematics and displays values to five sexagesimal places (roughly 8 decimals) for every minute of arc from 0° to 87° and values to six sexagesimal places (10 decimals) between 87° and 90°. Most of the tabular values are correct; less than half of them contain errors of +1 or −1 (incidentally +2 or −2) in the final sexagesimal position. However, there are some peculiar groups of errors starting from argument 87°, precisely where the number of sexagesimal places is increased from 5 to 6. As can be seen from Table 4, the sine values for multiples of 5′ in this part of the table are correct (as in the whole table), whereas the errors in between these multiples are alternately positive and negative. A plausible explanation of these characteristics is the use of quadratic interpolation between accurately calculated values for every 5 minutes of arc (note that in the case of linear interpolation the errors in all groups would have had the same sign).

Generally, the use of interpolation in a trigonometric table can be recognized from the tabular differences: linear interpolation leads to groups of roughly constant first-order differences, quadratic interpolation to groups of roughly constant second-order differences. However, quadratic interpolation on intervals of 5′ produces such a good approximation to the sine that under normal circumstances it is practically impossible to distinguish between accurately computed and interpolated values. For the same reason, the second-order differences in Ulugh Beg's sine table change so slowly that separate groups cannot be recognized, except between 87° and approximately 87°25′, where the differences fluctuate around $-1'''5^{iv}44^{v}$ (see

| arc | Ulugh Beg's sine | second-order differences | Ulugh Beg *minus* recomputation |
|---|---|---|---|
| 86;50 | 59;54,30,11, 0 | −1,5 | |
| 86;51 | 59;54,33,38,42 | −1,6 | |
| 86;52 | 59;54,37, 5,19 | −1,6 | |
| 86;53 | 59;54,40,30,50 | −1,6 | |
| 86;54 | 59;54,43,55,15 | −1,5 | |
| 86;55 | 59;54,47,18,34 | −1,6 | |
| 86;56 | 59;54,50,40,48 | −1,6 | |
| 86;57 | 59;54,54, 1,56 | −1,5 | |
| 86;58 | 59;54,57,21,58 | −1,5,46 | −1 |
| 86;59 | 59;55, 0,40,55 | −1,5,48 | |
| 87 | 59;55, 3,58,46,14 | −1,5,48 | |
| 87;1 | 59;55, 7,15,31,40 | −1,5,48 | +11 |
| 87;2 | 59;55,10,31,11,18 | −1,5,47 | +16 |
| 87;3 | 59;55,13,45,45, 8 | −1,5,48 | +15 |
| 87;4 | 59;55,16,59,13,11 | −1,5,40 | +10 |
| 87;5 | 59;55,20,11,35,26 | −1,5,38 | |
| 87;6 | 59;55,23,22,52, 1 | −1,5,40 | −8 |
| 87;7 | 59;55,26,33, 2,58 | −1,5,40 | −11 |
| 87;8 | 59;55,29,42, 8,15 | −1,5,38 | −11 |
| 87;9 | 59;55,32,50, 7,52 | −1,5,46 | −8 |
| 87;10 | 59;55,35,57, 1,51 | −1,5,46 | |
| 87;11 | 59;55,39, 2,50, 4 | −1,5,45 | +5 |
| 87;12 | 59;55,42, 7,32,31 | −1,5,46 | +7 |
| 87;13 | 59;55,45,11, 9,13 | −1,5,45 | +7 |
| 87;14 | 59;55,48,13,40, 9 | −1,5,42 | +5 |
| 87;15 | 59;55,51,15, 5,20 | −1,5,42 | |
| 87;16 | 59:55,54,15,24,49 | −1,5,42 | −3 |
| 87;17 | 59;55,57,14,38,36 | −1,5,41 | −5 |
| 87;18 | 59;56, 0,12,46,41 | −1,5,42 | −5 |
| 87;19 | 59;56, 3, 9,49, 5 | −1,5,45 | −3 |
| 87;20 | 59;56, 6, 5,45,47 | −1,5,45 | |
| 87;21 | 59;56, 9, 0,36,44 | −1,5,44 | +3 |
| 87;22 | 59;56,11,54,21,56 | −1,5,46 | +3 |
| 87;23 | 59;56,14,47, 1,24 | −1,5,44 | +4 |
| 87;24 | 59;56,17,38,35, 6 | −1,5,43 | +2 |
| 87;25 | 59;56,20,29, 3, 4 | −1,5,44 | |
| 87;26 | 59;56,23,18,25,19 | −1,5,42 | −1 |
| 87;27 | 59;56,26, 6,41,50 | −1,5,44 | −3 |
| 87;28 | 59;56,28,53,52,39 | −1,5,43 | −2 |
| 87;29 | 59;56,31,39,57,44 | −1,5,45 | −2 |
| 87;30 | 59;56,34,24,57, 6 | −1,5,44 | |
| 87;31 | 59;56,37, 8,50,43 | −1,5,45 | +1 |
| 87;32 | 59;56,39,51,38,36 | −1,5,45 | +1 |
| 87;33 | 59;56,42,33,20,44 | −1,5,44 | +1 |
| 87;34 | 59;56,45,13,57, 7 | −1,5,45 | +1 |
| 87;35 | 59;56,47,53,27,46 | −1,5,44 | |

**Table 4.** The use of second-order interpolation in the sine table of Ulugh Beg (the differences are given in thirds, fourths and, starting from 86°58′, fifths).

the third column of Table 4). So it is only thanks to the errors in this part of the table, apparently related to the change from 5 to 6 sexagesimal places, that we can recognize the use of second-order interpolation in the computation of Ulugh Beg's table of sines.

It can in fact be shown that a special type of interpolation, described in many Persian *zījes* from the 13th to 15th centuries and attributed to the tenth-century Iranian astronomer Abū Jaʿfar Muḥammad al-Khāzin [see Hamadanizadeh 1987], was used. With this type of interpolation, the tabular values for arguments between $x$ and $x+n\cdot\Delta x$ (with $n$ an integer) are calculated in such a way that they lie on the parabola through the points $(x-\Delta x, f_T(x-\Delta x))$, $(x, f(x))$ and $(x+n\cdot\Delta x, f(x+n\cdot\Delta x))$, where $f(x)$ and $f(x+n\cdot\Delta x)$ are accurately calculated tabular values for arguments $x$ and $x+n\cdot\Delta x$ and $f_T(x-\Delta x)$ is the result of the application of the same type of interpolation to the preceding interval. Thus the errors in Ulugh Beg's sine values for arguments $87°1'$ and greater result from the less precise value of $\sin 86°59'$ that had to be used for their calculation (use of the value 59;55,0,40,55,15 would have avoided the following groups of errors).

As far as I know, Ulugh Beg's sine table is the first in which the use of this particular type of interpolation has been demonstrated. Unpublished research by the present author indicates that his set of tables of oblique ascensions for geographical latitudes 1° to 50° also relies heavily on this type of interpolation.

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# A New Treatise by al-Kāshī on the Depression of the Visible Horizon

by MOHAMMAD BAGHERI

In this paper I present an edited Arabic text and an English translation of a short treatise, wrongly attributed to Bābā Afḍal al-Dīn al-Kāshī (7th/13th century), an Iranian mystic poet who lived in Kāshān, Iran. The treatise is about the determination of the visible horizon, a geometrical problem mentioned in two letters by Ghiyāth al-Dīn Jamshīd al-Kāshī (ca. 826/1422) to his father. I discuss the history of this problem and I show as well that the short treatise was written not by Bābā Afḍal but most likely by Jamshīd al-Kāshī.

Ghiyāth al-Dīn Jamshīd al-Kāshī, an eminent Iranian mathematician and astronomer of the 15th century, left his birthplace Kāshān for Samarkand in A.D. 1421 and joined the scientific circle of Ulugh Beg. From Samarkand he wrote letters in Persian to his father who lived in Kāshān and who apparently was familiar with mathematics and astronomy. In these letters, al-Kāshī described — among other things — the scientific problems which were discussed in the scientific meetings in Samarkand. One of these letters has been known since 1859, when a learned Qājār prince, I'tiḍād al-Salṭana, quoted a fragment of it in the report of his visit to the Maragha observatory in the company of the Qājār king Nāṣir al-Dīn. This letter has been published several times, and translations into English, Turkish, Arabic, Russian and Uzbek have also appeared with introductions and commentaries. Another letter was found recently. The text was published in my book *Az Samarqand be Kāshān: Nāmehāye Ghiyāth al-Dīn Jamshīd Kāshānī be pedarash* (From Samarkand to Kāshān: Letters of al-Kāshī to His Father), [Bagheri 1996]; and an English translation with an introduction and commentary can be found in [Bagheri 1997].

The problem of the depression of the visible horizon is mentioned in both letters among several other problems brought up by Ulugh Beg. This problem remained unsolved in Samarkand until al-Kāshī arrived there and managed to solve it. In the letter known since 1859, the problem is described as follows:

> Suppose that a man stands at a spot around which the curvature of the surface of the earth is quite accurate; let his height be three and a half *gaz*, by the *dhirā' of the arm* [i.e., 3.5 cubits]. At what distance will a visual ray, leaving his eyes and touching the surface of the earth tangentially, intersect with the true horizon and with what angle of depression will it reach the outermost sphere of the firmament? [Sayılı 1960: 97–98]

Al-Kāshī found it an easy problem and solved it completely in one day. Sayılı gives no geometrical explanation, but he says, "In the problem concerning the visual ray and the true horizon, the difficulty encountered probably concerned calculation and

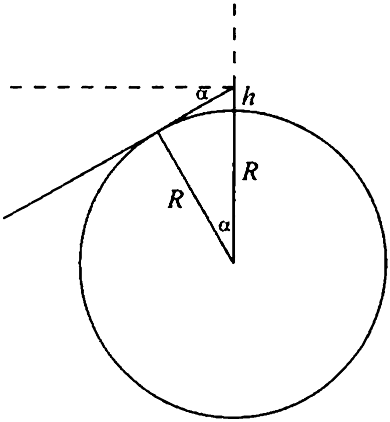



Figure 1

the derivation of numerical results with a given degree of accuracy" [Sayılı 1960: 47]. Prof. E.S. Kennedy, who also published an English translation of this letter, provides the following explanation of the problem without providing any figure: "If $R$ is the radius of the earth and $h$ the height of the person (measured in the same units), then the angular distance of the true horizon is arccos $[R/(R + h)]$, and this angle plus a quadrant is the true horizon's depression measured from the zenith" [Kennedy 1960: 208] (see Fig. 1).

In his commentary to the Russian translation of this letter, G. Sabirov gives another interpretation of the problem as shown in Fig. 2, where he calculates the complement of the angle which Kennedy calculated. Sabirov gives the formulas for the angle and distance in question as

$$\alpha = \arcsin \frac{R}{R+h} \qquad \text{and} \qquad l = \sqrt{h(2R+h)}$$

[Sabirov 1973: 204–205]. In the other editions and translations of this letter, there is no reference to what al-Kāshī may have had in mind regarding this problem.

The discovery of the other letter from al-Kāshī to his father sheds more light on this problem. In this letter, he writes:

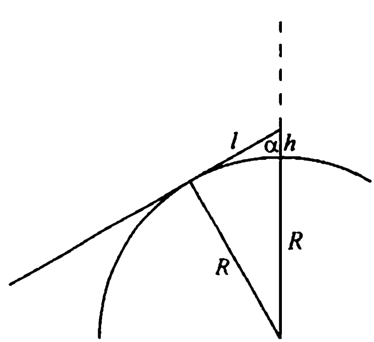



Figure 2

> During the period that the arena was theirs [his rivals' in Samarkand who wanted to test him], in the discussions held in the presence of His Royal Majesty [Ulugh Beg] they were confronted with some difficulties into which they had looked for a month or two or even for a year, but to which no solution had been found. For example, this problem: [Let us suppose] somebody is standing on a perfectly circular ground or on the sea surface, and the visual ray issuing from his eyes is tangent to that, and [then] reaches the sphere of the ecliptic [*falak al-burūj*, i.e., the sphere of the fixed stars[1]]. Now at which distance will [that ray] intersect the true horizon, and, where it reaches the sphere of the ecliptic, how much will it

1 This expression for the sphere of the fixed stars is found in the section on astronomical terminology in the commentary of Bīrjandī (d. A.D. 1527/8) on *Zīj-i Ulugh Beg* (Tehran, Sipahsalar Library, MS 680, fol. 61v).

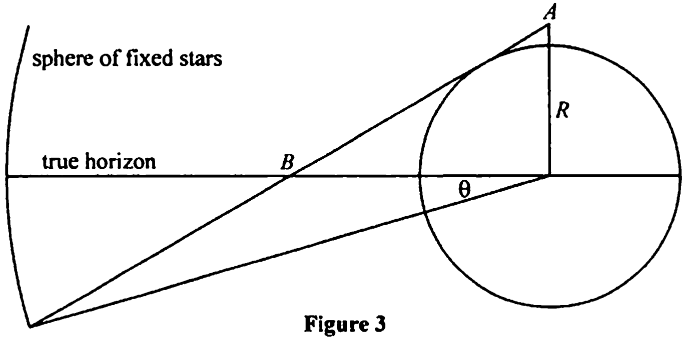



**Figure 3**

be depressed from the true horizon? And many other [problems] which were detailed [to you] previously.

In my commentary on this problem (see Fig. 3), I have used the celestial dimensions given by al-Kāshī in his *Sullam al-samā'* [al-Kāshī 1881/82: 34–36], where $R$, the radius of the earth, is 1272 *parasangs* (one *parasang* is 12,000 cubits) and the minimum distance of the sphere of the fixed stars is 26,328 $R$, and thus I have calculated $AB$ approximately equal to 1477 $R$ and θ approximately equal to 2′12″. In the letter known since 1859, al-Kāshī refers to "the higher sphere" (*falak-i a'lā*), the radius of which is equal to the maximum distance of the fixed stars, and he gives the value 26,340 $R$ for it in his *Sullam al-samā'*. Using this value, one obtains the same result, because it is very close to the minimum distance of the sphere of fixed stars.

In the astronomical texts of the Medieval Islamic period three types of horizon are defined (see Fig. 4): 1) the true horizon (*ufuq ḥaqīqī*), i.e., the great circle on the celestial sphere produced by the plane passing through the center of the universe; 2) the sensible horizon (*ufuq ḥissī*), i.e., the intersection of a plane tangent to the earth at the observer's position with the celestial sphere; and 3) the shield-like horizon (*ufuq tursī*, also *ufuq ḥissī* in a more general sense), i.e., the circle drawn on the celestial sphere by the visual rays issuing from the observer's eye tangent to the surface of the earth. Depending on the height of the observer and the place

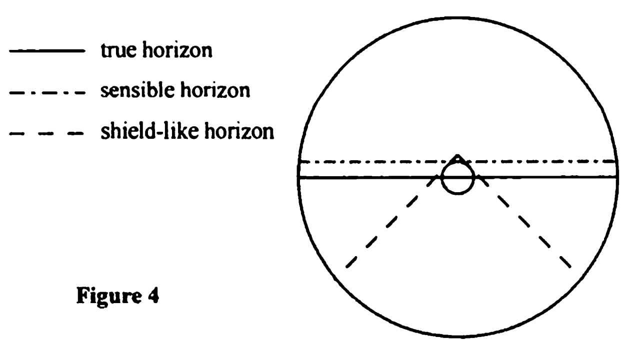



**Figure 4**

where he stands, this circle may be a small circle or a great circle coinciding with the true horizon. This is the actual boundary between the visible and invisible parts of the sky. The problem in al-Kāshī's letter actually refers to the position of the true horizon relative to the shield-like horizon.

In his commentary on Chaghmīnī's *al-Mulakhkhaṣ fī 'l-hay'a*, Qāḍī-zāda Rūmī defines the shield-like horizon without using the adjective *tursī* [Rūmī 1881/82: 44]. In his Arabic commentary on Ṭūsī's *Tadhkira*, Bīrjandī mentions the three types of horizon using the adjective *tursī* for the third one.

The depression of the visible horizon was discussed by Abū Sahl al-Kūhī (d. about A.D. 1014) in his treatise *Fī ma'rifat mā yurā min al-samā' wa-l-baḥr* (On knowing the visible parts of the sky and the sea). In this treatise al-Kūhī provides the geometric solution to the problem, but he gives no numerical value.[2]

This work of al-Kūhī is mentioned by Ibn al-Haytham (d. after 1040 in Cairo) in a fragment of his treatise *Risāla fī anna al-ẓāhir min al-samā' akthar min niṣfi-hā* (Treatise on [proving] that the apparent [part] of the sky is more than half of it) kept in Oxford Bodleian as Thurston 3 (fol. 116r) and Marsh. 713 (fols. 232r–232v). Sezgin mentions a third manuscript of this fragment kept in Alexandria [Sezgin 1978: 260]. In another fragment of this treatise, kept in Oxford Bodleian, Thurston 3 (fol. 104r), Ibn al-Haytham says that if the height of the observer's eyes is 3.5 cubits, the visible part of the sky will be 4'26" greater than its invisible part. In fact, if $\theta$ is the depression of the visible horizon, then the visible part of a great semicircle from the zenith to the nadir will be $90 + \theta$ degrees and its invisible part will be $90 - \theta$ degrees. Therefore the visible arc is $2\theta$ greater than the invisible arc. Ibn al-Haytham's result is very close to $2\theta$ as calculated by me (4'24") on the basis of al-Kāshī's parameters.

In his *Kashf 'awār al-munajjimīn* (Uncovering the shortcomings of the astronomers, MS Leiden Or. 98/1, fols. 75v and 84r) Samaw'al b. Yaḥyā (d. about 1174/5) refers to both Ibn al-Haytham's and al-Kūhī's works and quotes from them.

Kamāl al-Dīn al-Fārisī (d. A.D. 1317/18) wrote a commentary entitled *Tanqīḥ al-Manāẓir* (Revision of the *Optics*) on Ibn al-Haytham's *Manāẓir* (*Optics*). In the 7th Book of this work [al-Fārisī 1929, 2: 156], al-Fārisī says that the refraction of light affects our vision of the celestial bodies and the distances between them in two ways:

1) If we apply the correction for refraction, the distance of the moon is found to be less than that given in the astronomical tables, and therefore, the distances and sizes of the other celestial bodies are less than the values given in the tables;
2) Refraction also affects the part of the sky which is visible from a high point. Al-Fārisī adds that Abū Sahl al-Kūhī and others have written treatises about this problem.

2 I have seen the ms. of this work kept in the Holy Shrine Library (Mashhad, Iran) as MS 5412, copied in A.D. 1253/4. For other mss., see [Sezgin 1974: 320].

He also adds that refraction affects the visible part of the sky when the height of the eyes from a level ground or the surface of a sea is three ⟨and a half⟩ cubits. Al-Fārisī says that Ibn al-Haytham calculated the visible part of the sky for this case, without taking refraction into account.

In his *Taḥdīd nihāyāt al-amākin li-taṣḥīḥ masāfāt al-masākin* (Determination of coordinates of positions for the correction of distances between cities), Abū'l-Rayḥān al-Bīrūnī discusses the depression of the visible horizon in connection with the radius of the earth, but not in relation to the radius of the universe [al-Bīrūnī 1967: 183–186].

After my publication of the Persian text and the English translation of al-Kāshī's newly found letter, my colleague Tofigh Heidarzadeh pointed out to me that a short unpublished Arabic treatise in the former Senate library (Tehran), namely, MS 89/2, fols. 14–15, contains a solution to this problem. The treatise is attributed to Mawlānā Afḍal al-Kāshī, but in a marginal remark this authorship is doubted. I will show that this treatise is most probably by Jamshīd al-Kāshī himself, and not by Afḍal al-Kāshī, who is also known as Bābā Afḍal. He was an Iranian mystic poet of the 13th century, who was familiar with astronomy. His full name is Afḍal al-Dīn Muḥammad b. Ḥusayn Kāshānī (or Kāshī). The Arabic text of the short treatise, which consists of three parts, as well as my English translation are reproduced as an appendix to the present paper.

In the first part of the treatise, the numerical solutions are given as follows:

> If a person of 3.5 cubits height stands on a circular place on the surface of the earth, any ray issuing from his eyes, tangent to the surface of the earth extending towards the sky, will intersect the ⟨true⟩ horizon according to Euclid's ⟨parallel⟩ postulate. We ⟨may⟩ extend it towards the higher sphere. Then the distance between the center of the ⟨true⟩ horizon and the ⟨intersection⟩ point of the above-mentioned line of the ray will be approximately 22,566,113,141 cubits; and the above-mentioned line of the ray will join the higher sphere such that the Sine of the arc of the depression will be approximately 191,134,838 cubits.[3]

The methods for finding these values are explained in the second part. Using the notation of Fig. 5 on page 365 and taking $HI = h$, $EA = R$, we obtain the following modern formulas:

$$EI = \frac{R^2}{\sqrt{[R^2 - R^4/(R+h)^2]}}$$

and

$$ZN = R(EZ - EI)/EI.$$

The proofs of the validity of these methods are presented in the third part of the treatise. From the formula for $EI$ we conclude:

$$R \cong \sqrt[3]{2h \cdot EI^2}.$$

3 "Sine" stands for the sine in a base circle with radius $R$, i.e., $\mathrm{Sin}\,x = R \cdot \sin x$. Cosine and Cotangent used later in this paper are similarly defined by $\mathrm{Cos}\,x = R \cdot \cos x$ and $\mathrm{Cot}\,x = R \cdot \cot x$.

Using the two above-mentioned values $h = 3.5$ and $EI = 22,566,113,141$, we find $R \cong 15,275,796$ cubits $= 1272.93$ *parasangs*. In his *Sullam al-samā'*, al-Kāshī gives $R = 1272$ *parasangs*, which is quite close to what we have calculated here; the difference may be due to the approximations which we have made [al-Kāshī 1881/82: 34]. On the other hand, Bābā Afḍal gives the diameter of the earth equal to 2267 *parasangs*, which corresponds to $R = 1133.5$ *parasangs*, differing more than 10 % from the $R$ calculated above. Bābā Afḍal presents his value for the diameter of the earth in an appendix (*mulḥaq*) to his *'Araḍ Nāma* (Book on Accidents) published in his collected works [Bābā Afḍal 1987: 244].

From the formula for $ZN$, we conclude:

$$EZ \cong ZN\sqrt{(R/2h)}.$$

Substituting the numerical values for $ZN$ (from the treatise published here), $R$ (al-Kāshī's value from *Sullam al-samā'*), and $h = 3.5$ cubits, we find

$$EZ \cong 282,244,190,500 \text{ cubits} = 23,520,349.21 \textit{ parasangs}.$$

This value differs considerably from the value $26,328\,R$ or 33,509,188 *parasangs*, which is the minimum distance of the sphere of fixed stars according to al-Kāshī in his *Sullam al-samā'*. Since $23,520,349.21 : 33,509,188 = 1 : 1.425 \cong 1 : \sqrt{2} = 1 : 1.414$, the difference may be explained as follows: al-Kāshī used an erroneous method of computation, equivalent to the formula

$$EZ \cong ZN\sqrt{(R/h)}.$$

Thus his result could have been 33,262,796.84, which differs by only 0.74 % from 33,509,188. The difference may be due to rounding and approximation errors. If al-Kāshī had not made this mistake, he would have found the Sine of depression as $(191,134,838) \cdot \sqrt{2} = 272,433,466.5$. On the other hand, the value $EZ = 23,520,349.21$ *parasangs*, which we calculated above, is equivalent to $EZ \cong 18,490\,R$. But the minimum distance of the sphere of the fixed stars according to [Bābā Afḍal 1987: 245], which can be inferred from his description of the thickness of the celestial spheres, is about $43,800\,R$ or 49,647,300 *parasangs*. Therefore, the treatise presented here is not based on Bābā Afḍal's parameters.

There is more evidence in favor of al-Kāshī's authorship. If the problem had been discussed by Bābā Afḍal, who lived in Kāshān about two centuries before al-Kāshī, it could not have been a new problem for al-Kāshī and his father, and al-Kāshī could have referred to it as Bābā Afḍal's problem. Although Bābā Afḍal was familiar with astronomy, he could not be compared with al-Kāshī or Qāḍī-zāda Rūmī. So it is very likely that because of the similarity in names, the treatise has erroneously been attributed to Afḍal al-Kāshī instead of Jamshīd al-Kāshī. Moreover, we find the term *falak-i a'lā* or *falak al-a'lā* ("the higher sphere") in one of al-Kāshī's letters and also in the short treatise presented here, while Bābā Afḍal does not use this term in his writings.

Later, I found another treatise containing this solution in MS 3585/7 kept in the Malik Library in Tehran. In the table of contents at the beginning of the codex, the title is given as the treatise of Mīrzā Kāfī al-Qā'inī on the difference between the two horizons. Here the proof and description are similar to those in the treatise wrongly attributed to Afḍal al-Kāshī, but no numerical values are given. A marginal note presents an alternative method equivalent to

$$EI = \frac{R(R+h)}{\sqrt{(R+h)^2 - R^2}},$$

which is a simplified form of the above-mentioned formula for $EI$.

The note was added by a certain Muḥammad Mahdī al-Ḥusaynī al-Mūsawī, who says that this solution is simpler than that given by his grandfather's uncle in Quhistān. Quhistān was a historical region in the province of Khurāsān that extended from south Nīshābūr towards south-east Iran up to Sīstān and that included Būzjān in the 10th century A.D.; its capital was Qā'in. This town was a center of scientific activity in the Islamic period, and some mathematicians from Qā'in are known to us.

Another marginal remark states that $ZN$ is not really the desired Sine, because the arc $ZK$ is not centered at $E$.

Thus the problem of the depression of the visible horizon was proposed by al-Kūhī and Ibn al-Haytham, appeared in Samarkand four centuries later, was transmitted from Samarkand to Kāshān through al-Kāshī's correspondence with his father, and found its way also to Quhistān in eastern Iran.

## Acknowledgment

I thank Dr. Charles Burnett from the Warburg Institute (London) and Dr. Hans van de Velde from Leiden University Library for making copies of manuscripts available to me. I am also grateful to Dr. Jan P. Hogendijk for his precious comments on this paper.

## Appendix

On the following pages I present an edition of the Arabic text of the short treatise wrongly attributed to Bābā Afḍal al-Kāshī together with an English translation on facing pages. My additions to reconstruct the original text are given in angular brackets ⟨ ⟩. Abundant words of the text are shown in square brackets [ ]. Parentheses are used for my own numbering of the paragraphs.

## Arabic text

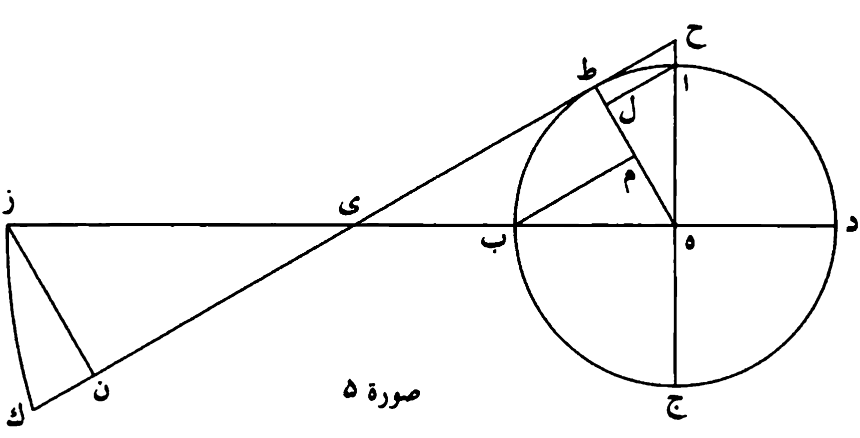
لمولانا أفضل الكاشى رحمه الله تعالى

بسم الله الرحمن الرحيم

(1) اذا قام شخص طول قامته ثلث أذرع و نصف فى موضع الإستدارة على وجه الأرض فالشعاع الخارج من بصره المماسّ لسطح الارض الممتدّ الى جهة السماء يقطع الأفق الحقيقى بحكم مصادرة اقليدس و نتعداه الى الفلك الأعلى فالبعد بين مركز الأفق و موضع الخط الشعاعى المذكور ٢٢٥٦٦١١٣١٤١ ذراعاً تقريباً و يصل[4] الخط الشعاعى المذكور الى الفلك الأعلى على إنحطاط قوس جيبها ١٩١١٣٤٨٣٨ ذراعاً تقريباً

(2) و طريق معرفة ذلك ان نقسم مربع ذراعات نصف قطر الارض على ذراعات نصف القطر مضافاً اليها طول قامة الشخص و نسقط مربع الخارج عن مربع نصف القطر ⟨و نقسم على جذر الباقى مربع نصف القطر⟩ حتى يحصل المقدار الأول فاذا نقصناه من نصف قطر العالم و ضربنا الباقى فى نصف قطر الأرض و قسمناه على المقدار الأول حصل المقدار الثانى

(3) و لنفرض لبيان ذلك ا ب ج ⟨د⟩ كرة الأرض على مركز ه و ه ز نصف قطر العالم و ا ح طول قامة الشخص و ح ط ى ك الخط الشعاعى المماس للأرض على نقطة ط القاطع للأفق على ى و للفلك الأعلى ⟨على⟩ ك ثم نصل بين المركز و نقطة ط بخط ه ط ⟨و نخرج من ا عمود ال على ه ط⟩ و من ب عمود ب م عليه و من ز عمود ز ن على الخط الشعاعى على هذه الصورة




صورة ٥

4 In the text, it is written نصل instead of يصل ,which suits the context better.

## Translation

By Mawlānā Afḍal al-Kāshī — may God the Almighty have mercy upon him!
In the name of God, the compassionate, the merciful.

(1) If a person of 3.5 cubits height stands on a circular place on the surface of the earth, any ray issuing from his eyes, tangent to the surface of the earth extending towards the sky, will intersect the ⟨true⟩ horizon according to Euclid's ⟨parallel⟩ postulate.[5] We ⟨may⟩ extend it towards the higher sphere. Then the distance between the center of the ⟨true⟩ horizon and the ⟨intersection⟩ point of the above-mentioned line of the ray will be approximately 22,566,113,141 cubits, and the above-mentioned line of the ray will join the higher sphere such that the Sine of the arc of the depression will be approximately 191,134,838 cubits.

(2) The way to find this ⟨is as follows:⟩ We divide the square of the radius of the earth in cubits by the radius of the earth plus the height of that person in cubits. Then we subtract the square of the result from the square of the radius ⟨and we divide the square of the radius by the square root of the remainder⟩, so that the first magnitude will be obtained. If we subtract it from the radius of the universe and multiply the remainder by the radius of the earth and divide it by the first magnitude, the second magnitude will be obtained.

(3) To explain this, we assume $ABGD$ as the sphere of the earth with the center $E$, $EZ$ as the radius of the universe, $AH$ as the height of the person, and $HTIK$ as the ray tangent to the earth at the point $T$, intersecting the horizon at $I$ and the higher sphere at $K$. Then we join the center and the point $T$ by the line $ET$ ⟨and from $A$ we draw $AL$ perpendicular to $ET$⟩, from $B$, $BM$, perpendicular to it, and from $Z$, $ZN$ perpendicular to the line of the ray, as shown in this figure:

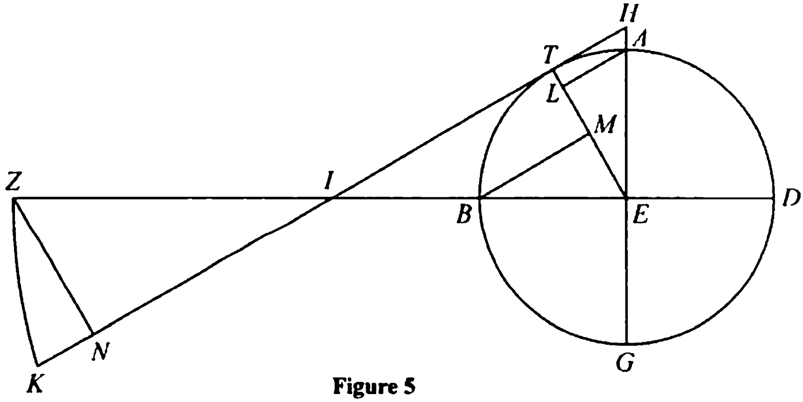



**Figure 5**

5 Postulate 5 in the *Elements* [Euclid 1956, I: 155].

فمثلثا ا ل ه ح ط ه متشابهان لإشتراك زاوية ه وكون ط و ل قائمتين فنسبة ح ه الى ا ه أعنى ه ط كنسبة ه ط الى ه ل بالرابع من سادسة الأصول فاذا ضربنا نصف القطر أعنى ه ط فى نفسه و قسمناه على ح ه يكون الخارج مقدار ه ل لما تقرر فى الأصول أن مسطح الوسطين فى الأربعة المتناسبة إذا قسم على أحد الطرفين يكون الخارج الطرف الآخر ثم إن ه ل جيباً لقوس ب ط لما تقرر أن ⟨جيب⟩ تمام اى قوس الى نصف القطر جيباً لتمام تلك القوس فيكون مساوياً لخط ب م و اذا نقصنا مربع ب م عن مربع ب ه و اخذنا جذرالباقى حصل مقدار ه م بإستبانة شكل العروس فنقول مثلثا ه ب م ه ى ط ايضاً متشابهان متساويا الزوايا كل لنظيرها و نسبة ه م الى ه ط أعنى ه ب كنسبة ه ب الى ه ى فنضرب ب ه فى نفسه و نقسم على م ه حتى يحصل ه ى و هو المقدار الأول

(4) ثم إذا نقصنا⟨ه⟩ عن ه ز نصف قطر العالم بقى ز ى معلوماً فمثلثا ز ن ى ه ط ى متشابهان متناسبان بمثل ما ذكرنا أعنى نسبة ه ط الى ن ز كنسبة ه ى الى ز ى و المجهول هيهنا أحد الوسطين فنضرب ز ى فى ه ط و نقسم الحاصل على ه ى فالخارج من القسمة ز ن و هو المقدار الثانى

(5) و بطريق آخر نقول بعد استخراج قوسى ا ط ب ط كما ذكر نقسم جيب احديهما على [تمام] جيب الأخرى ليحصل ظل كل واحد من القوسين أعنى الظل المصطلح لها و هو ح ط ى ط ثم نتمم التقريب بمثل ما ذكرناه و لو ظهر تفاوت بين ما إستخرجناه و ما إستخرجه غيرنا فلعله لإهمال بعض الكسور في أحد الحسابين و الأمر فيه سهل إذا المقصود فى أمثال هذه التعريف لا التحقيق و التدقيق و الله ولى التوفيق تم

Then, the triangles *ALE* and *HTE* are similar because they have the angle *E* in common, and *T* and *L* are right angles. Then the ratio of *HE* to *AE*, that is *ET*, is equal to the ratio of *ET* to *EL* according to *Elements* VI.4. If we multiply the radius, i.e. *ET*, by itself and divide it by *HE*, the result will be the magnitude of *EL*, because it is established in the *Elements* that for any four proportionals, if we divide the product of the means by one of the extremes, the other ⟨extreme⟩ will result.[6] Then, *EL* is the Sine of the arc *BT*, because it is established that the Cosine of any arc ⟨reaching⟩ up to the radius, is the Sine of the complement of that arc, and thus it is equal to the line ⟨segment⟩ *BM*. If we subtract the square of *BM* from the square of *BE* and take the square root of the remainder, the magnitude of *EM* will result according to the Pythagorean theorem.[7] Then we say that the triangles *EBM* and *EIT* are also similar and their corresponding angles are equal. The ratio of *EM* to *ET*, that is *EB* is equal to the ratio of *EB* to *EI*. We multiply *BE* by itself and divide by *ME* so that *EI* results, and it is the first magnitude.

(4) Then, if we subtract it from *EZ*, the radius of the universe, the remainder *ZI* will be known. The triangles *ZNI* and *ETI* are similar and proportional as we have said, i.e., the ratio of *ET* to *NZ* is equal to the ratio of *EI* to *ZI*, and here the unknown is one of the means. We multiply *ZI* by *ET* and divide the result by *EI*, the quotient is *ZN* and this is the second magnitude.

(5) In another way, we say: After finding the arcs *AT* and *BT* as mentioned, we divide the Sine of one of them by the Sine of the other to find the Cotangent of any of the two arcs, i.e., the Cotangent conventionally used for it: *HT* and *IT*. Then we complete the approximation as we have explained. Should there appear any discrepancy between what we found and what someone else found, it may be due to neglecting some fractions in one of the calculations. This is an easy matter because in such cases a ⟨general⟩ characterization is intended, not an accurate and precise ⟨numerical⟩ result. God provides success. The end.

6 See *Elements* VII.18 [Euclid 1956, II: 318–319].

7 Proposition of the Bride given in *Elements* I.47 [Euclid 1956, I: 349–350].

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# On the Origins of the *Toan phap dai thanh* (Great Compendium of Mathematical Methods)

by ALEXEI VOLKOV

The transfer of mathematical knowledge within one ancient tradition as well as from one tradition to another may take a multitude of forms, ranging from the transfer of texts to direct instruction. One way to trace the diffusion of mathematical knowledge is through the study of the itinerary of "earmarked" problems found in treatises written in "problem-solution" format, which demonstrate strong similarities as far as their texts, structures, numerical data, or algorithms are concerned. Recently the author discovered in Hanoi a Vietnamese mathematical treatise, *Toan phap dai thanh*, attributed to the Vietnamese scholar Luong The Vinh (1441–?). This paper provides a preliminary analysis of the contents of the treatise in comparison with earlier and contemporary Chinese mathematical books, in order to suggest an hypothesis concerning the date of compilation and the possible origins of the Vietnamese treatise.

## Introduction

*Modern Scholarship on Vietnamese Mathematics.* The history of mathematics and mathematics education in Vietnam have not been subjects of interest to historians of science for some time. Study of the extant historical documents has been undertaken by only a few modern scholars, and none of the original mathematical treatises preserved in Hanoi and (partly) in Paris have as yet been published or studied.

General works by colonial French scholars, such as P. Huard and M. Durand [1954], or those by modern Vietnamese authors [Ta 1979], contain only scant and sometimes unreliable information on the subject, and no more than a short paragraph was devoted to the topic in the recently translated and re-edited book, *A history of Chinese mathematics*, by J.-C. Martzloff [1997]. I am not aware of any effort made by Western scholars to locate, publish, or study the corpus of surviving Vietnamese documents concerning science, and, more specifically, mathematics.

The first attempt to study the extant materials on Vietnamese mathematics was made by the outstanding Chinese mathematician and historian of science, Zhang Yong (1911–1939).[1] In 1938 Zhang visited Hanoi and explored the collection of books at the French School of the Far East (École française d'Extrême-Orient). He discovered a dozen mathematical books, and, as Li Yan reports,

1 In this part on Zhang Yong's work on the history of mathematics, I draw mainly on Li Yan's paper [1954].

copied or purchased some of them. Unfortunately, Zhang Yong died prematurely without publishing his findings. Li Yan provides the list of twelve books studied by Zhang Yong; ten of them Li marked as "copied [by Zhang Yong]," and two, "purchased." However, my attempts to locate these books in the archive of Li Yan's materials concerning Vietnamese mathematics failed.[2] The collection contains only a set of incomplete copies of Vietnamese mathematical books mentioned in [Li 1954]; the copies consist of 4–6 manuscript pages each and represent the very first pages of the originals, that is, they include the preface (if any), the table of contents, and, in several cases, some other parts of the treatises, in particular, lists of so-called "large numbers" (that is, terms used to note the powers of 10 larger than $10^4$), multiplication and division tables, etc. I was unable to determine whether the copies in Li's collection are the original ones made by Zhang Yong in Vietnam, or only copies of his copies, presumably, made by Li Yan.

A remarkable effort to systematize the up-to-date information on the history of mathematics in Vietnam was recently undertaken by Han Qi [1991]. In his paper he introduced the history of mathematics education in Vietnam by drawing mainly on the book of Zhu Yunying [1981], and provided a brief introduction to the extant Vietnamese mathematical texts, written mainly on the basis of his study of the above-mentioned partial copies preserved in Beijing. However, the original Vietnamese texts remained unavailable to him.[3]

*The Toan phap dai thanh* (*Great Compendium of Mathematical Methods*). In 1998 and 1999 I had the opportunity to work with the original manuscripts and block-prints stored at the Institute of Han-Nom Studies (Hanoi). I was particularly interested in two manuscript copies of the mathematical treatise entitled *Toan phap dai thanh* (Great Compendium of Mathematical Methods, hereafter referred to as *TPDT-A* and *TPDT-B*). The treatise, as the first pages of both copies suggest, is ascribed to Luong The Vinh, a famous literatus and functionary who flourished in the mid-15th century. Other texts found at the Han-Nom Institute, and works in other Vietnamese libraries, will be discussed elsewhere; here discussion is devoted exclusively to the *Toan phap dai thanh*.

A major problem concerning the surviving text is the question of authorship. It is impossible to know whether it was indeed written by the 15th-century scholar Luong The Vinh (an historical figure holding a high official post), or whether this attribution was made later by the copyists or editors of what was an anonymous treatise. It is also possible that the treatise was compiled later than the 15th century, and that in order to promote the newly-written book, its anonymous compiler(s) intentionally ascribed it to Luong The Vinh, a famous author who was well-known in his day as an expert in mathematics. In order to provide at least tentative answers to questions about the date and authorship of the text, it is necessary to consider first the historical context of Vietnamese mathematics, and

2 The collection is currently found in the Institute for History of Natural Sciences, Academia Sinica, Beijing.

3 Han Qi, private communication, October 1998.

then to offer a more detailed, yet still preliminary, study of the mathematical contents of the treatise.

## Mathematics Education in Vietnam: An Historical Survey

Any study of the Vietnamese mathematical tradition first requires a consideration of cultural context and political connections with China. Even the earliest stages of Vietnamese history show close cultural ties with its northern neighbor. According to legendary accounts, the first king (*vuong*) of Vietnam, Kinh Duong, who founded the Hong Bang dynasty in 2879 B.C., was a descendant of Shen Nong, one of the mythical emperors of China. After eighteen legendary kings, Kinh Duong's line ended in 258 B.C. [Chapuis 1995: 8–12]. Modern scholarship does not confirm this legendary record of the first kings of Vietnam. However, the people living in the North of Vietnam in the third and second millennia B.C. belonged to the so-called "triangle of affiliated cultures" occupying a large territory from the modern Chinese provinces Fujian and Zhejiang in the East to Yunnan province in the West, including the Sichuan province in the North as well as Guangdong and Guangxi provinces and a part of Northern Vietnam in the South [Pozner 1994: 44, 554; Chapuis 1995: 5–6; Taylor 1983: 43].[4] In the first millennium B.C. a proto-Vietnamese kingdom, Vanlang, ruled by the Hong Bang dynasty and featured in traditional Vietnamese historiography, embraced a part of the territory of the modern Northern Vietnam; it is possible that it included parts of the Chinese provinces of Guangxi and Guangdong [Pozner 1994: 57, 558–559].[5] This kingdom may have had stable political and cultural connections with the Chinese kingdoms Yue in Zhejiang, Chu in the central part of the basin of the Yangzi (Yang-tse) river, and Ba and Shu in Sichuan and Yunnan [Pozner 1994: 52]. In 258 B.C., Thuc Phan, a descendant of the rulers of the Chinese state of Shu in modern Sichuan province, under the title of King An Duong (An Duong vuong), became the ruler of Vanlang and founded a new Thuc (Chinese "Shu") dynasty (257–208 B.C.) with its new capital situated to the North-West of modern Hanoi [Taylor 1983: 17–23; Pozner 1994: 69–88].

In 207 B.C., the Chinese general Zhao Tuo (Trieu Da in Vietnamese) founded the Trieu (Chinese Zhao) dynasty (207–111), which controlled parts of the territories of modern Northern Vietnam, Chinese Guangdong and Guangxi provinces, with its capital Phien-ngu (Chinese reading Panyu) in Guangdong [Taylor 1983: 23–37]. After a series of military campaigns launched by the Han Empire, Vietnam was annexed by China in 111 B.C. However, it appears that the Chinese presence in the country was basically formal: Vietnam had to pay a regular tribute to China, but otherwise enjoyed a high degree of political and cultural independence.

4 The ancient Vietnamese and the peoples of ancient southeastern China belonged to the same language group [Taylor 1983: 43].

5 Compare with the localization of Vanlang in [Taylor 1983: 2, map 1; Pozner 1994: 47–48, 52; Chapuis 1995: 13, map 3; 14, map 4]. On the Hong Bang dynasty, see [Taylor 1983: 1–13; Chapuis 1995: 12].

The actual Chinese domination in Vietnam began only after the suppression of the so-called rebellion of Sisters Trung in A.D. 40–44, which was actually a full-sized military campaign led by the general Ma Yuan, after which Vietnam fell under full Chinese military and political control for the first time.[6]

In China, the system of state education and examinations was created no later than the Han dynasty (206 B.C.–A.D. 220), and there are historical documents from the first and second centuries A.D. that record a group of candidates from Vietnam who successfully passed the examinations in China [Tran 1938: 18; Pozner 1994: 279, 351]. At that time the examination curricula did not include mathematics, but general demands of the times may have heightened interest of the Vietnamese in Chinese mathematics, calendrical astronomy, and related disciplines. The transmission of scientific knowledge in the first millennium A.D. may also have been facilitated through Buddhist networks[7] and individual scholars.[8] Mathematics examinations and state educational institutions were established in China during the Sui (581-618) and Tang (618-907) dynasties.[9] Even though at

6 This oversimplified sketch does not reflect the complexity of the problems related to the early history of Vietnam. For a pathbreaking reconstruction of Vietnamese history (based, however, mainly on Chinese historical materials and thus seen mainly from a Chinese historiographical perspective), see the work of H. Maspero [1916–18]. L. Aurousseau [1923] paid more attention to Vietnamese sources and proposed a different interpretation (for an assessment of some of Aurousseau's statements, see [Taylor 1983: 314–315]). Later, an important impact on the study of early Vietnamese history was made, among others, by such studies as [Tran 1932; Gaspardone 1934; Dao 1957; Taylor 1983; Pozner 1980a, 1980b, 1994].

7 In the third–fifth centuries Northern Vietnam was one of the important centers for translation of original Buddhist texts into classical Chinese. There were connections between these centers of translations of Buddhist materials in Northern Vietnam and in China; one example is provided by the biography of a monk named Hui (?–280). His ancestors were from Kang (Sogdiana) and lived for several generations in India; his father was a merchant who moved to Vietnam. Hui spoke six languages and worked at the court of the ruler of the Kingdom of Wu (modern Jiangxi and Zhejiang Provinces), where he translated a large number of Buddhist treatises. Worth notingly, Hui was versed "in astronomy/astrology and diagrammatic apocrypha" (?) (*tianwen tuwei*) *before* he arrived in China [*GSZ*, *juan* 1: 14–19]. Even though this case exemplifies the transmission of a particular scholarly expertise from India and its cultural neighbors to China rather than in the opposite direction, it suggests the existence of a channel of information that may have worked in both directions. In [Volkov 1994], the possible conceptual ties between the theory of indefinite generation of large numbers and the Buddhist theory of reincarnation found in the treatise *Shushu jiyi*, conventionally attributed to the authorship of Xu Yue (born not later than 185 — died not earlier than 227), were tentatively depicted. This treatise contains several technical terms of Buddhist/Indian origin. Interestingly enough, the presumed author of the treatise lived in the Kingdom of Wu where the monk Hui worked a few decades later. In this connection it is also worth noting that Buddhist monks transmitted mathematical and astronomical expertise from Korea to Japan, at least at the earliest stages [Volkov 1997a].

8 As Tran Van Giap points out, by the end of the second century A.D., Northern Vietnam had become a safe heaven for educated people who escaped from China during the military chaos at the end of the Han dynasty [Tran 1932: 215; 1938: 17].

9 For a description of mathematics education and the examination procedures of the Sui and Tang dynasties, see [Siu & Volkov 1999].

that time Vietnam was part of the Tang Empire, there is no evidence that in Vietnam public and private educational institutions where mathematics was taught did exist and were as active and numerous as in China, given that the recruitment of civil servants functioned in Vietnam differently from China. The highest level functionaries were sent to Vietnam from China, whereas lower-level officials belonged to local clans and were educated in local Buddhist monasteries [Taylor 1983: 126–131; Pozner 1994: 351-353]. Under these circumstances one cannot be sure that Vietnamese educational institutions strictly followed the mathematical curriculum adopted in China in A.D. 656,[10] yet it is rather likely that mathematical books written in China in the first millennium A.D. may well have circulated among the Vietnamese intellectual elite, officials, and Buddhist clergy. Some of these books may have been the same as those used in contemporary educational institutions in Korea and Japan, but it is not known exactly which, if any, mathematical books actually reached the province.[11]

After a new series of wars that put an end to the short periods of *de facto* independence in 541–546 and 555–602, the relative political stability of the province lasted from the early seventh century until 880, when the Tang dynasty army left the province and the local élites started fighting for power. Vietnam finally gained independence in A.D. 939; the country was united by one of the local clans and was proclaimed an Empire in 968. At that time the newly-born Vietnamese state took shape based upon the blueprint of its Chinese counterpart; the institution of state functionaries imitating the bureaucratic system of the Chinese Song dynasty (960–1279) was established in 1006 [Pozner 1994: 373]. Apparently, this was related to a formal request sent in 1007 by the Vietnamese government to the ruling Chinese Song dynasty for canonical Confucian and Buddhist books, which were promptly sent to Vietnam that same year. A State University (*Quoc tu giam*, Chinese *Guozi jian*, literally Directorate of Education for Sons of the State), an equivalent of the Chinese University of the Tang dynasty, was established early in the 11th century, and mathematics was included in the curriculum [*KDST*b: 1313 ff.]. There are records about mathematical examinations that took place in 1077, 1179, 1261, 1363, 1404, 1437, 1472, and 1505 [*KDST*b: 697, 984, 1293; *KDST*a, 12: 4b; *DYSJ*: 342, 484; Zhu 1981: 127, 131, n. 28, 31, 33; Han 1991: 6]. Information is currently lacking for mathematics examinations which may have been held after the beginning of the 16th century; the next time they are again mentioned is in a record of 1762, when the examinations were ordered to be held every 15 years [*KDST*a, 42: 13b; *DVSK*: 1151].

As for China, the official mathematics examinations were suspended by the end of the Tang dynasty (618–907) and reintroduced during the Song dynasty (960–1279). The mathematical textbooks were re-edited and block-printed in 1084

10 On the curriculum, see [des Rotours 1932; Siu & Volkov 1999].

11 The books used for mathematics instruction in Korean and later in Japanese State Universities were different from the twelve treatises used in China from 656 on; this is clear from a comparison of the lists of textbooks found in historical sources; see [Fujiwara 1940; Hérail 1977: 276–277; Kim 1994].

[Li 1955b]; the examinations were held in 1104, 1106, 1109, and 1113 [Hucker 1985; Martzloff 1997: 82]. On the basis of this information it may be assumed that in Vietnam the mathematics education and examination system most probably functioned, at least in the 11th century, following the old, Sui-Tang tradition, although one cannot rule out the possibility that some local modifications had been made. In particular, the textbooks used by Vietnamese scholars in the 11th century were most probably those edited during the Tang dynasty or their earlier, unofficial versions, together with other treatises circulating in the country, and not the Chinese re-editions of 1084. It is possible that later some of the Song editions of the mathematical classics (1084 and 1212–1213) also reached Vietnam and were used for mathematics instruction.

However, one cannot be sure whether by the mid-15th century (the time when the *Toan phap dai thanh* was presumably compiled) the old pre-Song versions of the mathematical treatises or their later Song editions were still in use in Vietnamese state education. One of the reasons for this uncertainty is that during the war with Champa a fire occurred in the Vietnamese capital in 1371 and destroyed a large number of books [Tran 1938: 43]. Moreover, the restored collections suffered a great loss soon after when the Chinese general Zhang Fu sent all Vietnamese archives and a large number of ancient and contemporary books to Nanjing during a short-term Chinese occupation in 1413–1427.[12] It is difficult to estimate the scale of the damage caused to Vietnamese governmental and private collections by the Chinese invasion; according to [Tran 1938: 44–45], more than fifty percent of the books were lost.[13] If at least some mathematical books from the state libraries were also moved to China, it is possible that the compilation of one or several mathematical treatises was undertaken in the late fifteenth century in order to provide state educational institutions with new textbooks to compensate for the loss of the old ones. This conjecture is indirectly corroborated by the fact that in 1460–1497 two successive Imperial orders prescribed a search for ancient books all over the country and delivery of them to the court [Tran 1938: 45]. Official scholars of that time, such as Luong The Vinh, thus may have obtained access to the newly found mathematical books, if any, and may have used them to produce their own compilations.

12 The order to search the Vietnamese books and to send them to China was issued by the Chinese Emperor Chengzu (Yongle) in 1418 or in 1419 [Tran 1938: 43, n. 3]; see also [Cadière & Pelliot 1904: 619, n. 3].

13 I am not aware of any considerable collections of books from 15th-century Vietnamese libraries actually found in China. The only exception is the *Short history of the Great Viet[nam]* (*Dai Viet Su Luoc*; Chinese: *Da Yue shilüe*) brought to China during the invasion. In the late 18th century, this work was found in the collection of the Governor of Shandong province and published in the Chinese Imperial Encyclopedia, *Siku quanshu* [Pozner 1980: 277–278; Nikitin 1993].

## The Treatise *Toan phap dai thanh* and Its Author

*The Extant Copies*. There are two manuscript copies of the treatise *Toan phap dai thanh*, both found in the Han-Nom Institute (Hanoi), manuscript A (*TPDT-A*) and manuscript B (*TPDT-B*).[14] When manuscript B was produced is unknown (but certainly prior to 1934), while manuscript A on the opening page notes the date when the copy was made: year Giap Than of the era Bao Dai, that is, 1944. Neither manuscript has a preface or a postface, or any other data which would date the treatise or positively identify the author(s). The name "Doctor Luong The Vinh" is written only on the first page of each manuscript next to the title. Manuscript A contains several copyists' mistakes and is written with a script which in many cases is difficult to read if compared with manuscript B. The first impression is that both copies were made on the basis of the same original, which, by the time of reproduction, had several defects (a large section is missing at the beginning of the text which neither copyist has noticed; probably one or more pages of the original were missing). It is also possible that manuscript A is a copy of manuscript B, made by an inexperienced copyist. However, further textual studies are needed to confirm this hypothesis.

*Modern Scholarship on the Personality of the Presumed Author and the Time of Compilation.* Luong The Vinh is widely known in Vietnam as a skillful mathematician. However, traditional Vietnamese and Western scholarship provides little information concerning this author and his mathematical work. The available data may be summarized as follows:

In *Connaissance du Viet-Nam* [Huard & Durand 1954], we are told that:

> [A]n arithmetical treatise, the *Toan phap dai thanh*, is credited to Luong The Vinh (doctor in 1463); this appears to be a reworked treatise by Vu Huu, a contemporary of the latter, devoted to use of the abacus.[15]

The same authors also inform us that:

> Vu Huu, who obtained his doctorate in 1463, compiled a *Dai thanh toan phap*,[16] which was edited by his contemporary Luong The Vinh (doctor in 1478).[17] This is one of the rare works on mathematics mentioned in a bibliography by Le Quy Don.[18]

14 The call numbers as well as the seals on several pages of each manuscript suggest that these manuscripts used to belong to the collection of the École française d'Extrême-Orient.

15 "On attribue à Luong-the-Vinh (docteur en 1463), un ouvrage d'arithmétique intitulé *Toan phap dai thanh* (Méthode très complète de calcul) qui aurait été un remaniement d'un ouvrage de Vu-Huu, un de ses contemporains, traitant de l'emploi de l'abaque" [Huard & Durand 1954: 144]. On the use of the abacus see below.

16 Note that the treatises supposedly written by Vu Huu and Luong The Vinh had *different* titles.

17 One can notice a contradiction concerning the dates when Luong The Vinh received his doctoral degree.

18 "Vu-Huu, reçu docteur en 1463, a composé un *Dai thanh toan phap*, mis au point par son contemporain Luong-the-Vinh (docteur en 1478). C'est un des rares ouvrages de mathéma-

According to these brief remarks, the biographies of the two individuals who are considered as authors of the treatise are almost identical. Both, it is said, were born in the same year (1441) and obtained their degrees also in the same year (1463); the books they supposedly wrote have virtually the same titles (compare *Toan phap dai thanh* and *Dai thanh toan phap*), and the book by Vu is said to have been edited by Luong. However, as we shall see later, the Vietnamese sources depict the activities and personalities of these two scholars in very different terms.

Han Qi also mentions Vu Huu as the author of the *Dai thanh toan phap*, later reworked by Luong The Vinh who, Han Qi adds, "introduced the Chinese mathematical methods to Vietnam" [1991: 8]. The statement well may be true, but the author does not provide any evidence for this assertion.[19] Theoretically, the authors of the Vietnamese mathematical works of the 15th century may have drawn exclusively upon the rich tradition that already existed in Vietnam from the late first–early second millennium A.D. (which certainly originated, at least partly, from Chinese mathematics).

Jean-Claude Martzloff in his book *A History of Chinese Mathematics* depicts the mathematical activity in 15th-century Vietnam rather differently. As he reports,

> Vu-Van-Lap's *Nam su tap bien* (1896) mentions two Vietnamese mathematical works entitled *Cuu chuong toan phap* (Chinese: *Jiuzhang suanfa*, Mathematics in Nine Chapters) and *Lap thanh toan phap* (Chinese: *Licheng suanfa*, Ready Reckoner). [...] Both were composed in 1463 by two successful candidates in the civil service recruitment examination.[20]

We shall return to Martzloff's account later.

*Traditional Vietnamese Scholarship on the Personality and Work of Luong The Vinh.* A study of extant medieval Vietnamese texts undertaken by the Vietnamese scholar Le Quy Don (1726–1783/4),[21] whose work was completed and published by Phan Huy Chu (fl. ca. 1821), is the most important source on the mathematical treatises by Luong The Vinh. The bibliography compiled by Le and Phan was thoroughly investigated by E. Gaspardone [1934] and later translated into French by Tran Van Giap [1938]. One bibliographical entry that may be relevant to the present study reads as follows (the English translation is mine):

tiques mentionnés dans la bibliographie de Le-qui-Don" [Huard & Durand 1954: 120]. The bibliography mentioned is probably chapter 6 of the *Kien-van tieu-luc* (Little Records of What Was Seen and Heard) by Le Quy Don (1726–1783/4), referred to in [Huard & Durand 1954: 6].

19 When making this statement, Han Qi refers to a work by P. Huard and M. Durand with the English title "The Beginning of Science in Vietnam," which I have been unable to find; the above-mentioned book in French [Huard & Durand 1954] does not mention that Luong The Vinh introduced any specifically Chinese mathematical methods to his compatriots.

20 [Martzloff 1997: 110]. Martzloff refers to [Tran 1938: 93], but the specified page does not contain the information he mentions.

21 On Le Quy Don see [Gaspardone 1934: 18–24; Durand 1998: 238–240].

> *Dai thanh toan phap*, "General Notions of Mathematical Theories." A work in two volumes (*quyen*, lit. "rolls"), written by Vu Quynh [...].[22] [...] In the *Nam su tap bien* of Vu Van Lap [...], two works on mathematics are mentioned: the first entitled *Cuu chuong toan phap* [is] by Luong The Vinh, *hieu* [appellation] Thuy-Hien, who was born in Cao-Huong village of the *huyen* [district] Thien-Ban (present-day Vu-Ban [district] in Nam-Dinh [province]), and was the first in the doctoral examinations in the fourth year [of the era] *quang-thuan* (1463); the second [is] the *Lap thanh toan phap* by Vu Huu, who was born in Mo-Trach village, and obtained the doctoral degree in the same year as Luong The Vinh (1463). According to the biography of Luong The Vinh [by] Phan Huy Chu [...], the former author edited the text of the *Dai thanh toan phap*, while Vu Huu, according to the *Cong du tiep ki* [...], was a true mathematician, well known in his time. This latter compiled the *Dai thanh toan phap*, which he used to teach the methods of measuring the rice fields. The Emperor gave him the task of constructing the gates of the fortresses. One can conclude that the *Dai thanh toan phap* was written by Vu Huu and edited by Luong The Vinh. [The entry] is not, therefore, about Vu Quynh, and here we deal with an omission made by Le Quy Don and Phan Huy Chu [Tran 1938: 97–98 (bibliography # 111)].[23]

This passage can be briefly summarized as follows:

(1) The treatise mentioned by Le Quy Don was entitled *Dai thanh toan phap* (unlike the text found in the Han-Nom Institute, the *Toan phap dai thanh*), and consisted of two volumes (*quyen*, part or volume, literally "rolls," Chinese: *juan*), while the *Toan phap dai thanh* does not show any visible subdivision into two parts. The author of the *Dai thanh toan phap* was Vu Quynh, a famous official. The intention of Tran is (a) to prove that the attribution made by Le Quy Don is erroneous and the treatise was written by Vu Huu, and (b) to reconcile this conclusion with the statement made by Phan Huy Chu that this treatise was "edited" by Luong The Vinh.

22 Vu Quynh was born ca. 1452 in the same Mo-Trach village as Vu Huu (see below), and died after 1516 [Gaspardone 1934: 76–77].

23 The original text reads: "*Dai thanh toan phap* 'Notions générales des théories mathématiques'. Ouvrage en 2 *quyen*, écrit par Vu-Quynh (voir: *supra*, L.Q.D., n° 62). P.H.C. (q. 45, f° 162), IV, 219. Il est fait mention dans le *Nam su tap bien* de Vu-van-Lap (q. 3, f° 44 et 45) de deux ouvrages de mathématiques: le premier intitulé *Cuu chuong toan phap* par Luong-the-Vinh, *hieu* Thuy-hien, qui, originaire du village de Cao-huong du *huyen* de Thien-ban (Vu-ban actuel à Nam-dinh) fut reçu premier au concurs du doctorat en la 4° année *quang-thuan* (1463); le second, *Lap thanh toan phap* par Vu-Huu qui, originaire du village de Mo-trach, fut reçu docteur en la même année que celle où fut reçu Luong-the-Vinh (1463). D'après la biographie de Luong-the-Vinh dans le P.H.C. (q. 6, f°108), le premier auteur 'a mis au point les textes du *Dai thanh toan phap*'; tandis que Vu-Huu, d'après le *Cong du tiep ki* (q. 1, f° 7 r°), fut un véritable mathématicien fort réputé en son temps. Ce dernier a composé le *Dai thanh toan phap* par lequel il a enseigné la méthode pour mesurer les rizières. Il fut chargé par l'empereur du calcul concernant la reconstruction des portes des citadelles. Il en résulte que le *Dai thanh toan phap* avait été établi par Vu-Huu et qu'il fut mis au point par Luong-the-Vinh. Il n'est donc pas de Vu Quynh, et il y aurait là une omission par L.Q.D. et P.H.C."

(2) In a bibliographical work of a certain Vu Van Lap, two works are mentioned: *Cuu chuong toan phap* (Chinese: *Jiuzhang suanfa*) by Luong The Vinh, and the *Lap thanh toan phap* (Chinese: *Licheng suanfa*) by Vu Huu. But Vu Van Lap, unlike Le Quy Don, does not credit either Luong The Vinh or Vu Huu as the author of a book entitled *Dai thanh toan phap*.

(3) Tran turns to the biography of Luong (provided by Phan Huy Chu, a commentator of Le Quy Don who to a certain degree followed the latter's opinion), which provides a compromise: Phan informs the reader that Luong edited the treatise, while in another source the same treatise was credited to the authorship of Vu Huu, who was a "real mathematician." Tran concludes that both Le and Phan are wrong, that the actual author of the *Dai thanh toan phap* was Vu Huu (and thus the name of Vu Quynh should be changed to Vu Huu), and Luong was a later editor of the treatise.[24]

This entire matter was treated slightly differently by the French scholar E. Gaspardone, who writes as follows:

> *Dai thanh toan phap* (Very complete method of calculation [sic — A.V.]). In 2 q[uyen = chapters]. By Vu Quynh.
>
> The biographies of Vu Quynh that I consulted [...] do not mention this work at all. One MS (A 2931),[25] damaged at the beginning, has on the first pages replacing [the lost original pages] the title *Toan phap dai thanh* and indicates: "Compiled by the Doctor of State Luong The Vinh."[26] Another MS (A 1584) of an analogous [= mathematical] work written by a more careful copyist, limits itself in evoking the tradition: "[this is] a vulgar transmission of [the method] for which the origin was established by the winner of the First Prize [of doctoral examinations] of the Southern State [= Vietnam], [Mr.] Luong The Vinh,"[27] and [this remark] is related to only a part of its contents.[28] The biography of Luong The Vinh [...] does not speak about this work.[29] The *Dai thanh toan hoc chi minh*, a recent work by Phan Gia Ki, who lived in the first part of the 19th century, provides no more than a similar title (MS A 1555). Phan Huy Chu, at the end of ch. 45, reproduces Le Quy Don [Gaspardone 1934: 149].[30]

24 Martzloff [1997: 110] mistook the year of examination of Luong and Vu for the year of compilation of their books, and mixed up the doctoral and recruitment examinations.

25 This is manuscript B.

26 My translation of the Sino-Vietnamese characters in the work of Gaspardone. — A.V.

27 Translation of the phrase in classical Chinese is mine. — A.V.

28 The treatise which Gaspardone mentions is catalogued under the title *Toan phap ki dieu*, and on the first page the reign of the Emperor Minh Menh (reigned 1820–1840) is mentioned, that is, this copy must have been produced after 1820. However, the treatise appears to be based on earlier sources. The remark about Luong is made concerning only one arithmetical method, at the very beginning of the treatise.

29 [Gaspardone 1934: 139]. Interestingly enough, here a preface by Luong to a Buddhist treatise is mentioned.

30 The original text reads thus (I omit Sino-Vietnamese characters): "*Dai thanh toan phap* [...] (*Méthode très complète de calcul*). 2 q. Par Vu Quynh. Les biographies de Vu Quynh que j'ai consultées (cf. n° 1) ne font aucune allusion à cet ouvrage. Un ms. (A. 2931), mutilé au début, porte sur les premiers f[oli]os qui le remplacent le titre de *Toan phap dai thanh* et l'indi-

Thus, Gaspardone informs us that:

(1) the biographies of Vu Quynh available to him do not mention a treatise entitled *Dai thanh toan phap*;
(2) a treatise A 2931 (i.e., manuscript B) is credited to Luong The Vinh, but the name of Luong is written only on the pages replacing the missing first part and may well have been added later;
(3) the received biography of Luong The Vinh does not mention this treatise; however, another mathematical manuscript (A 1584) mentions Luong as an authority in mathematics.[31]

Gaspardone does not mention the treatise written by Vu Huu at all, and, apparently, believes that Luong, indeed, was skillful enough and that he may well have written a mathematical treatise with the title resembling that mentioned by Le Quy Don; however, he does not reject the possibility of an *a posteriori* attribution of treatise A 2931 (manuscript B) to Luong.

The first page of the manuscript A of the *Toan phap dai thanh* also features the name of Luong The Vinh, as well as his doctoral title, but this cannot prove that Luong was indeed the author, since this copy was not made until 1944, that is, *after* Gaspardone had seen manuscript B (it was remarked previously that manuscript A may be a copy of manuscript B). Thus Gaspardone's question about the authorship of manuscript B has yet to be answered.

Interestingly, in the text of both manuscripts there is a reference to a certain text entitled *Dai thanh*. This may be a shortened form of either the *Toan phap dai thanh* or the *Dai thanh toan phap*. More importantly, this *cannot* be a reference to the treatise of manuscripts A and B itself, since the author quotes the "Method of the 'Great Compendium' (*Dai thanh*)," and then suggests his own "new method." However, this reference may be explained on the hypothesis that Luong edited and commented on the text of one of his predecessors (supposedly, Vu Huu).

What conclusions can be drawn from these data?

First, the attribution to Vu Quynh of a mathematical text (in two *quyen*) entitled *Dai thanh toan phap* does not exclude the possibility that an earlier version of the text was actually written by Vu Huu.[32] Probably, this book (attributed by various authors, as we have seen, to Vu Huu, Vu Quynh, and Luong The Vinh) is not the same as the *Toan phap dai thanh* found in the Han-Nom Institute. Not

cation: [...]. Un autre ms. (A. 1584) d'un ouvrage analogue et d'un copiste plus prudent, se borne à invoquer la tradition [...], et pour une part seulement de son contenu. La biographie de Luong The-Vinh (cf. n° 142) ne parle pas d'un tel ouvrage. Le *Dai thanh toan hoc chi-minh* [...], ouvrage récent par Pham Gia-Ki [...], qui vécut dans la premiere moitié du XIX$^{e}$ s., ne fournit guère qu'une confirmation du titre (ms. A. 1555). Phan Huy-Chu, q. 45 *in fine*, reproduit Le Qui-Don."

31 At this point, Gaspardone and Tran slightly disagree, since Tran apparently believes that Luong was less skillful in mathematics than Vu Huu.

32 They both were born in the same village, and it may be conjectured that they belonged to the same clan. It is not impossible, therefore, that a mathematical book circulated within the clan of Vu, and that Vu Quynh edited the book written some time earlier by his relative.

only do their titles differ, even though they consist of the same four characters, but the latter is not subdivided into *quyen*, while the former is said to be comprised of two *quyen*.

Second, there is some evidence of Luong The Vinh's mathematical expertise. Luong was a contemporary of Vu Huu, and also a distinguished literatus and state functionary. It is possible that he compiled a book with the same or a slightly different title than that written by Vu. One source credits him with the authorship of the *Dai thanh toan phap*, which again differs from the title of the extant treatise, while another suggests that he instead wrote a book entitled *Cuu chuong toan phap*. These two statements do not contradict each other, since it is not impossible that Luong authored two or even more mathematical books.

A preface to the *Cuu chuong lap thanh toan phap* (Chinese: *Jiuzhang licheng suanfa*) by Pham Huu Chung (1722)[33] quotes the first chapter of the *Dang khoa luc* where Luong is credited with the authorship of the *Dai thanh toan phap*; meanwhile, Pham attributes to Luong the *Cuu chuong lap thanh toan phap*, saying that he, Pham, reworked Luong's treatise and translated it into "National language," i.e., *Nom* (he probably added the terms "*lap thanh*" to its title). One can conclude that by the early 18th century Luong was indeed credited with the authorship of both books, the *Dai thanh toan phap* and *Cuu chuong toan phap*.

## The Contents of the Treatise and its Origins

In the following, several "earmarked" mathematical problems and methods found in the *Toan phap dai thanh* will be discussed in order to suggest the possible origins of the contents of the treatise.

*The Structure of the Treatise and its Contents.* The treatise is compiled in the traditional "Chinese" way, that is, it is designed as a collection of problems with numerical answers given along with the procedures (algorithms) for their solution. However, there are procedures that do not correspond to any particular problem; this most probably is the result of omissions of parts of the text containing the corresponding problems and answers. The problems often begin with the traditional "Now we have..." (Vietnamese *kim huu*, Chinese *jin you*) and the algorithms are usually introduced as "methods" (Vietnamese *phap*, Chinese *fa*), yet this format varies slightly from one problem to another. In some cases the problem is not explicitly stated, but is introduced with a diagram of a geometrical figure with given dimensions. In one case neither numerical data nor a question is found, yet this may well be a fragment from the famous "Chinese remainder theorem" (# 136, see below). The total number of problems in the treatise thus amounts to 138. The text of the treatise can be provisionally subdivided into 8 parts, as follows:

33 A manuscript copy of this treatise is attached to the fourth chapter of the *Chi minh lap thanh toan phap* (Chinese: *Zhiming licheng suanfa*), call no. A 1240, found in the École française d'Extrême-Orient in Paris.

Part 1 (Problems 1–35). These problems are all devoted to partitioning, and, in particular, to division. For example, Problems 1 and 2 are as follows: $y = ax$, $z = bx$, $x + y + z = N$, find $x$, $y$, and $z$. In Problems 11–19, $S$ monetary units are to be equally distributed among $N$ people. In Problems 21–35, the reader is asked to divide an amount of money $S$ by $N$ (the number of people); here $S$ is always equal to 123,456,789, and $N$ subsequently is equal to 1, 2, ... , 9, 11, 12, ... ,19.

Part 2 (Problems 36–42). These problems are devoted to the calculation of the areas of plane figures: a square (# 36), a rectangle (# 37), a figure approximated by the area of a trapezium (# 38), a circle (# 39), and a segment of a circle (# 41). The author(s) adopted the ratio Circumference : Diameter = 3 : 1 and used the classical Chinese formulas $S = C^2/12$, $S = 3D^2/4$, and $S = CD/4$ for the area of the circle (where $C$ is the circumference, and $D$ the diameter), and $S = [(h + b)/2]h$ for the area of the segment (where $h$ is the height of the segment, and $b$ the chord).

Part 3 (Problems 43–69). These problems are devoted to proportions, the Rule of Three, and the Rule of Double False Position, as well as to rather simple cases of multiplication and division. Here one finds a method for calculating the height of an object when the height of another object and the length of the shadows of both objects are given (# 43; see below for further discussion). In Problems 57, 58, and 61, $N_i$ objects of the $i$-th category cost $A_i$ monetary units, $i = 1, \ldots, n$, where here $n = 2$ and 3; the reader is asked how many objects one can buy for a given amount of money $S$. Problems 59 and 60 feature the Method of Double False Position: $Ax = N + a$, $Bx = N - b$, $A = B + 1$. The algorithms are as follows: $x = a + b$, $N = A(a + b) - a$ (# 59), $N = B(a + b) + b$ (# 60).

Part 4 (Problems 70–85). This group is devoted to root extraction (## 70–81), and to an auxiliary algorithm used for the conversion of one type of monetary units to another.

Part 5 (Problems 86–93) is a sequel to Part 3. The reader is asked to solve problems on the calculation of interest and on multiplication/division. However, there is a problem devoted to the calculation of the volume of a solid figure referred to as a "boat" (# 88), as well as a fragment on divination placed beween Problems 88 and 89.

Part 6 (Problems 94–131) is devoted to various subjects, in particular, to the calculation of the areas of various figures. Here one finds such shapes as rectangles, circular segments, a "horn of the bull," circles, "drums," ellipses, rings, an "eye-lid" (or "eye-brow," that is, the intersection of two circles), an isosceles triangle, a rectilinear figure composed of several adjacent trapezia, a trapezium, a quadrilateral with four given sides $a$, $b$, $c$, $d$ (the formula provided is $S = ([(a + c)/2][(b + d)/2])$, and the figure formed by two adjacent squares. The remaining problems in this group are devoted to the extraction of square roots, calculation of the volumes of rectilinear solids, and to the conversion of metrological units.

Part 7 does not contain mathematical problems; this is a large independent text devoted to land taxation.

Part 8 (Problems 132–138) embraces various problems devoted to "numerical divination" (## 133-134), the calculation of the height of a tree when the length of its shadow is given (# 135), a rhymed solution to a famous problem of indeterminate analysis from the *Sunzi suanjing* (# 137), and the calculation of the area of a quadrilateral (once again, using the formula $S = [(a + c)/2][(b + d)/2]$).

*The Use of Counting Instruments.* In China the abacus was used as early as the 12th century;[34] however, for a long time it remained an instrument used mainly for accounting and financial transactions, while professional mathematicians continued to use the counting rods as late as the 14th and 15th centuries. The abacus is briefly mentioned in some mathematical books of that time.[35] Many Chinese books published after this date, and, in particular, the influential *Suanfa tongzong* (1592) by Cheng Dawei, make extensive use of the instrument. It remains unknown when the abacus and related books were known in Vietnam, yet if the *Toan phap dai thanh* was compiled relatively late (say, in the 18th century or later), the operations with the instrument most likely would be mentioned by its author(s). Conversely, absence of references to the abacus would suggest an earlier date of compilation. Unfortunately, in certain cases the description of arithmetical operations performed with the abacus can look very similar to those performed with counting rods, given that many of the arithmetical operations for the abacus originated from operations with counting rods, and the same terms were used for certain operations performed with both. This is why sometimes one cannot be certain whether the author of a given medieval treatise is speaking about operations performed with the abacus or with counting rods. However, preliminary analysis of the *Toan phap dai thanh* suggests that, contrary to the opinion of Huard and Durand quoted above, who insisted that Luong The Vinh's book describes operations with the abacus, the compiler(s) of the *Toan phap dai thanh* were more likely describing algorithms for counting rods. The evidence supporting this hypothesis is as follows:

34 This statement is supported by the image of an abacus found in the famous Chinese scroll, *Qingming shanghe tu* (Picture of [Sailing] Up the River [during] the Qingming Festival) by Zhang Zeduan (fl. ca. 1126). Martzloff [1997: 216] doubts that the painted object is an abacus, but his objections are probably due to a misunderstanding, since the picture to which he refers is only a modified copy of the original scroll made in the 18th century. For references to the abacus in literary sources see [Hua 1987].

35 The *Jiuzhang suanfa bilei daquan* (1450) by Wu Jing [*JSBD*] contains phrases that can be interpreted as references to the instrument [Hua 1987]. The earliest mathematical book which provides a picture of the abacus is the *Zhiming suanfa* (Computational Procedures with Leads and Clarifications) by Xia Yuanze (1439) [Kodama 1970]. Another early book especially devoted to operations with the abacus is the *Panzhu suanfa* (Counting Procedures for Pearls on a Plate) by Xu Xinlu published in 1573 (the term "Pearls on a Plate" refers to the abacus).

(1) The abacus is never explicitly mentioned in the text of the treatise, and there is no picture of any such instrument. This makes the treatise look very different from all other Chinese and Vietnamese books featuring application of the abacus, which usually include several pictures of the instrument at the beginning of the book, and devote at least several pages to explaining how the arithmetical operations should be performed.
(2) When an algorithm involving two operands is described, they are always said to be put in the upper part (Vietnamese *thuong*, Chinese *shang*) and in the lower part (Vietnamese *ha*, Chinese *xia*), and not "to left" and "to right" as one might expect in a text referring to an abacus (this argument, however, is not conclusive, given the fact that the left and the right parts of the abacus were also referred to as the "upper" and "lower" parts in some Chinese texts).
(3) In several cases the reader is told to set a third number on the counting instrument apart from the two numbers already mentioned. This again suggests that the author(s) had in mind counting rods rather than the abacus.
(4) In the treatise such expressions explicitly occur as "spread the counting rods" (Vietnamese *bo toan*, Chinese *bu suan*) (Problem 2) and "the counting rods" (Vietnamese *toan tu*, Chinese *suanzi*) (Problem 134 about determining the sex of a new-born child).
(5) The treatise contains so-called "tables of division." These tables appeared in China as early as the 11th to 12th century, when the abacus was not yet in use. Later the same tables were used for computations with the abacus [Yamazaki 1959; Hua 1987: 229–265]. Their presence thus cannot prove that the compiler(s) necessarily had use of the abacus in mind.

To conclude, preliminary analysis of the treatise suggests that most probably the *Toan phap dai thanh* was compiled *before* Vietnamese mathematicians were acquainted with any Chinese books devoted to abacus calculations (one can suggest that this happened no later than the mid-18th century). If the book was compiled later than this time, the compilation must have been done exclusively on the basis of early Chinese and Vietnamese mathematical books that do not discuss explicitly operations with the abacus.

*The Table of Multiplication.* The treatise contains the multiplication table beginning with $9 \times 9$ and descending to $1 \times 1$. In China the tables of this kind were called *jiujiu*, that is, "nine nines" or "nine [times] nine." In the history of Chinese mathematics, two types of multiplication tables are known. The first type occurs in texts prior to the Southern Song dynasty (1127-1279). The tables of this kind begin with "nine [times] nine [is] eighty one" (this line gave the name to the table), and finish with "two [times] two [is] like 4" (before the fifth century) or "one times one [is] like 1" (after the fifth century).[36] The tables of the second kind appear in texts of the Southern Song (1127-1279) dynasty or later. They begin

36 I translate the word *ru* as "like" ("to be alike"), which occurs only when the product has only one figure; the term is used, probably, to make the lines in the table fit within the same format.

with "one [times] one [is] like one" and go up to "nine [times] nine [is] eighty one."[37]

The multiplication table in the Vietnamese treatise begins with "nine nines is eighty one" and ends with "one times one is like one." This suggests that the original (Chinese) treatise from which the table was borrowed was written not earlier than the fifth century A.D. and not later than the beginning of the 13th century. On the basis of further evidence provided below, it is likely that the table was borrowed from a version of the *Sunzi suanjing* that reached Vietnam probably no later than the end of the first millennium A.D. As we shall also see below, this hypothetical version of the *Sunzi suanjing* was probably *different* from the earliest extant version printed in China in 1212–1213 (the latter has been conventionally considered identical with the edition of 1084).

*The "Large Numbers."* The Vietnamese treatise contains a fragmentary list of so-called "large numbers," that is, technical terms for the powers of 10 higher than 4.[38] This system appears for the first time in the classic Chinese mathematical texts *Shushu jiyi*, *Wujing suanshu*, and *Sunzi suanjing*.[39] The Vietnamese treatise contains a very brief explanation of the system which reads:

> [*Commentary*:] The [formation] rule for [large] numbers [is as follows]:
> 1, 10, 100, 1,000, 10,000 (*van*; Chinese *wan*);
> 10 [*van*], 100 [*van*], 1,000 [*van*], [*van van*, that is,] *ti* (Chinese: *yi*);

37 See, for example, [Li & Du 1987: 13–15]. To my knowledge, this fact was discussed in detail for the first time in the paper of Mikami Yoshio [1921]. Mikami found the earliest tables of the second type in two treatises, the *Suanfa tongbian benmo* by Yang Hui (mid-13th century) and the *Suanxue qimeng* by Zhu Shijie (1299). Mikami's discussion concerns a Japanese treatise entitled *Kuchizusami* (preface of A.D. 970), in which a table of the first type (ending with "one times one") is found; for the table from the *Kuchizusami*, see [*KZ*: 83]. Li Yan in a paper originally printed in 1925 repeated certain ideas of Mikami's paper and, moreover, compared the table in the Japanese treatise with those found in the *Sunzi suanjing* and in a mathematical text from Dunhuang, see [1955a: 172–173]. The table in the extant version of the *Sunzi suanjing* ends with "one times one is like one," while the table from Dunhuang ends with "two times two is like four." A comprehensive study of the history of those tables in China prior to the Song dynasty is provided in a recent paper by [Ou *et al.* 1994]. On the "Nine nines" multiplication tables in Dunhuang MSS, see also [Libbrecht 1982: 218, 228].

38 Interpretations of the numerical values associated with the names of the "large numbers" in the *Shushu jiyi* are incorrect in the works of several scholars (including J. Needham, Qian Baocong, Ho Peng-Yoke, J.-C. Martzloff). Some of their mistakes are remarked upon and corrected in [Brenier 1994: 98]; later the correct interpretation was published in the English edition of the book by J.-C. Martzloff [1997: 99], although Martzloff's original French version followed Needham.

39 The *Shushu jiyi* has traditionally been said to have been written at the beginning of the third century A.D.; however, some authors suggest that the treatise was actually written by its commentator, Zhen Luan (fl. ca. 560). In either case, the extant version of the treatise is *not* the one used in the Tang dynasty mathematics college, but a version (probably defective) found in the early 13th century [Volkov 1994]. The *Sunzi suanjing* was compiled, as suggested by Wang Ling, at some time between A.D. 280 and 473; see the reference in [Libbrecht 1982: 212, n. 53]. See also [Yan 1937; Berezkina 1963: 5–6; Lam & Ang 1992: 4–7].

10 [*ti*], 100 [*ti*], 1,000 [*ti*], *wan* [*ti*];
10 [*van ti*], 100 [*van ti*], 1,000 [*van ti*], [*van van ti*, that is,] *trieu* (Chinese: *zhao*).

This is one of the three systems described in the *Shushu jiyi*, the so-called "medial numbering system."[40] According to this system, a new term (*yi*, *zhao*, ...) for the power of ten is used every time when the previous term is multiplied by $10^8$. The excerpt above can be re-written in modern notation as follows:

$10^0$, $10^1$, $10^2$, $10^3$, $10^4$ [ = *van*];
$10^5$ [ = $10^1$ *van*], $10^6$ [ = $10^2$ *van*], $10^7$ [ = $10^3$ *van*], $10^8$ [ = *van* times *van*] = *ti*;
$10^9$ [ = $10^1$ *ti*], $10^{10}$ [ = $10^2$ *ti*], $10^{11}$ [ = $10^3$ *ti*], $10^{12}$ [ = *van* times *ti*];
$10^{13}$ [ = $10^1$ *van ti*], $10^{14}$ [ = $10^2$ *van ti*], $10^{15}$ [ = $10^3$ *van ti*],
$10^{16}$ [ = *van* times *van ti*] = *trieu*.

This table establishes a pattern for forming the higher powers of ten. Unfortunately, the extant part of the list of "large numbers" used to note the powers $10^{8k}$, $k = 3, ..., 11$ found in the Vietnamese treatise is incomplete; it represents only the very end of the full list. Apparently, the previous page (or pages) of the original text was (were) lost, and the copyists who made the two extant copies did not notice this.[41] The extant part reads as follows:

*van van* [times] *nhuong* (Chinese: *rang*) is one *gian* (Chinese: *jian*);
*van van* [times] *gian* is one *chanh* (Chinese: *zheng*);
*van van* [times] *chanh* is one *tai* (Chinese: *zai*);
*van van* [times] *tai* is one *cuc* (Chinese: *ji*) [*TPDT-A*: 1a].

This is a fragment of the Chinese medial system, but with one mistake made by the copyists: they mistook the character *cau* (Chinese: *gou*) for the character *nhuong* (Chinese: *rang*).

Another reference to the "large numbers" is found only in manuscript B [*TPDT-B*: 201]; in manuscript A it was probably incidentally omitted by the copyist. It reads as follows:

*Van van* units is one *ti*; *van van* [times] *ti* is one *trieu*; [similarly, one has] *kinh* (Chinese: *jing*), *cai* (Chinese: *gai*), *nhuong*, *cau*, *gian*, *chanh*, *tai*, and *cuc*.

Here the list is almost complete (only one "large number," *ti*, Chinese *zi*, equal to $10^{40}$, is missing; it should have been placed between *cai* and *nhuong*), yet some characters are written slightly differently from their counterparts found in Chinese texts. Since manuscripts A and B probably stem from the same prototype, or manuscript A is a copy of manuscript B, it is likely that the original (now lost) text of the treatise must have included the complete list of the "large numbers," although possibly written with slight graphical modifications.

40 For more on the three systems see [Volkov 1994].
41 The loss of a few pages at the beginning was remarked upon by [Gaspardone 1934], who, most probably, inspected the type of paper and paleographical data, rather than the mathematical contents of the text.

The descriptions of the system of "large numbers" found in various early Chinese sources sometimes provide different numbers of elements disposed in different orders;[42] one possible explanation for this phenomenon is that at the early stages the system was not well established, whereas at later stages the differences were probably due mainly to mistakes of copyists. The latter might also be the case for the Vietnamese text, where one of the "large numbers" is missing. As for the unorthodox forms of several terms in the Vietnamese treatise, the Dunhuang manuscripts also contain characters which differ from the ones found in the *Shushu jiyi*, *Wujing suanshu*, and the *Sunzi suanjing*.[43]

The most important detail is that the list contains the "large number" expressed with the character *cuc* (Chinese *ji*). This term appears in the Chinese texts for the first time in the context of the "medial series of large numbers" in one of the Dunhuang manuscripts produced no later than the ninth–tenth century. Thus the time when the term appears in mathematical texts in China can be roughly estimated as sometime between the seventh and ninth centuries, since it is not found in the extant texts of the collection of mathematical texts *Suanjing shishu* (edited in the mid-seventh century);[44] nor is it found in the list provided by the abovementioned Japanese treatise, *Kuchizusami* [*KZ*: 83]. We can therefore set the lower boundary for the time when the list was transmitted to Vietnam from China as the very end of the first millennium A.D.

*Rhymed Algorithms for Computation of Areas.* There is a rhymed summary of methods at the beginning of the section devoted to the computation of areas (Problems 36–42 of the *Toan phap dai thanh*) and providing formulas for computing the areas of figures of various shapes, including a rectangle, a right-angle triangle, a circle, a ring, a trapezium, a "bull's horn," a "sail," etc. Very similar rhymed formulas can be found in the *Suanfa tongzong* (1592) by Cheng Dawei [*SFTZJS*: 226–227]. These two lists are textually very close, and in many cases are identical or differ only in a few characters. The order of formulas, however, differs. The question is: does this mean that the Vietnamese compiler had the text of Cheng Dawei at hand? If the answer is affirmative, then the final compilation of the Vietnamese treatise could not have taken place prior to the early 17th century.

42 [Brenier 1994: 94–95] lists the differences.

43 [Martzloff 1997: 99] provides a table displaying the characters found in the Dunhuang manuscripts as the principal forms, and the characters found in other treatises as optional ones, even though the author claims that the table contains the "nomenclature for large numbers as found in the *Shushu jiyi*." Moreover, for the character *zheng* he provides only the form found in the Dunhuang manuscripts, without mentioning the form which occurs in all of the other sources.

44 Yan Dunjie notices that in a phrase from the Han dynasty treatise, *Fengsu tong*[*yi*] (Thorough [Investigation] of Popular Customs) quoted in the collection *Taiping yulan* (Imperial Survey [of Books Compiled in the Era] Taiping [976–984]), the term *ji* (Vietnamese: *cuc*) is used in one occasion in the context of the so-called "lower series of large numbers" (see *Taiping yulan*, *Sibu congkan* edition, vol. 51: 3b); however, Yan himself suggests that this is but a later emendation [Yan 1937: 23].

However, Cheng Dawei's rhymed rules themselves followed a long-standing tradition. The first four lines of his set of formulas are quoted from the *Jiuzhang suanfa bilei daquan* (1450) by Wu Jing [*JSBD*: 411]. Moreover, well before Wu Jing these same lines appeared in the *Suanfa quanneng ji* by Jia Heng [*SFQNJ*: 1339], and in the *Xiangming suanfa* by An Zhizhai [*XMSF*: 1391] (both active during the Yuan dynasty, 1279–1368). This means that the presence of these lines in the Vietnamese treatise does not necessarily imply that they were borrowed from Cheng Dawei's book. Thus the early date of the compilation of the Vietnamese manuscript cannot be rejected on the sole basis of the parallels with the *Suanfa tongzong*, since the rhymed formulas may well have been created as early as the Song dynasty (960–1279) and adopted by later authors.

*The Problems of Remote Surveying.* The Vietnamese treatise contains three problems (43, 44, and 135) devoted to calculating the height of an object provided the length of the shadow of the object. These read as follows:

> [43] The poem for measuring the shadow says:
>
> The method of measuring the shadow and gauging the height is the most remarkable. Set apart [a rod of] one *truong* (Chinese: *zhang*), [let] the shadows follow each other.[45] Take the value of the shadow of the pole [of unknown height] to be the "modifiable value," divide it by the shadow of the [one] *truong* [rod], and [the value sought] can [be] revealed!
>
> Now, suppose that there is a high pole, its shadow is 40 *truong*. Separately set [a rod of] one *truong* at the bottom end (?) of the shadow of the pole. The shadow of it is 1 *truong* 2 *xich* (Chinese: *chi*). The question: what [is the amount of] the *truong*, *xich*, and *thon* (Chinese: *cun*) of the length of the pole?[46]
>
> Method: set [on the counting surface] forty *truong* of the shadow of the pole, it will be the modifiable operand. Using 1.2, divide it. [The result] corresponds to the question. [...][47]
>
> Answer: the length of the pole is 33 *truong* 3 *thon* 3 *phan* 3 *li*.

> [44] Now, suppose that there is a betel palm tree, [its] shadow length is 10 *truong*. Separately set a one *truong* [rod] at the bottom of the palm. The shadow [of the rod] is only 9 *xich*. What [is the amount of] the *truong*, *xich*, *thon* of the length [*sic*] of the palm?
>
> Method: set [on the counting surface] 10 *truong* of the length of the shadow of the palm, it will be the modifiable operand. Use nine to divide it. [The result] corresponds to the question. [...][48]
>
> Answer: the length of the palm is 11 *truong* 1 *xich* 1 *thon* 1 *phan* 1 *li*.

45 That is, the shadow cast by the pole whose height is to be measured, and the shadow of the rod.

46 1 *truong* = 10 *xich* = 100 *thon* = 1000 *phan* (Chinese: *fen*) = 10000 *li* (Chinese: *li*).

47 The numerical computations provided in the text are omitted.

48 The computations are omitted.

[135] Now, suppose that there is a tree [of unknown] height, the length of the sun's shadow on the ground is 3 *truong*. The question: what is the height of the tree?

Method: take a pole of 1 *truong*, set it next to the tree, see that the length of the shadow on the ground is 7 *xich* 5 *thon*, set [with the counting rods]: in the upper [part] 3 *truong* of the shadow of the tree, in the lower [part] [set the length of the shadow of the pole], divide [what is in the upper part by what is in the lower part], establish [the value]. [...]

Divide it by [3 and multiply by?] 4, establish the height of the tree, 4 *truong*. [The answer] corresponds to the question.

The computation of the height of the pole recalls the "measurement with heart-mind," one of the fourteen "methods of computing/numbering" (*shu shu*) enumerated in the *Shushu jiyi* by Xu Yue (see above); the latter is called *ji* which can be tentatively rendered as "counting method [based on] evaluation." The text of the Chinese treatise does not specify that this "method" is related to remote surveying;[49] however, Zhen Luan (fl. ca. 560) in his commentary provides seven examples that he presents as fitting within the category of the method of "evaluation." These include three problems concerned with remote surveying (measuring the width of a river; the height of a gnomon; the depth of a well),[50] two problems concerned with a special case of solving simultaneous equations with two unknowns,[51] and two problems involving indeterminate simultaneous equations with three unknowns (two cases of the famous "one hundred fowls problem"; see below for further discussion). The second of the three problems on remote surveying reads as follows:

Someone asked: Let us suppose that [we] have a long bamboo pole, [and we] don't know its height. Since [we] "do not use counting tallies and rods," how can [we] "evaluate" and know it [ = the height]?

Answer: let us consider the shadow of the pole, whatever length it has, and draw [a line] on the ground to mark it. Suppose that [we] have at hand an object of 3 *chi* [length], [and we] set it up vertically, as [the pole]. Take the length of the shadow cast by the object and use it to measure the shadow of the pole; [multiply the result by 3 *chi*], obtain [the height]! [*SJSS*, 2: 546].[52]

This problem is similar to Problem 25, Chapter 3, of the *Sunzi suanjing* in which in order to measure the height of a pole one measures its shadow and also measures the height and the shadow of a smaller auxiliary gnomon [*SJSS*, 1: 317; Berezkina 1963: 37; Lam & Ang 1992: 120, 178]. Some authors suggest that the solution of the latter problem was given on the basis of the similarity of right-angle triangles

49 For a translation of the relevant excerpt, see [Volkov 1997b].

50 The contents of these three problems correspond to the three classical types of remote surveying: measuring the parameters of a remote object situated (1) on the same level as the observer, (2) higher, or (3) lower than the observer.

51 They are $\{x + 1 = y - 1; y + 1 = k(x - 1)\}$; for the first problem $k = 1.5$, for the second, $k = 5$.

52 For a discussion of two other problems, see [Feng 1995: 83–85].

[Berezkina 1963: 62–63; Lam & Ang 1992: 120; Feng 1995: 80–82]. Another interpretation could be provided on the basis of the equality of areas S(PFAN) and S(AGEC) in Figure 1. It is also possible that the problems were solved on the basis of the notion of simple proportion without any specific geometrical consideration: if an object of $A$ units in height has a shadow of $B$ units, then an object of $C$ units in height has a shadow of $(A/C) \cdot B$ units.

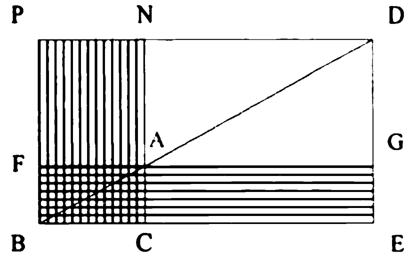


**Figure 1**

The earliest case of the problem in which proportions of this kind occur is found in the *Zhoubi suanjing* (compiled not later than the first century A.D. on the basis of older materials). In the latter treatise the diameter of the sun is calculated with a similar technique on the basis of observations of the sun with a sighting tube [*SJSS*, 1: 28; Cullen 1996: 178]. Examples that slightly antedate the commentaries on the *Shushu jiyi* are provided in Problems 12 and 14, Chapter 1, of the *Zhang Qiujian suanjing* (written most probably in the fifth century) [*SJSS*, 2: 339–340; Berezkina 1969: 31–32]. The former method is explained by the commentator Zhao Shuang (ca. 285); as for the two latter problems, the solutions can either be reconstructed on the basis of the proportionality of the sides [Berezkina 1969: 64–65, notes 29, 31; Feng 1995: 82–83], or by using the methods of "geometrical algebra."[53]

During the Song dynasty, Yang Hui (fl. ca. 1265) provided a justification for more complicated methods of remote surveying using "geometrical algebra," while the Yuan dynasty mathematician and astronomer Zhao Youqin (1271–1335?) in his treatise *Gexiang xinshu* (New Writing on the Image of Alteration) suggested a formula based on the idea of a linear functional relationship between the height and the shadow of the remote object. After the Yuan dynasty, problems of the calculation of the height of an object occur in treatises of the Ming dynasty (1368–1644) mathematicians Wang Wensu (ca. 1465–?) and Cheng Dawei (1533–1606). Wang Wensu in his *Suanxue baojian* devotes several pages to problems of remote surveying. He provides a discussion of four problems that are similar to that of the *Sunzi suanjing* mentioned above, and quotes the procedure used in Problem 25, Chapter 3 [*SXBJ*: 778–779]. Cheng Dawei in the 12th chapter of his *Suanfa tongzong* (1592) explains methods of remote surveying beginning with two problems

53 In [Volkov 1997b] I suggested that as early as the sixth century A.D. there existed a class of mathematical methods referred to as "calculation with heart-mind," and that at least some methods of the category of "remote surveying" were considered part of this class. It is difficult to say exactly how Zhen Luan defined the class of methods called "calculation with heart-mind," but the problems he provided as examples give the impression that the methods dealt with proportions rather than with geometrical algebra. Zhao Youqin (see below) also applied the generic name "measurements with heart-mind" to the problems of remote surveying.

also similar to the above-mentioned problem from the *Sunzi suanjing* [*SFTZJS*: 772].[54]

To conclude, it seems likely from the above discussion that the three problems on remote surveying found in the Vietnamese treatise belong to a tradition that existed in China as early as the fifth–sixth century; in particular, very similar problems and methods of solution can be found in the *Sunzi suanjing* and *Zhang Qiujian suanjing*.

*Problems Related to Numerical Divination.* This section of the *Toan phap dai thanh* is devoted to a group of problems related to methods of divination involving simple numerical operations; this group of problems provides the most interesting evidence concerning the origins of the treatise.

*The Sex of an Unborn Child.* The Vietnamese treatise contains the following algorithm (Problem 134):

> Method of computation [of the gender] of the embryo.
> [*Commentary*:] add the number 49 to the month [when the] embryo [was conceived?], subtract the age of the [future] mother; to avoid [any] doubts, eliminate/subtract [from it the numbers] from one to nine [step by step], [if at a step something still] remains, subtract [again] the [next] number [in the series]. If [at the end] an even [number] is encountered, it will be a girl; [if it is] an odd number, it will be a boy.[55]
>
> The computational protocol (*cao*) reads:
> First of all, dispose [on the counting surface?] "seven [times] seven, [i.e.] 49," add the day [a mistake for "month"?] of conception, eliminate [= subtract] the year of the birth of the mother [= the age of the mother?], then add 19 [a mistake for 29?];[56] for the Number of Heaven, eliminate [= subtract] one; for the number of the Earth, [subtract] two; for the number of Man, three; [continue with] four, five, six, seven, eight, nine; eliminate [= subtract] 45 counting rods [altogether taken from] each [of the Nine] Palaces.
>
> [*Commentary*:] [if] the [number in the] position [of result?] is odd, a boy will be born; [if] the [number in the] position is even, a girl will be born.

This problem immediately recalls the last problem of the *Sunzi suanjing*:

> Now, let us suppose that there is a pregnant woman, aged 29, in difficulty (= pregnant) [since] the ninth month,[57] unknown is the [gender of the baby] to be born.

54 For more on these methods see [Volkov 1997b].

55 This algorithm is textually quite close to the one found in the *Suanfa tongzong* (1592) of Cheng Dawei (see below).

56 Some support for the hypothesis that the original text contained the number 29 is provided below. In the written language, the characters for "one" and "two" differ in only one stroke and may easily have been confused by the copyists.

57 The possibility of reading the text as "pregnant for nine months" cannot be rejected, at least theoretically, since this problem is concerned with lunar months (the gestation period thus lasts for 10 months). However, if this reading is adopted, then the result of divination would

> The answer is: [she] will give birth to a boy.
>
> The procedure: set forty-nine, add the months of "difficulty"; what is to be eliminated [is]: for Heaven, eliminate one, [for] the Earth, eliminate two, [for] Man, eliminate three, [for] the Four Seasons, eliminate four, [for] the Five Phases, eliminate 5, [for] the 6 Pitches, eliminate six, [for] the Seven Stars [of the Dipper], eliminate seven, [for] the Eight Wind Directions, eliminate 8, [for] the Nine Islands [= the world in traditional Chinese cosmology], eliminate 9.[58] [Take] what is not exhausted; if [it is] odd, it will be a boy, [if it is] even, it will be a girl.[59]

A version of this problem is also found in the *Jiuzhang suanfa bilei daquan* by Wu Jing (1450) [*JSBD*: 49]. It reads slightly differently:

> Now, let us suppose that there is a pregnant woman, aged 28, in difficulty (= pregnant) [since] the eighth month. The question: the [baby] to be born will be a boy or a girl?
>
> The answer is: [she] will give birth to a boy.
>
> The procedure:
> Firstly, set [the number] [*commentary*: "forty-nine"], add [the months of] "difficulty" [*commentary*: "eight months"], [obtain] the total [*commentary*: "fifty-seven"]; subtract the age [*commentary*: "twenty-eight"], [obtain] the remainder [*commentary*: "twenty-nine"]. Then subtract: [for] Heaven, subtract ... [*commentary*: "one"], [for] the Earth, subtract ... [*commentary*: "two"], [for] the Man, subtract ... [*commentary*: "three"], [for] the Four Seasons, subtract ... [*commentary*: "four"], [for] the Five Phases, subtract ... [*commentary*: "five"], [for] the Six Pitches, subtract ... [*commentary*: "six"], [for] the Seven Stars [of the Dipper], subtract ... [*commentary*: "seven"]. What remains is 1, therefore a male [will be born].
> Moreover, if the [remaining] number is [still] large, subtract again. [For] the Eight Wind Directions, subtract ... [*commentary*: "eight"], [for] the Nine Islands [= the world in traditional Chinese cosmology], subtract ... [*commentary*: "nine"]. [Take] what remains; if [it is] odd, it will be a male, [if it is] even, it will be a female.

The problem and the method from the latter treatise were reproduced by Cheng Dawei in his *Suanfa tongzong*; however, the procedure there is not stated in a general case: it stops at the subtraction of 7.[60]

depend on the month of pregnancy in which the divination was performed, which would be rather odd.

58 Modern scholars believe that the subtraction of the numbers 1, 2, ... , 9 stops as soon as the (partial) result is negative; see [Lam & Ang 1992: 182; Martzloff 1997: 138]. The method found in the *Jiuzhang suanfa bilei daquan* (see below) justifies this suggestion.

59 The Chinese text quoted here is from [Li 1955a: 174]. Compare this with the text in the edition of the [*SJSS*]. For translation and discussion, see [Lam & Ang 1992: 182; Martzloff 1997: 138]. The translation of the *Sunzi suanjing* of E.I. Berezkina excludes the translation of this problem because of its "purely astrological character" [1963: 7].

60 Although there is an additional line saying "if the [resulting] number is large, again subtract 8 for the Eight Wind directions" [*SFTZJS*: 990], the subtraction of 9 is not mentioned.

The function $P$ described in the latter two treatises maps the pairs of integers {Age, Month} onto the set of two elements $\{m, f\}$. This function, despite its elementary description, is rather interesting. Generally, the values of the function alternate: $P(i, j+1) = m$ or $f$ if and only if $P(i, j) = f$ or $m$, respectively. However, one can see that for certain values of (Age, Month) the values $P$(Age, Month) of the function cease to alternate, that is, $P(i, j) = P(i, j+1)$; in several cases $P(i, 12) = P(i+1, 1)$. Moreover, the actual computations involved were even more complex given that, according to the traditional Chinese calendrical system, some years may have contained an additional (intercalary) month.

As Yan Dunjie showed [Yan 1937: 26–27; Martzloff 1997: 138], this method is based on traditional numerological ideas already found in the *Neijing suwen*, a classic of traditional Chinese medicine,[61] as well as in some Daoist treatises.[62] Moreover, the numerical constant used in all of these problems, 49, is the number used in the mantic procedure described in the *Xici zhuan*, the so-called Great Commentary on the *Yi jing* (The Book of Changes), one of the most venerated Chinese classics originally devoted to divination.[63]

The comparison of the texts of these problems found in the Vietnamese and three Chinese treatises reveals their striking similarity. The numerical data in the *Sunzi suanjing* on the one hand, and the two other Chinese treatises differ slightly; however, in both cases the result of the operations $R = 49 +$ (months of the age of the embryo) – (years of the age of the mother) is the same and equals 29. This makes it likely that the Vietnamese treatise also originally contained the number 29 (miswritten by the copyists as "19") in referring to this partial result. The algorithms were therefore the same, yet the original Vietnamese text was corrupted: instead of "then add 19 (or '29')" one should read "then set 29."[64]

In his paper published in 1921, the Japanese historian of mathematics Mikami Yoshio drew attention to the similarity between the problem from the *Sunzi suanjing* and the one found in the Japanese treatise *Kuchizusami* (preface written in A.D. 970).[65] The Japanese problem reads as follows:

> Let [us suppose that there is] a pregnant woman who may give birth to a child; [the following is] the method to know whether [it] will be a boy or a girl.
>
> The procedure reads: set [on the counting surface] the age of the woman [*Commentary*: from her birth till the year of conception], add [12 for] 12 spirits; this will be the modifiable operand. One can eliminate [one unit] for Heaven, two

61 Yan mentions *juan* 14 of the *Suwen* section and *juan* 12 of the *Lingshu* section.

62 For the original text of the *Taishang Laojun kaitian jing* (The Classic of the Most High Old Lord on the Beginning of the Heaven [and Earth]) see [*YJQQ*, *juan* 2: 14a]. Here the nine numbers are explicitly associated with the magic square of order 3.

63 More references to the role played by the number 49 (related in Chinese numerology to the so-called "Number of Grand Expansion" *dayan shu*) in the scholarly tradition related to divination can be found in [Volkov 1994].

64 The characters for "add" (Vietnamese *gia*, Chinese *jia*) and "set / dispose [counting rods?]" (Vietnamese *liet*, Chinese *lie*) could easily have been mixed up by the copyists.

65 This observation of Mikami was mentioned in a paper of Li Yan [1955a], published for the first time in 1925.

> [for] the Earth, three [for] Man, [four for] the Four Seasons, [five for] the Five Phases, [six for] the Six Spirits,[66] [seven for] the Seven Stars [of the Dipper?], [eight for] the Eight Wind Directions, [nine for] the Nine Palaces. If the remainder is one, three, five, seven ... [*Commentary*: it is the Yang [numbers], [the baby will be] male.] If the remainder is two, four, six, eight ... [*Commentary*: it is the Yin [numbers], [the baby will be] female.][67]

In his analysis, Mikami suggests that even though one cannot know whether the problem in the *Kuchizusami* stemmed directly from the *Sunzi suanjing*, the problem from the Japanese treatise certainly followed the Chinese tradition represented by the problem found in the *Sunzi* [Mikami 1921: 108–109].[68] This is also true for the problem in the Vietnamese treatise. The textual similarity of the problem in the *Toan phap dai thanh* with the problems from the Chinese treatises suggests that its compilers had at hand a text originating from a Chinese prototype. Was it a version of the *Sunzi* that circulated in Vietnam in the late first–early second millennium, or was it one of the above-mentioned books of the 15th-17th centuries? The following section may provide additional information for the solution of this problem.

*Calculations for Medical Purposes.* Speaking about the problem of determining the sex of an unborn child, Mikami Yoshio remarks that the example found in the *Sunzi suanjing* is the sole mathematical problem related to numerical divination to be found in the extant Chinese mathematical treatises, while the *Kuchizusami* contains two other procedures of the same kind. These read as follows:[69]

> There is a sick person; it is unknown if (s)he will die or live.
> Set [on the counting surface] "nine times nine — eighty one." Add [twelve for] the twelve spirits to obtain ninety three. Also add the age of the sick person. [What is] obtained [after] addition, divide [it] by three. If there is a remainder, a man will die and a woman live. If there is no remainder, a woman will die, and a man live.[70]

> A procedure to know whether a given person will die or live.
> Set [on the counting surface] eighty one, add [twelve for] the Twelve Spirits, also add [twelve for] the Twelve Months. Then also add the age of the sick person. Divide the sum by three. If there are remaining counting rods, [the person] will not die; if there is nothing left, [the person] will die.[71]

66 This is most probably a copyist's error unnoticed by all later editors; it certainly should be read as "Six Pitches" (Chinese *liu lü*).

67 This is based on the critical Japanese edition of 1982 [*KZ*: 84]; note that the texts as given in [Mikami 1921: 108; Li 1955a: 174] differ slightly.

68 [Mikami 1921: 108–109]. Since the result of the divination depends only on the age of the woman and does not depend on the month, it is likely that the Japanese treatise contains a simplified (or, probably, a defective) version of the method stemming from the same origin as the *Sunzi suanjing*.

69 I follow [*KZ*: 84]; the text in [Mikami 1921: 109] reproduced in [Li 1955a: 174] is slightly different.

70 Note that the chances for a woman surviving are twice as great as they are for a man.

71 If we take into consideration the previous rule, the sick person is a woman.

As in the method related to determining the sex of the child, the algorithms related to illness are not sensitive to the time when the illness was contracted, nor to the birthdate of the sick person. Thus the given procedure will produce the same results for all sick persons of the same age and gender. Mikami points out that the consecutive addition of 81 and 12 is not necessary, since the divisor is 3, but he does not provide any suggestions as to what could be the relationship between these two methods and the *Sunzi suanjing* [Mikami 1921: 109]. On the other hand, Li Yan suggests that the two problems may have been part of the Chinese treatise by the time when it was transmitted to Japan, and were preserved there, while in China the original text underwent drastic changes. Similar problems, he conjectures, probably followed the problem of the pregnant woman in the archaic version of the *Sunzi suanjing*. The fact that they were lost in China may have been because they were placed at the very end of the text.

Another procedure that is found in the Vietnamese treatise closely resembles the method from the Japanese treatise. It reads as follows:

> The method for "calculating the illness."
> [*Commentary*:] For a sick man, one wants to know if his illness is mild or grave. Set [on the counting surface] the age of the sick person, [add] the month and day [when the sickness] began. Multiply [the sum?] by three; when it is done, divide by nine. [If the remainder is] three, [the illness] is mild; [if the reminder] is six, [the illness is] grave; [if the reminder is] nine, [it] will be difficult to cure.
>
> The calculational protocol reads: Firstly, add the age to make the constant "model value" (*fa*), then add the month, day, and hour (?). To obtain [the prediction about] the illness, multiply by three and divide by nine, [consider] the remainder.
> [*Commentary*:] If three, [the illness is] mild; if six, grave; if nine, difficult to cure.

The multiplication by three with the subsequent division by nine is mathematically equivalent to the division by three in the Japanese text. The lethal condition is the divisibility by three in the Vietnamese treatise as well as in the second problem from the Japanese book, yet the Vietnamese rule has an additional prescription specifying whether the illness is serious or not. More importantly, the Vietnamese method takes into consideration the time when the person became sick, and thus, unlike its Japanese counterpart, produces different results for persons of the same age.

As already mentioned, Mikami believed that the method for calculating the seriousness of a disease was not to be found in any extant Chinese text. However, an almost similar text is found in the *Jiuzhang suanfa bilei daquan* (1450) by Wu Jing. It is remarkably close to the Vietnamese treatise and reads as follows:

> The method for "performing divination on the illness."
> First of all, set [on the counting surface] the amount in years of the age of the person, then add the month and the day, and [thus] obtain the "hour" [or "time"?] of contracting the illness. Multiply [the sum?] by three, divide by nine.

[If] the remainder is three, [the illness] is mild; [if the reminder] is six, [the illness is] grave; [if the reminder is] nine, [it] will be difficult to cure.

[An example:] Now, let a sick man be 47 *sui* old.[72] He got sick on the 9th day of the third month. What is the prediction?
The method states: first of all, set [on the counting surface] 47, the age of the sick man, add the "3rd month" and the "9th day" when he got sick; altogether it will be 59. Multiply this by three, obtain 177. Wishing to divide by 9, first remove 90 [from the number], next remove 81, the remainder is 6. Therefore, the illness is grave. If it were the 8th day [of the month] when he got sick, the remainder would be 3, and therefore the illness would be mild. If it were the 10th day [of the month] when he got sick, the remainder would be 9, and then [the illness] would be difficult to cure [*JSBD*: 49].

To summarize, our preliminary consideration of the problems on numerical divination found in the extant version of the *Sunzi suanjing* (edited in 1084), the Japanese text written around the year 970, the Chinese books of 1450 and 1592, and the Vietnamese treatise suggests that one of the possible reasons for the similarity of the methods is that the three latter treatises drew — directly or indirectly — on the same Chinese prototype, presumably a version of the *Sunzi suanjing* which, at the time when it entered Japan, contained a set of problems different from those found in the version of the text subsequently edited and printed in 1084 and 1213. In particular, it contained a series of problems devoted to "medical" calculations similar to the problems found in the Japanese and Vietnamese treatises.

The earliest Vietnamese state examinations on mathematics of which we are aware took place in 1077, that is, *before* mathematics instruction and examinations were re-established in China and a new set of textbooks was edited (1084). The Vietnamese officials may therefore well have used another set of books for instructional purposes containing the "old" Tang edition of the *Sunzi suanjing* imported to Vietnam from China in the seventh-ninth centuries. Another possibility is that the compilers of the Vietnamese treatise had at their disposal the sources (also related to the *Sunzi suanjing*) upon which Wu Jing drew in his treatise of 1450, if not a copy of his treatise itself.

*Indeterminate Analysis.*

*The "One-Hundred Fowls Problem."* The name "one-hundred fowls problem" comes from a famous problem from the *Zhang Qiujian suanjing* (Mathematical Treatise of Zhang Qiujian) that is equivalent to solving the simultaneous equations $\{5x + 3y + z/3 = 100;\ x + y + z = 100\}$ [*SJSS* 2: 402–404; Berezkina 1969: 59–60]. The problem and the method of its solution have been the subject of publications of several modern authors [Van Hée 1913; Berezkina 1969: 81, n. 90; Libbrecht 1973: 276–280; Martzloff 1997: 308–310]. The problem is a particular case of the simultaneous equations $\{ax + by + z/c = N;\ x + y + z = N\}$, where $a$, $b$, $c$, $x$, $y$, $z$ and $N$ are all positive integers. The *Toan phap dai thanh* contains one problem (# 137) of this type. In this problem one is asked to calculate the number of

72 The age counted in years from the moment of conception.

elephants, horses, and lambs, as well as the number of people of three categories taking care of them, if it is known that the total number of animals is 1,000, the total number of people is 1,000, and that nine men take care of one elephant, four of one horse, and one man of 20 lambs. The original text of the problem reads as follows:

> The counting method for the problem about the number of elephants, horses, together with lambs and shepherds.
> Elephants and horses, with the added lambs, together are one thousand. The number of men [taking care of each kind of animals] is unknown. [One] wishes that the numbers of the men of the different [kinds] all together be [equal to] 1,000. [*Commentary*: "only this is said."] [For each] male/battle elephant [there should be] 9 men together, a horse [needs] three people to be nourished. Each [shepherd] takes care of 20 lambs. To test,[73] [I am] asking those of all four sides [of the world] who know counting: how many are elephants, horses [and lambs], and how many men are there [for each kind of animal]?
>
> The answer reads:[74]
> The number of soldiers [who ride the elephants] is 540;
> The number of men [giving] grass [to horses] is 420;
> The number of lamb shepherds is 40;
> 60 male/battle elephants;
> 140 male/battle horses;
> 800 lambs.
>
> The computational protocol reads: first of all, add three nines, obtain 27, this will be the unmodifiable operand (Vietnamese *phap*; Chinese *fa*). Multiply this by 20 lambs, the number of the soldiers [riding the elephants thus] will be established. Divide this by nine, establish the number of the elephants. Dispose 27 [on the counting surface], subtract 9, add 3, multiply by 20, and [thus] establish the number of men [giving] grass [to the horses]. Divide this by 3, [thus] establish the [number of] horses. Again dispose [on the counting surface] 1,000 men, subtract from this the number of the soldiers [riding he elephants and the number of the men taking care of the horses], [thus] establish the number of the shepherds, multiply this by 20, establish the number of lambs.

In modern terms, one has to solve the simultaneous indeterminate equations:

$$\begin{aligned} x + y + z &= 1000 \\ 9x + 3y + z/20 &= 1000. \end{aligned}$$

The answer given in the text, $x = 60$, $y = 140$, $z = 800$, is correct. The general solution (which is not given) is: $x = 1 - 59u$, $y = 319 + 179u$, $z = 680 - 120u$, where $u = 0, -1$, that is, the problem has only two positive integer solutions, and one is given by the author of the treatise, whereas the second, $x = 1$, $y = 319$, $z = 680$, is not given.

73 Or "in trying" [the candidates? the reader?]. This is the reading in manuscript A; the (graphically close) reading of the manuscript B is "to know, [I am] asking ..."; this is, probably, a mistake of the copyist.

74 The answer is written with smaller characters and looks like a commentary.

However, the algorithm provided is not valid in the general case. In modern terms, the author suggests that the two simultaneous equations $\{ax + by + z/c = N;\ x + y + z = N\}$ be solved as follows:

$$\begin{aligned} x &= (abc)/a \\ y &= [(ab - a + b)c]/\mathrm{b} \qquad (*) \\ z &= c(N - ax - by). \end{aligned}$$

This "solution" is obviously incorrect in the general case (this is obvious at first glance since the values of $x$ and $y$ do not depend on $N$). Nevertheless, what useful information can be drawn from the fact that the method works in this particular case?

*The Classical "Hundred-fowls" Problem and "Pseudo-Solutions."* The classical "hundred fowls" problem (Zhang Qiujian, *juan* 3, Problem 38) reads as follows:

> Let us now suppose [that] one cock costs 5 coins,[75] one hen costs 3 coins, 3 chickens cost 1 coin. For the total of 100 coins [one] buys 100 fowls. The question is: how many each of cocks, hens, and chickens [are bought]?
> The answer reads:
> Cocks are 4, the price is 20;
> hens are 18, the price is 54;
> chickens are 78, the price is 26.
> Another answer:
> Cocks are 8, the price is 40;
> hens are 11, the price is 33;
> chickens are 81, the price is 27.
> Another answer:
> Cocks are 12, the price is 60;
> hens are 4, the price is 12;
> chickens are 84, the price is 28.
> The procedure reads:
> For the cocks, each [time] add four; for the hens, each [time] subtract seven; for the chickens, each time add three, then obtain [the result] [*SJSS*, 2: 402–404; Berezkina 1969: 59–60].

The commentary of an unidentified author states:[76]

> [The reason] why it is so is that the excesses and deficits compliment each other and are blended up together in one and the same price. Therefore, no procedure can thoroughly correspond to the inner structure (*li*) of it [*SJSS* 2: 401].[77]

75 The original text reads *qian* ("monetary units," "money") which I have rendered as "coins."

76 The commentary was certainly written by the same person who commented on the extension of the text of 1084 quoted below which explained the method and mentioned the Tang dynasty editor Li Chunfeng (602–670) and the late Tang–early Song (?) mathematician Xie Chawei (see below). In other words, the author of the commentary is most probably one of the editors of the Song edition of the classic.

77 [*SJSS*, 2: 404]; see also the original Song edition in [*SKSJLK*]. Consequently, there is no general formula for the solution to this problem.

A long anonymous passage follows; it was written, most probably, by the Song-dynasty (960–1279) editors. It reads as follows:

> As for this problem, if one performs calculations according to the above-mentioned procedure, it is difficult to understand [its rationale]. However, [the passage in question] is the same in all the extant versions [of the text that we have] compared; probably, some [part of the] text was lost. Since the text has circulated already for a long time, [we] do not have the possibility of studying [the original?] and of justifying [the reading]. Beginning from the Han and Tang dynasties, [such famous scholars as] Zhen Luan and Li Chunfeng added their commentaries, yet there was no comprehensive discussion. Now [we] are going to add anew the method(s) and the calculational protocol(s) proposed by professor(s) of the Mathematical College and[78] by Xie Chawei.[79]

The author goes on to quote an algorithm suggested by some scholars. Their names are not specified either in the extant edition of 1212–1213 (reprinted from the 1084 edition), or in the critical edition of Qian Baocong. However, Van Hée and Libbrecht both attribute the quoted excerpts to Zhen Luan (fl. ca. 560), Liu Xiaosun (active in the late sixth century?), Li Chunfeng (602–670), and Xie Chawei (active in the late 10th–early 11th century?), respectively. Questions about these attributions and discussion of this paragraph would lead us far beyond the scope of the present paper, but it should be noted that the method described here is as incorrect as the one found in the Vietnamese treatise. It amounts to the following formulas:

$$\begin{aligned} y &= \lfloor N/3c \rfloor \\ x &= 3c - (N - 3cy) \qquad (**) \\ z &= N - x - y. \end{aligned}$$

Both sets of formulas (*) and (**) are wrong in the general case. However, interestingly enough, they provide correct solutions of the simultaneous equations in some particular cases, and not only for the unique combinations of the parameters provided in the problems. For example, the set of formulas (*) provides correct solutions for $N = 12$, $a = 3$, $b = 1$, $c = 2$; $N = 20$, $a = 5$, $b = 1$, $c = 2$; $N = 24$ for three triplets of $(a, b, c)$: $a = 4$, $b = 1$, $c = 4$; $a = 5$, $b = 1$, $c = 3$; $a = 6$, $b = 1$, $c = 2$; and so on (however, these formulas do not work in the case of $N = 100$, that is, they could not be used to solve the problem of one hundred fowls). The set of formulas (**) works for even a larger number of cases: for example, for $9 < N < 360$, $1 < a < 21$, $b \leq a$, and $c < 42$, there exist 748 triplets $(a, b, c)$ such that the set of formulas (**) provides correct answers. Thus, both sets of formulas for problems of this particular type are incorrect in general, but are nevertheless applicable in

78 This "and" does not look appropriate here, even though it is found in the original Song edition [*SKSJLZ*]. Most probably it was added by mistake, and the text mentions only one professor, Xie Chawei. Qian Baocong in his edition omits the word "and"; see [*SJSS*, 2: 405, n. 1].

79 In Western literature the excerpt was discussed by L. Van Hée [1913: 442–447] and U. Libbrecht [1973: 276–282]. E. Berezkina [1969: 60] in her translation omitted this paragraph. I provide my own translation.

certain cases. I suggest such incomplete solutions be called "pseudo-solutions," since they provide solutions for the problems in some cases, but are not valid in the general case.[80] It is not possible to say on what grounds they were devised, but they certainly were appreciated and transmitted, as suggests their presence in both the Chinese and Vietnamese books. As a preliminary hypothesis, it may be conjectured that these kinds of formulas were used in the context of mathematical instruction in China and, presumably, also in Vietnam. If a student mechanically memorized the "pseudo-solution" formula for a problem, certain difficulties when facing a similar problem with modified numerical parameters could always arise. How the student then proceeded in such situations, might well have been a criterion for evaluating the student's mathematical abilities.[81]

*The "Hundred Fowls" Problem in Other Mathematical Traditions.* A preliminary inquiry into the history of the "one hundred fowls problem" suggests that the parameters found in the Vietnamese treatise ($a = 9$, $b = 3$, $c = 20$, $N = 1{,}000$) are not found in any other medieval treatise in the world. As mentioned above, the *Shushu jiyi* also contains two problems of this kind, for the first problem, $a = 5$, $b = 4$, $c = 4$, $N = 100$; for the second, $a = 4$, $b = 3$, $c = 4$, $N = 100$.[82] Among medieval mathematicians who dealt with the problem, one of the earliest and most outstanding was Alcuin (ca. 735–804). Seven problems in his treatise, *Propositiones ad acuendos iuvenes*, are related to the "Hundred fowls problem."[83] Similar indeterminate problems (involving four unknowns) were studied by the Indian mathematician Śrīdharācārya (ca. 850–950). Two versions of the "hundred fowls problem" are also known from an oral tradition in Morocco [Oudadess 1979: 5].[84] In one of these, $a = 1$, $b = 10$, $c = 5$, $N = 100$, and in the other, $a = 4$, $b = 1/2$, $c = 1/4$, $N = 20$ (i.e., $a = 8$, $b = 1$, $c = 2$, $N = 40$); in neither case are solutions provided. Nor does it seem possible to assign any certain dates to these problems. As for parameters of the equations and the numerical solutions found in the medieval works, they are shown in the following table (* indicates that the work concerned presents all positive solutions of the problem):

80 The "method" provided in the text of Zhang Qiujian strongly suggests that the author of the treatise had at his disposal the correct general formula. However, discussion of this topic, once again, requires a separate paper.

81 On the hypothesis concerning the problems with modified parameters as constituting the core of examination papers, see [Siu & Volkov 1999].

82 Martzloff's recent book fails to mention Zhen Luan's problems, and contains a misprint in the formula for the solution of the problem of Zhang Qiujian ($y = 25 - 4t$ instead of $y = 25 - 7t$) [Martzloff 1997: 308].

83 The text of the treatise was transmitted in 13 manuscripts, eight of them written about 1000 A.D. or in the 11th century. The oldest (Vat. Reg. lat. 309) is from the end of the ninth century (written in St-Denis, near Paris). A critical edition was recently published by M. Folkerts [Alcuin 1978] and translated in [Folkerts & Gericke 1993]; see also the discussion of the text in [Folkerts 1993]. In his commentary, Folkerts mentions the problems of Zhang Qiujian and Abū Kāmil (ca. 850–930); he also mentions diplomatic contacts between the courts of Charlemagne and Hārūn al-Rashīd [Folkerts 1993: 278].

84 I am thankful to John Denton who kindly offered me a copy of the unpublished work [Oudadess 1979].

| Author | Parameters | Solutions provided in the text |
|---|---|---|
| Zhang Qiujian | $a=5, b=3, c=3, N=100$ | $x=4, y=18, z=78$<br>$x=8, y=11, z=81$<br>$x=12, y=4, z=84$ * |
| Zhen Luan | $a=5, b=4, c=4, N=100$ | $x=15, y=1, z=84$ * |
| Zhen Luan | $a=4, b=3, c=3, N=100$ | $x=8, y=14, z=78$<br>[The only other positive solution:<br>$x=16, y=3, z=81$] |
| Alcuin 5 | $a=10, b=5, c=2, N=100$ | $x=1, y=9, z=90$ * |
| Alcuin 32 | $a=3, b=2, c=2, N=20$ | $x=1, y=5, z=14$ * |
| Alcuin 33 | $a=3, b=2, c=2, N=30$ | $x=3, y=5, z=22$ * |
| Alcuin 33a | $a=3, b=2, c=2, N=90$ | $x=6, y=20, z=64$<br>[The other positive solutions:<br>$x=3, y=25, z=62$;<br>$x=9, y=15, z=66$;<br>$x=12, y=10, z=68$;<br>$x=15, y=5, z=70$] |
| Alcuin 38 | $a=3, b=1, c=24, N=100$ | $x=23, y=29, z=48$ * |
| Alcuin 39 | $a=5, b=1, c=20, N=100$ | $x=19, y=1, z=80$ * |
| Alcuin 47 | $a=2, b=1/2, c=4, N=12$<br>[i.e., $a=4, b=1, c=2, N=24$] | $x=5, y=1, z=6$ |
| Abū Kāmil | $a=5, b=1/20, c=1, N=100$<br>[$a=5, b=1, c=20, N=100$] | [For transposed $y$ and $z$, positive $x, y, z$:<br>$x=19, y=1, z=80$.<br>This is Alcuin's # 39.] |
| Luong The Vinh | $a=9, b=3, c=20, N=1000$ | $x=60, y=140, z=800$<br>[The only other positive solution:<br>$x=1, y=319, z=680$] |

The means by which the problem was transmitted from one region to another remain unclear. The Vietnamese problem certainly has a strong local taste due to the mention of "battle elephants," but exactly when it was introduced and adopted by local mathematicians remains obscure. It may have been transmitted to Vietnam from China, or possibly from other cultural areas (from India or the countries of Islam). The only evidence for ties with Chinese mathematics is the "pseudo-solution" found in the Vietnamese text, since similar "pseudo-solutions" for such problems apparently existed in the Chinese mathematical tradition from the second half of the first millennium A.D.

*The Chinese Remainder Theorem.* Probably the most famous problem of indeterminate analysis comes from the Chinese treatise *Sunzi suanjing* (Mathematical Treatise of Master Sun). This is Problem 26 of juan 3 [*SJSS*, 1: 318; Berezkina 1963; Lam & Ang 1992] and, in modern notation, is equivalent to the following simultaneous congruencies:

$$N \equiv r_1 \pmod 3$$
$$N \equiv r_2 \pmod 5$$
$$N \equiv r_3 \pmod 7,$$

where $r_1 = 2, r_2 = 3, r_3 = 2$. The Chinese treatise provides the correct solution:

$$N = 70r_1 + 21r_2 + 15r_3 - 105k,$$

where $k$ is a non-negative integer chosen to make the answer $N$ have the minimal positive value.

The Vietnamese treatise does not contain this or any generic problem. However, there is a stanza entitled "Method of marking up/counting (?) soldiers" (Vietnamese *diem binh phap*, Chinese *dian bing fa*), the text of which is most probably corrupted and reads as follows:

> [If?] **three** men are walking together, [then] **seventy** [are] pushed (away?);
> For **five** trees, [there are] **21** branches of plum-blossoms;
> **Seven** men walking together received (?) **half a month** [= **15**];
> Subtract **100**, subtract **5**; what is established is the time.
> [*TPDT*-A: 64a; *TPDT*-B: 230]

The stanza precedes the above-mentioned problem of 1,000 animals (# 137), yet is not related to it. I argue that the stanza is a part of the missing problem borrowed from the treatise of Sunzi. Indeed, it is textually very close to the stanza found in the *Suanfa tongzong* (1592) by Cheng Dawei, which may approximately be rendered as follows:

> **Three** men walking together, [then] **seventy** are rare/dispersed (?);
> **Five** trees, **21** branches of plum-blossoms;
> **Seventh** [month's] gathering is in the **middle of the month** [= **15th** day];
> Subtract **105**, and then [you] can know [the answer] [*SFTJS*: 430].[85]

The literal meaning of this text is again highly obscure, yet its main purpose is to provide all the numerical values necessary for the solution of the problem of Sunzi: it specifies the coefficients 70, 21, and 15 ("the middle of the month" = 15) corresponding to the moduli 3, 5, and 7, as well as the constant 105. Does the fact that such a similar stanza appears in the Vietnamese treatise mean that the text

85 U. Libbrecht [1973: 291–292] translates the passage as follows: "Three septuagenarians in the same family is exceptional./ Twenty-one branches of plum-blossom from 5 trees./ Seven brides in ideal union precisely the middle of the month./ Subtract 105 and you get it." Martzloff [1997: 161] slightly modifies Libbrecht's rendering in adding question and exclamation marks (that probably were not intended by the Chinese author) in two lines.

was compiled on the basis of Cheng Dawei's treatise, that is, not earlier than the late 16th century? Not necessarily, since as Libbrecht has noted, the last *juan* (volume) of the treatise *Zhiyatang zachao* (1290) by Zhou Mi (1232–1308?) contained the algorithm for the solution of the "Sunzi problem" in the form of a slightly different stanza:

> A child of **three** years old [when the father?] is **seventy** [years old?], it is rare;
> **Five** leave **twenty one**,[86] it is even a more unusual thing;
> At seventh move [= month?] [during] the Shangyuan [festival = the **15**th day of the month?], [the Shepherd and Weaving Lady?] meet each other.
> "Cold meal" [is?] on the Qingming Festival [**105**th day?]; then [you] will get it.[87]

Clearly this and similar rhymed formulas were known to mathematicians as early as the 13th century. Cheng Dawei mentions that one of the names of the stanza is "Han Xin's marking up/counting soldiers" (*Han Xin dian bing*) that can be compared with the title of the poem in the Vietnamese treatise. However, this fact again does not provide sufficient evidence for a direct connection between the Vietnamese stanza and the work of Cheng Dawei, since a similar name for the method of Sunzi, "The Prince of Qin's method of secretly counting soldiers" (*Qin wang an dian bing*),[88] can be found as early as Yang Hui's treatise, *Xugu zhaiqi suanfa* (1275) [Lam 1977: 151–153, 322–323].

## Conclusions

This preliminary exploration of the contents of the Vietnamese treatise *Toan phap dai thanh* does not permit as yet a clear picture of its origins; however, it allows to formulate a number of hypotheses and directions for future research.

The treatise found in two copies in the Han-Nom Institute in Hanoi does not contain any information to establish that it was indeed authored by Luong The Vinh, the 15th-century literatus and official. The title of the book is not the same as that of the treatise *Dai thanh toan phap* traditionally credited to his authorship, and its attribution to Luong may have been due to later editors or copyists. Although the time of its compilation remains uncertain, the book may well have been written exclusively on the basis of treatises compiled in China prior to the

86 A very obscure line; see Libbrecht's translation in footnote 87. One can also suggest that it may be related to financial transactions or some popular contemporary game, thus using a technical meaning of the verb "to leave" (*liu*).

87 I am thankful to Liu Yan for a stimulating discussion of the stanza. The translation in [Libbrecht 1973: 286] reads: "A child of 3 years when 70 is rare;/ At 5 leave behind the things of 21 (or: From 5 [things], to leave behind 21) is still more rare;/ At 7 one celebrates the Lantern Festival. Again they meet together./ *Han-shih* (= 105), *ch'ing-ming* (= 106). Then you will get it." Neither translation seems entirely satisfactory, but the main intention of the author was certainly to present a set of numbers in a rhymed form, probably having no particularly deep meaning. I quote the stanza of Zhou Mi from [*SFTZJS*: 442, n. 60].

88 I adopt J. Needham's and U. Libbrecht's interpretation of the title; see [Libbrecht 1973: 283, n. 80].

late 15th century (if those were also available in Vietnam), and at least some parts of these materials (such as early versions of the *Sunzi suanjing* and *Zhang Qiujian suanjing*) may have circulated in Vietnam as early as the first millennium A.D. The compilation of the treatise, however, involved a substantial "localization," that is, an adaptation of the problems and methods to local measure units, currency, taxes, as well as to the names of specific local objects mentioned in the problems (plants, drugs, kinds of food, animals, etc).[89]

The seeming similarity between certain methods in the *Toan phap dai thanh* and in the Chinese treatise *Suanfa tongzong* (1592) by Cheng Dawei (in particular, the stanza containing the numerical parameters for the problem of Sunzi) can be explained by the fact that Cheng Dawei based his manual on mathematical texts available to him compiled prior to the 16th century, firstly and most importantly, on the treatises of Yang Hui (fl. ca. 1275) and Wu Jing (fl. ca. 1450). It is not impossible that the compilers of the *Toan phap dai thanh* also had access to these treatises, or to even earlier treatises that were, in turn, used by Yang Hui and Wu Jing.

One can conjecture that in the Ming dynasty Chinese mathematics lost the ideological and material support of the state it had enjoyed during the first and early second millennium A.D., and thereafter was practiced and perpetuated mainly by low level state officials, merchants, artisans, etc. as a "practical" discipline. Conversely, in contemporary Vietnam mathematics remained embedded in the framework of the traditional Chinese-style state education system directly linked to the bureaucratic hierarchy via the institution of mathematics examinations. However, both mathematical traditions appear to have employed documents of a similar kind, that is, the corpus of "practical" or "popular" mathematical knowledge that was fairly distinct from the high-level mathematical methods including the higher degree polynomial algebra of the late Song–early Yuan dynasties. Thus, one promising direction for future study of the *Toan phap dai thanh* would be a closer comparison of this treatise with the available Song, Yuan, and early Ming sources (in particular, the treatise of Wu Jing) devoted to "popular applied mathematics." This sort of study may provide new clues about the identities and professional positions of the compilers of the treatise.

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89 A thorough study of the history of metrology and taxation in Vietnam might lead to more definitive conclusions concerning the time when the treatise was compiled, given that such information occurs in a large number of problems.

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*GSZ* = *Gaoseng zhuan* (Biographies of Highest Monks), by Shi Huijiao. Beijing: Zhonghua shuju, 1992.

*JSBD* = *Jiuzhang suanfa bilei daquan* (The Great Complete [Collection] of the Generic [Methods Related to] the Computational Methods of Nine Categories), by Wu Jing. In [*ZKJDTSJ*, 2: 5-333].

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*KDST*b = *Kham dinh Viet su thong giam cuong muc*. Taibei, 1969.

*KZ* = *Kuchizusami* (Counting Rhymes). In *Zoku gunsho ruiji*, Vol. 64 (32a), pp. 61–85. Tokyo: Heibunsha, 1982.

*SFQNJ* = *Suanfa quanneng ji* (Collection of Omnipotential Counting Methods), by Jia Heng. In [*ZKJDTSJ*, 1: 1317-1346].

*SFTZJS* = *Suanfa tongzong jiaoshi* (Systematic Treatise on Counting Methods, with Emendations and Explanations), by Cheng Dawei. Mei Rongzhao & Li Zhaohua, Eds. Hefei: Anhui jiaoyu, 1990.

*SJSS* = *Suanjing shishu* (Ten Mathematical Canons), Qian Baocong, Ed., 2 vols. Beijing: Zhonghua shuju, 1963.

*SKSJLZ* = *Song ke suanjing liu zhong* (Six Computational Treatises Blockprinted During the Song [dynasty]). Beijing: Wenwu, 1980.

*SXBJ* = *Suanxue baojian* (Precious Mirror of Mathematical Learning), by Wang Wensu. In [*ZKJDTSJ*, 2: 337–971].

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*TPDT*-B = *Toan phap dai thanh* (Great Compendium of Mathematical Methods), attributed to Luong The Vinh. Manuscript B, call no. A 2931. Hanoi: Institute of Han-Nom Studies.

*XMSF* = *Xiangming suanfa* (Counting Methods Explained in Detail), by An Zhizhai. In [*ZKJDTSJ*, 1: 1349–1397].

*YJQQ* = *Yunji qiqian* (Seven Lots from the Bookbag of the Clouds), compiled by Zhang Junfang. Beijing: Xinhua shudian, 1992. I follow the translation of the title suggested by Judith M. Boltz in *A Survey of Taoist Literature, Tenth to Seventeenth Centuries*. Berkeley: Institute of East Asian Studies, University of California, 1987: 229.

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## Glossary

| | |
|---|---|
| An Dương [vương] | 安陽[王] |
| An Zhizhai | 安止齋 |
| Bảo đại [era] | 保大 |
| bố toán | 布算 |
| bu suan | see *bố toán* |
| cai | 核 |
| cao | 草 |
| câu | 溝/冓 |
| chánh | 正 |
| Cheng Dawei | 程大位 |
| Chengzu | 成祖 |
| chi | see *xích* |
| *Chỉ minh lập thành toán pháp* | 指明立成算法 |
| cun | see *thốn* |
| cực | 極 |
| *Cửu chương lập thành toán pháp* | 九章立成算法 |
| *Cửu chương toán pháp* | 九章算法 |
| *Đại thành toán pháp* | 大成算法 |
| *Đại Việt sử lược* | 大越史略 |
| *Đại Việt sử kí toàn thư* | 大越史記全書 |
| *Đăng khoa lục* | 登科錄 |
| dayan shu | 大衍數 |
| *Da Yue shilue* | see *Đại Việt sử lược* |
| dian bing fa | see *điểm binh pháp* |
| điểm binh pháp | 點兵法 |
| fa | see pháp |
| fen | see phận |
| *Fengsu tongyi* | 風俗通義 |

| | |
|---|---|
| gai | see *cai* |
| *Gaoseng zhuan* | 高僧傳 |
| *Gexiang xinshu* | 革象新書 |
| gia | 加 |
| giản | 簡 |
| Giáp Thân [year] | 甲申 |
| gou | see *câu* |
| Guozi jian | see *Quốc tử giám* |
| hạ | 下 |
| Han Qi | 韓琦 |
| *Han Xin dian bing* | 韓信點兵 |
| Hồng Bàng | 鴻龐 |
| Hui | 會 |
| ji | see *cực* |
| jia | see *gia* |
| Jia Heng | 賈亨 |
| jian | see *giản* |
| jing | see *kinh* |
| jin you | see *kim hữu* |
| ji shu | 計數 |
| *jiujiu* | 九九 |
| *Jiuzhang suanfa bilei daquan* | 九章算法比類大全 |
| *Jiuzhang xiangzhu bilei suanfa daquan* | 九章詳注比類算法大全 |
| Kang | 康 |
| *Khâm định Việt sử thông giám cương mục* | 欽定越史通鑑綱目 |
| kim hữu | 今有 |
| kinh | 京 |

| | |
|---|---|
| Kinh Dương [vương] | 涇陽[王] |
| *Kuchizusami* | 口遊 |
| *Lập thành toán pháp* | 立成算法 |
| Lê Quý Đôn | 黎貴惇 |
| li | 厘 |
| li$_1$ | 理 |
| Li Chunfeng | 李淳風 |
| Li Yan | 李儼 |
| *Licheng suanfa* | see *Lập thành toán pháp* |
| lie | see *liệt* |
| *liệt* | 列 |
| *Lingshu* | 靈樞 |
| liu | 留 |
| liu lü | 六律 |
| Liu Xiaosun | 劉孝孫 |
| Lương Thé Vinh | 梁世榮 |
| Ma Yuan | 馬援 |
| Minh Mệnh | 明命 |
| *Nam sử tập biên* | 南史輯編 |
| *Neijing* | 內經 |
| nhương | 穰/禳 |
| Nôm | 喃 |
| *Panzhu suanfa* | 盤珠算法 |
| Phạm Hữu Chung | 范有鍾 |
| phận | 分 |
| pháp | 法 |
| Phiên-ngu | 番禺 |
| qian | 錢 |
| *Qingming shanghe tu* | 清明上河圖 |
| Qin wang an dian bing | 秦王暗點兵 |
| Quốc tử giám | 國子監 |
| quyển | 卷 |
| rang | see *nhương* |
| ru | 如 |
| shang | see *thượng* |
| Shen Nong | 神農 |
| *shu shu* | 數術 |
| *Shushu jiyi* | 數術記遺 |
| *Siku quanshu* | 四庫全書 |
| *Song ke suanjing liu zhong* | 宋刻算經六種 |
| *Suanfa dacheng* | see *Toán pháp đại thành* |
| *Suanfa quanneng ji* | 算法全能集 |
| *Suanfa tongbian benmo* | 算法通變本末 |
| *Suanfa tongzong* | 算法統宗 |
| *Suanfa tongzong jiaoshi* | 算法統宗校釋 |
| *Suanjing shishu* | 算經十書 |
| *Suanxue baojian* | 算學寶鑑 |
| *Suanxue qimeng* | 算學啓蒙 |
| suanzi | see *toán tử* |
| sui | 歲 |
| *Sunzi suanjing* | 孫子算經 |
| *Suwen* | 素問 |
| tải | 載 |
| *Taiping yulan* | 太平御覽 |
| *Taishang Laojun kaitian jing* | 太上老君開天經 |
| thốn | 寸 |
| Thục | 蜀 |
| Thục Phán | 蜀泮 |
| thượng | 上 |
| ti | 億 |
| ti$_1$ | 秭 |
| tianwen tuwei | 天文圖緯 |
| *Toán pháp đại thành* | 算法大成 |
| *Toán pháp kì diệu* | 算法奇妙 |
| toán tử | 算子 |
| triệu | 兆 |
| Triệu | 趙 |
| Triệu Đà | see Zhao Tuo |
| Trưng | 徵 |
| trượng | 丈 |
| vạn | 萬 |
| Vănlang | 文郎 |
| Vũ Hữu | 武有 |
| vương | 王 |
| Vũ Quỳnh | 武瓊 |
| wan | see *vạn* |
| Wang Wensu | 王文素 |
| Wu | 吳 |
| Wu Jing | 吳敬 |
| *Wujing suanshu* | 五經算術 |
| xia | see *hạ* |
| *Xiangming suanfa* | 詳明算法 |
| Xia Yuanze | 夏源澤 |
| xích | 尺 |
| *Xici zhuan* | 繫辭傳 |
| Xie Chawei | 謝察微 |
| *Xinji tongzheng gujin suanxue baojian* | 新集通證古今算學寶鑑 |
| *Xugu zhaiqi suanfa* | 續古摘奇算法 |
| Xu Xinlu | 徐心魯 |

| | |
|---|---|
| Xu Yue | 徐岳 |
| Yang Hui | 楊輝 |
| yi | see *tỉ* |
| *Yi jing* | 易經 |
| *Yongle* | 永樂 |
| *Yunji qiqian* | 雲笈七籤 |
| zai | see *tải* |
| zhang | see *trượng* |
| Zhang Fu | 張輔 |
| *Zhang Qiujian suanjing* | 張邱建算經 |
| Zhang Yong | 章用 |
| Zhang Zeduan | 張擇端 |
| zhao | see *triệu* |
| Zhao Shuang | 趙爽 |
| Zhao Tuo | 趙佗 |
| Zhao Youqin | 趙友欽 |
| zheng | see *chánh* |
| Zhen Luan | 甄鸞 |
| *Zhiming suanfa* | 指明算法 |
| *Zhiyatang zachao* | 志雅堂雜鈔 |
| *Zhongguo kexue jishu dianji tonghui* | 中國科學技術典籍通彙 |
| *Zhoubi suanjing* | 周髀算經 |
| Zhou Mi | 周密 |
| Zhu Shijie | 朱世傑 |
| Zhu Yunying | 朱雲影 |
| zi | see *tỉ* |

# Regiomontanus' Role in the Transmission of Mathematical Problems

by MENSO FOLKERTS

Johannes Regiomontanus (1436–1476) played an important role in the transmission of mathematical knowledge from Italy to Central Europe. Even in Vienna, as a student of, and collaborator with, Georg Peurbach, and especially during his stay in Italy (1461–1467) — where he was in contact with Bessarion, Bianchini and other humanists — Regiomontanus became acquainted with theoretical texts and with the mathematical knowledge of the Italian practitioners. There are three manuscripts in Regiomontanus' own handwriting containing mathematical problems. The article lists the groups of problems treated by Regiomontanus. Most of them go back to Italian sources (e. g. Leonardo Fibonacci, Paolo Gerardi). Many of these problems became known to Fridericus Amann (d. 1464 or somewhat later), who lived in a monastery near Regensburg, and to Johannes Widmann, who taught algebra at Leipzig University in the 1480s, and were later to be found in "Rechenbücher" in Germany and elsewhere.

One of the central figures in 15th-century mathematics was Johannes Regiomontanus (1436–1476). He is well known for his contributions to astronomy (including astronomical instruments), algebra and trigonometry: his *De triangulis omnimodis libri quinque* (Five Books on Triangles of Every Kind), which was published posthumously in 1533 in Nuremberg, marks the beginning of a new phase in trigonometry in the West. But it is not so generally known that Regiomontanus also played an important role in the transmission of mathematical problems from Italy to Central Europe. This is the topic of the present paper. But first a few facts about his life.[1]

Regiomontanus was born in Königsberg (Franconia). At the early age of 11 he became a student at Leipzig university. There he acquired sufficient knowledge to write an almanac for 1448. From 1450 to 1461 he lived in Vienna. These years were decisive for his intellectual development. In 1452 he became *baccalaureus* and in 1457 *magister* of the faculty of arts. At that time Vienna was a centre of astronomical and mathematical studies, mainly under the influence of John of Gmunden (before 1385–1442) and Georg Peurbach (1423–1461). Although John of Gmunden had died before Regiomontanus went to Vienna, Regiomontanus was certainly acquainted with his works on astronomy and trigonometry, since John had left his books and instruments to the Vienna library. The other important astronomer and mathematician in Vienna, Peurbach, became Regiomontanus' teacher and friend.

1 The most detailed biography is [Zinner 1968]. A survey of Regiomontanus' mathematical contributions is given in [Folkerts 1977]. Some aspects of his role in the transmission of mathematical knowledge are discussed in [Folkerts 1985, 1986].

Peurbach was best known for his planetary theory, but he also wrote a treatise on sines and chords, which was heavily dependent on John of Gmunden's work, and a treatise on elementary arithmetic, *Algorismus*. Regiomontanus studied these and other writings in Vienna and copied some of them; the so-called Vienna *Rechenbuch*, in his hand, from the years 1454–1462 (most of it was finished in 1458) is still extant.[2] This manuscript contains works on arithmetic, theory of proportions, calculations involving polygons, circles and spheres, but no recreational mathematics or special mathematical problems.

When Cardinal Bessarion came to Vienna in 1460, he met Peurbach and Regiomontanus and persuaded them to go to Italy in order to finish an abridgment of the *Almagest* to replace George of Trebizond's inferior translation of it. Since Peurbach died shortly before their departure, only Regiomontanus accompanied Bessarion to Italy in 1461. The next four years he spent partly in the cardinal's entourage and partly traveling on his own. During both phases of this period he studied mathematics and astronomy and learned Greek; furthermore he finished Peurbach's compendium of the *Almagest* and wrote the first four books of his *De triangulis*.

In 1467, at the latest, Regiomontanus left Italy for Hungary, where he was attached to the court of King Matthias Corvinus. During his years in Hungary he was occupied mainly with astronomical questions and the drawing up of trigonometrical tables. From 1471 to 1475 he lived in Nuremberg and *inter alia* prepared books for printing on his own press. In 1475 he went to Rome to lay the foundations of the currently planned reform of the calendar, but in 1476 he died suddenly.

Regiomontanus was one of the few humanists actively interested in mathematics and astronomy, but not in philosophy. As a young man, he had the chance to broaden his knowledge of mathematical subjects at the university of Vienna. Later, through his contact with Bessarion he had at his disposal the considerable library that the cardinal had accumulated. He was thereby introduced into the circle of Italian humanists with an interest in mathematics and astronomy. One of them was Giovanni Bianchini (after 1400–after 1469).

Regiomontanus used Latin translations from the Arabic and redactions of them together with Greek manuscripts to elucidate what he thought was the true meaning of Greek mathematical texts. Reconstruction of original texts as such was not his aim, but only the restoration of the authors' meaning. Further, on the basis of such texts he went on to derive his own results. For this purpose he made full use of the works of mediaeval and contemporary writers, such as Jordanus Nemorarius and Peurbach.

Although Regiomontanus' *Nachlaß* was scattered in the 16th century, we have three catalogues of books that belonged to his library, and some of the manuscripts from his collection are still extant. In addition, there is a list printed on a single page — we will call it his *Programme*[3] — containing the titles of texts that Regiomontanus wanted to publish on his printing press. He had the idea of publishing the

2 Wien, Österreichische Nationalbibliothek, MS 5203.

3 Published in [Zinner 1968, Abb. 45].

mathematical and astronomical works that seemed to him to be the most important, but because of his early death, he could not carry out this plan. From all of these sources we are able to make some deductions about his place in the transmission of mathematics in general and of mathematical problems in particular.

We know that Regiomontanus was very interested in algebra. In 1463 he discovered in Venice a Greek manuscript containing the first six books of Diophantus. Before that, the work was apparently unknown in the West. In his lectures, which Regiomontanus gave at the university of Padua in 1464, he praised among the works on arithmetic (which tacitly includes algebra) not only Diophantus' treatise, but also two works by Jordanus Nemorarius: *Arithmetica* and *De numeris datis*. These two treatises are also mentioned in his *Programme*. There are only two other mediaeval writers whose treatises appear in the *Programme*: Johannes de Muris' *Quadripartitum numerorum* and a work entitled *Algorithmus demonstratus*. This last is probably the 13th-century treatise by a certain Gernardus, which was very popular in the later Middle Ages and which was copied by Regiomontanus in Vienna; on the basis of his copy it was published by Johannes Schöner in 1534.

Because the *Algorithmus demonstratus* belongs to the *algorismus* tradition and does not deal with special mathematical problems, we need not treat it here, but I should like to make some remarks on the other texts mentioned.

The *Arithmetica* of Jordanus Nemorarius was one of the most important works on arithmetic in the West.[4] It became the standard source for theoretical arithmetic in the Middle Ages and was used by Campanus, Roger Bacon, Oresme and others. About one half of the work is devoted to the treatment of the arithmetical books VII–IX of Euclid's *Elements*. One of the many original propositions in the *Arithmetica*, III.31, is on solving in positive integers the indeterminate equation $ax - by = v$, for integral values of $a, b, v$ ($a, b$ prime to one another and $a > b$). Jordanus' solution is in principle the same as that given by the Indian mathematicians. Book VIII treats the theory of figured numbers; some original conclusions can be found here, as elsewhere. In one manuscript[5] there is an additional proposition: the sum of the cube numbers beginning from the unit is a square number. Because Regiomontanus' manuscript of Jordanus' *Arithmetica* seems to have been lost, we do not know how deeply he studied the work. The same is true of Jordanus' *De numeris datis*, which was probably used as a university textbook for algebra. In it Jordanus gave the problems he treated an abstract form and he used a special system of symbols, which is not very suitable and does not make the understanding easier.

In the 14th century, Johannes de Muris treated algebra in great detail in his *Quadripartitum numerorum*. This treatise, written in 1343, was Johannes' principal work.[6] It contains, apart from arithmetic, music and mechanics, a detailed treatment of algebra, which is heavily dependent on al-Khwārizmī, though there are also practical problems in the style of Leonardo Fibonacci. Regiomontanus' manuscript

4 Edited in [Busard 1991].

5 Basel, Universitätsbibliothek, F II 33.

6 Edited in [L'Huillier 1990].

of the *Quadripartitum numerorum* is still extant.[7] It was not written by Regiomontanus, but there are corrections in his hand and also many annotations. In several places he converts algebraic solutions into geometric solutions. I shall not go into details here, because the content of his annotations has been described by [L'Huillier 1980].

Now I come to mathematical problems that exist in Regiomontanus' own handwriting and that do not belong to a foreign text by a known author. They are from three sources:

a. Manuscript New York, Columbia University, Plimpton 188. This manuscript starts with Johannes de Muris' *Quadripartitum numerorum* (see above). There follows, on fol. 73r–82v, a copy of al-Khwārizmī's algebra, in the translation by Gerard of Cremona, apparently in Regiomontanus' hand. On fol. 82v–96r it contains a collection of 64 arithmetic, algebraic and geometric problems, most of which were written by Regiomontanus, apparently in 1456. These problems have not yet been edited.
b. The autograph of his *De triangulis*, now in St. Petersburg (early 1460s).[8]
c. Correspondence with Giovanni Bianchini (1463–64), Jacob von Speyer (1465) and Christian Roder (1471) (autograph now in Nuremberg).[9] In these letters Regiomontanus suggested a series of mathematical and astronomical problems and gave the solutions of some of them and hints for others.

Of special interest are the problems in the Plimpton manuscript.[10] Some correspond to problems found partly in the older and partly in the more recent algebraic texts of the thirteenth to the fifteenth century. The older writings belong to the tradition of the *libri d'abbaco*, which are based upon Leonardo Fibonacci's *Liber abbaci*. There are some hundreds of texts that were written in Italian from the 13th to the 15th centuries; they were first investigated systematically in the last few decades, especially by Warren Van Egmond and the Siena group of historians of mathematics (R. Franci, L. Toti Rigatelli and their students). A book of this type, the *Libro di ragioni*, written in Montpellier in 1328 by Paolo Gerardi of Florence, seems to have been the first algebra in the vernacular.[11] By "the more recent algebraic texts" of the period I am referring to the development in Southern Germany. Though algebra enjoyed a secure place in Italian mathematical teaching from the 13th century, it did not come to Germany before the 15th. The decisive period was the second half of this century, and the most important places were Vienna, Regensburg and Leipzig. Of special importance was Fridericus Amann (d. 1464/5), a monk from the Emmeram monastery near Regensburg, who in 1461 copied, besides other mathematical texts,

7 New York, Columbia Library, Plimpton 188, fol. 1r–70v.
8 The work was not printed until 1533 [Regiomontanus 1533].
9 Published by Curtze [1902].
10 Information about the content is given in [Folkerts 1985].
11 Edited in [Arrighi 1987]; edition of the algebraic part with English translation in [Van Egmond 1978].

an algebraic treatise in the Italian tradition and many problems in arithmetic, algebra and geometry.[12]

The problems of the Plimpton manuscript constitute an important testimony to the transmission of mathematical problems from Italy to Central Europe. Even if it is not clear which problems were Regiomontanus' own work and which he had simply copied from older sources, his importance for the transmission of algebraic and geometrical texts must now be seen in a new light.

I should like to take a brief look at the most interesting groups of problems, most of which belong to the text in the Plimpton collection. I start with problems that might be derived from the practical arithmetic of the merchants. Then I proceed to problems of theoretical arithmetic and geometry.

## 1. Exchange of Goods (Barter)

Since the time of Leonardo Fibonacci the custom had been established among merchants that two prices were settled for goods to be exchanged, one for cash-payment ($b$) and the other for barter ($s$).[13] In the simplest case the exchange was acceptable if the ratio of $b : s$ was the same for both articles. In several problems a merchant demands cash for some fraction $\frac{1}{n}$ of his goods. In this case the formula is

$$\left(b_1 - \frac{s_1}{n}\right) : \left(s_1 - \frac{s_1}{n}\right) = b_2 : s_2 .$$

One such problem (Plimpton collection, no. 24) leads to the equation

$$\left(9 - \frac{x}{4}\right) : \left(12 - \frac{x}{4}\right) = 10 : x ,$$

so that a quadratic must be solved.[14]

## 2. Compound Interest

*2.1.* No. 22 in the Plimpton collection is a problem in compound interest: if 20 dinars increase to 30 dinars in two years, how much is the rate of interest? This problem is treated twice by Fridericus,[15] and it is also to be found in Gerardi [Van Egmond 1978: 167–168, no. 4].

*2.2.* Another problem of compound interest occurs in the Plimpton collection and in the correspondence: to find the interest by which an amount of 100 *florins* comes

12 Edited in [Curtze 1895]. It should be noted that the name "Fridericus Gerhart", as given by Curtze and all later historians of mathematics, must be corrected to Fridericus Amann; see [Gerl 1999: 1–2].

13 On the barter problems see [Tropfke 1980: 519–527 (esp. p. 522); Vogel 1978: 68–73].

14 This problem corresponds with München, Bayerische Staatsbibliothek, Clm 14908, fol. 141v–142v: [Curtze 1895: 54–55].

15 On fol. 140v–141r and fol. 149r–150v; see [Curtze 1895: 54, 61–62].

up to 265 *florins* within three years.[16] This problem leads to the cubic equation $x^3+300x^2+30{,}000x = 1{,}650{,}000$. Regiomontanus solves this problem by using a three-dimensional *gnomon*;[17] this may perhaps be seen as a step towards the solution of the general cubic found by the Italians in the first half of the 16th century. But in this case Regiomontanus was not original. His solution only works in a special case of the cubic equation $x^3 + ax^2 + bx = c$, namely if $a^2 = 3b$. In this case we find $x = \sqrt[3]{c + (b/a)^3} - b/a$. We know now that Master Dardi of Pisa had already solved this special problem in the second half of the 14th century [Van Egmond 1983: 417–418], and we find it somewhat later in anonymous texts.[18]

*2.3.* In 1464 Bianchini suggested another problem in compound interest: 100 *florins* become 900 *florins* in 6 years. Regiomontanus answered that the interest of the first year is $\sqrt[6]{9 \cdot 10^{12}} - 100$, but he added: *Labyrinthus maximus*.[19] Furthermore, he gave an example of a fourth-degree equation, $5x^4 + 3x^3 + 8x^2 = 260x + 5$, adding *quanta sit ipsa res, nondum habeo compertum*.[20]

## 3. People Buying a Horse

*3.1.* In problem 27 of the Plimpton collection three people buy a horse. None of them can pay for the horse by himself, but each can do so with help from one of the others. The way they can do it may be expressed in the equations

$$x + \frac{y}{2} = 100\,, \quad y + \frac{z}{3} = 100\,, \quad z + \frac{x}{4} = 100\,.$$

*3.2.* A similar problem is no. 33 of the Plimpton collection, which results in the following system of equations:

$$x + \frac{y}{3} = 100\,, \quad y + \frac{z}{4} = 100\,, \quad z + \frac{x}{5} = 100\,.$$

This system is solved with the help of the rule of double false position.

These problems had been, since the time of Leonardo Fibonacci, a standard item in recreational mathematics.[21]

16 Plimpton 188, no. 64. The same problem in the *Correspondence*: [Curtze 1902: 219, no. 6; 236].

17 See [Folkerts 1980: 205; Tropfke 1980: 443–444].

18 For correct formulas to solve one particular class of cubic equations in Italian manuscripts of the 14th century, see [Franci & Toti Rigatelli 1988], especially pp. 19–20.

19 Bianchini's problem: [Curtze 1902: 238, no. 4]; Regiomontanus' answer and comment: [Curtze 1902: 256, 280]. See [Kaunzner 1980: 138; Folkerts 1980: 207–208].

20 [Curtze 1902: 256; Kaunzner 1980: 138].

21 See [Tropfke 1980: 608–609]. The same problem is in München, Clm 14908, fol. 155rv: [Curtze 1895: 70–71].

## 4. "Give and Take"

From the 64 propositions in the Plimpton collection, about 15 are of the "give and take" type.[22] I present here no. 3:

*4.1.* There are 3 *socii*. The first says: If I had 10 from your (money), I should have as much as both of you and even 2 more. The second says: If I had 8 from you, I should have as much as both of you, half of it and even 4 more. The third says: If I had 6 from you, I should have the double of both of you minus 3.
This leads to the system of equations

$$\begin{aligned} x + 10 &= (y + z - 10) + 2 \\ y + 8 &= (x + z - 8) \cdot \tfrac{3}{2} + 4 \\ z + 6 &= (x + y - 6) \cdot 2 - 3 \,. \end{aligned}$$

The solution is original. First, one puts $x + y + z = R$. Then another unknown, $a$, is introduced. If the first *socius* has $\frac{1}{2}R + a$, then the other two have together $\frac{1}{2}R - a$. From the first equation, which becomes $\frac{R}{2} + a = \frac{R}{2} - a + 2$, one finds that $a = 1$. Hence the first *socius* has $\frac{R}{2} + 1$ after the exchange, or $\frac{R}{2} - 9$ before the exchange. A similar treatment of the second and third equations yields $\frac{3}{5}R - \frac{32}{5}$ and $\frac{2}{3}R - 7$. From the three individual results and the sum $R$ the three values $x$, $y$, $z$, i. e. the amounts of money in possession of the three *socii*, may be determined. This remarkable type of algebraic reasoning is otherwise unknown to me, but it might exist in some Italian text. It may be noted that the idea of considering the sum of the constituents is to be found in Leonardo Fibonacci's treatise [Boncompagni 1857: 190–203].

*4.2.* There is a similar problem (Plimpton collection, no. 30) which leads to the linear system

$$\begin{aligned} x + 7 &= (y + z - 7) \cdot 3 \\ y + 9 &= (x + z - 9) \cdot 4 \\ z + 11 &= (x + y - 11) \cdot 5 \,. \end{aligned}$$

The same problem is also given by Fridericus Amann[23] and in the 1480s (with the same method of solution) by Johannes Widmann in Leipzig.[24] This seems to be an indication that this problem was brought to Germany by Regiomontanus.

*4.3.* There are two problems in the Plimpton collection (nos. 11, 12) where the first *socius* has no money, but debts; these debts he can only repay when he gets his share from the other two. In no. 11 we have the following system:

$$\begin{aligned} -x + 132 &= y + z - 132 \\ y + 66 &= (-x + z - 66) \cdot 2 \\ z + 22 &= (-x + y - 22) \cdot 3 \,. \end{aligned}$$

22 For this type, see [Tropfke 1980: 609–611].

23 München, Clm 14908, fol. 146rv (without proof) and fol. 155v–156r: [Curtze 1895: 58, 71–72].

24 As part of the so-called *Latin Algebra*; see [Wappler 1899: 543–544].

## 5. To Find a Number

In the Plimpton collection there are some problems in which a number is to be found. Most of them serve as examples for the six types of equations given by al-Khwārizmī[25] (nos. 16–32). It is worth mentioning that the Latin text in the Plimpton manuscript, which describes the six forms of equations, agrees word-for-word with the German translation that Fridericus Amann wrote five years later,[26] and that all problems in this group are also to be found in this manuscript.[27] This part of MS Plimpton 188 may thus be the source of the examples that Fridericus took over in 1461 to illustrate the six types of equations.

*5.1.* In no. 17 of the Plimpton collection two numbers are to be found whose sum is 10 and whose quotient is 5, i. e. to solve

$$\frac{10 - x}{x} = 5 .$$

This problem had already been given by Gerardi [Van Egmond 1978: 166] and turns up later in the collection of mathematical texts in a manuscript which originated in Leipzig.[28]

*5.2.* In no. 19 a problem is given that leads to the equation

$$\left(x - \frac{x}{3} - \frac{x}{4}\right)^2 = 16 .$$

This, too, is treated by Fridericus[29] and is also to be found in some form in the 14th-century Italian *Columbia Algorismus*[30] and in the so-called *Deutsche Algebra* written in 1481.[31] Interestingly enough, Gerardi had treated the same problem, with 12 instead of 16 [Van Egmond 1978: 166, no. 2].

*5.3.* Again, problem no. 20 is given in similar form by Gerardi [Van Egmond 1978: 167, no. 3] and by Fridericus.[32] It leads to the system of equations

$$x + y = x \cdot y , \qquad x : y = 2 : 3 .$$

There are other problems of finding numbers in Regiomontanus' correspondence:

*5.4.* $x - y = \sqrt{5}$, $x^2 \cdot y^2 = 250$.[33] Regiomontanus rejects Bianchini's assertion that this is insoluble because 250 is not a perfect square [Curtze 1902: 236], but he gives no solution.[34]

25 $ax = b, ax^2 = b, ax = bx^2, ax^2 + bx = c, ax^2 + b = cx, ax + b = cx^2$.

26 In a manuscript now in Munich (Clm 14908). Edited in [Curtze 1895: 50].

27 Edited in [Curtze 1895: 50–73].

28 Now Dresden, C 80, fol. 352v.

29 On fol. 139rv. Edited in [Curtze 1895: 53].

30 $(x - \frac{x}{3} - \frac{x}{4})^2 = x$. See [Vogel 1977: 31–32].

31 $(x - \frac{x}{3} - \frac{x}{4})^2 = 2x$. See [Vogel 1981: 36–37].

32 München, Clm 14908, fol. 139v–140v: [Curtze 1895: 53–54].

33 [Curtze 1902: 219, no. 5], in a letter to Bianchini.

34 [Curtze 1902: 253]: *Quesivi duos numeros huiusmodi non in unitatibus, scilicet in actu tantum, sed in potentia etiam, si opus fuerit.*

*5.5.* Another problem of finding two numbers is disguised in a traditional purchase problem.[35] The problem is: $x$ *brachia* cost 15 *ducati*, 27 *brachia* cost $y$ *ducati*, and $x + y = 100$. This leads to the quadratic equation

$$15 : x = (58 - x) : 27 \,.$$

Regiomontanus gives a detailed solution and finds, correctly, that $x = 29 - \sqrt{436}$.
*5.6.* Two numbers $x, y$ must be found whose sum is 10 and such that $\frac{x}{y} + \frac{y}{x} = 25$.[36] This results in the quadratic equation

$$250x - 25x^2 = 2x^2 + 100 - 20x \,.$$

*5.7.* A number $x$ is sought for which $\frac{100}{x} + \frac{100}{x+8} = 40$.[37] Here the quadratic equation is

$$40x^2 + 320x = 200x + 800 \,.$$

*5.8.* Another problem in the correspondence is stated without further discussion: $x + y = 100$, $\sqrt{x} \cdot \sqrt[3]{y} = 25$ [Curtze 1902: 262, no. 20]. We note that this is equivalent to a fifth-degree equation.

## 6. A Special Arithmetical Problem

In problem 15 in the Plimpton collection somebody wants to go as many miles as he has dinars. After every mile the dinars he possesses are doubled, but he loses 4 dinars. At the end he has 10 dinars.

After 1 mile he has $2x - 4$ *denarii*.
After 2 miles he has $2 \cdot (2x - 4) - 4 = (2^2 \cdot x - 2 \cdot 4) - 4$.
After 3 miles he has $(2^3 \cdot x - 2^2 \cdot 4 - 2 \cdot 4) - 4$.
After $n$ miles he has $2^n \cdot x - 4(2^{n-1} + 2^{n-2} \ldots + 2 + 1) = 2^n \cdot x - 4(2^n - 1)$.

When $n = x$, the equation to be solved is $2^x \cdot x - 4(2^x - 1) = x$. This implies $x \cdot (2^x - 1) = 4 \cdot (2^x - 1)$, so that $x$ is 4. In the margin is the remark *prepostero solvitur* ("it is solved in a reversed order"). Perhaps this means "assuming the answer and trying it in the problem" — i. e. trial and error.

## 7. Water and Wine

Problem 34 of the Plimpton collection is about a vessel containing 12 measures of wine. By removing one measure of the vessel's contents and adding one measure of

35 Problem: [Curtze 1902: 296, no. 9]; answer by his correspondent Jacob von Speyer: p. 300; comment by Regiomontanus: pp. 304–305, with calculations on pp. 317–319.

36 Problem given by Bianchini: [Curtze 1902: 209]; answer by Regiomontanus: p. 216; calculations: pp. 232–234. See [Kaunzner 1980: 136 (with Plate 4)].

37 Problem given by Bianchini: [Curtze 1902: 237–238]; Regiomontanus' answer: p. 256; calculations: p. 278. See [Kaunzner 1980: 136 (with Plate 5)].

water, and repeating the procedure, the wine is diluted. The question arises: when does the vessel contain only water? The reasoning given for its never coming about is correct.

## 8. Weight Problem

No. 40 of the Plimpton collection concerns the so-called Bachet's weight problem, in which every weight up to 2560 units must be represented with 10 standard weights, each of which is an integral number of units. As usual, the first seven powers of three, $3^0 = 1, 3^1 = 3, \dots, 3^6 = 729$ are chosen, and then — surprisingly — the extension to 2560 is achieved by adding three weights of 489 units each.

This problem was first given by the Persian mathematician Abū Jaʿfar Muḥammad ibn Ayyūb Ṭabarī (11th c.)[38] and is found later in Italian texts (Leonardo Fibonacci, *Columbia Algorismus*). As far as I know, it does not appear in Central Europe before manuscript Plimpton 188.[39]

## 9. Indeterminate Analysis

*9.1.* In the Plimpton collection there are three problems which belong to indeterminate analysis. In no. 57 the system of equations

$$\begin{aligned} x + 13y + 14z &= 391 \\ x + y + z &= 40 \end{aligned}$$

is solved by subtracting the equations. This gives

$$12y + 13z = 351 .$$

From 351 the various multiples of 12 — i.e. 12, 24, … —, are subtracted, and at each stage the difference is tested to see if it is a multiple of 13. This happens with 195, since $351 - \mathit{13} \cdot 12 = \mathit{15} \cdot 13$. Hence $y = 13, z = 15, x = 12$.

*9.2.* This method does not work with no. 58:

$$\begin{aligned} x + 4y + 10z &= 389 \\ x + y + z &= 100 . \end{aligned}$$

For the result of subtracting the equations is $3(y + 3z) = 289$, and 3 does not divide 289. Therefore there is no solution in integers.

*9.3.* In the next problem (no. 59) the author tries to find a general method for solving such problems when they are soluble.

38 See [Hermelink 1975].

39 See [Tropfke 1980: 633–636].

*9.4.* In the correspondence Regiomontanus treats three indeterminate problems. One is linear, two are quadratic. The linear system is

$$x + y + z = 240\,, \quad 97x + 56y + 3z = 16047\,.$$

This problem comes twice in the correspondence.[40] The method of solution is not given, but when Regiomontanus mentions it first, he writes the solution (114, 87, 39) in the margin.[41]

*9.5.* The first quadratic system is

$$x + y + z = 214\,, \quad x^2 - y^2 = y^2 - z^2\,.$$

No solution is given [Curtze 1902: 262, no. 19].

*9.6.* The second quadratic system is

$$x + y + z = 116\,, \quad x^2 + y^2 + z^2 = 4624\,.$$

No solution is given [Curtze 1902: 262, no. 21].

## 10. Remainder Problem

*10.1.* Among the indeterminate problems the most important is the remainder problem which is treated as no. 60 in the Plimpton collection. It is equivalent to

$$N \equiv 2 \bmod 3 \equiv 4 \bmod 5 \equiv 1 \bmod 7\,.$$

The solution follows the Chinese *ta yen* (*da yan*) method[42]: the auxiliary numbers $70 = 2 \cdot 5 \cdot 7$ ($\equiv 1 \bmod 3$), $21 = 3 \cdot 7$ ($\equiv 1 \bmod 5$) and $15 = 3 \cdot 5$ ($\equiv 1 \bmod 7$) are formed. From this we see that Regiomontanus knew not only special solutions, but also the general method of solving the remainder problem. We know that Leonardo had already discussed a similar problem[43] and later Fridericus Amann treats exactly the same problem with the same auxiliary numbers.[44] Once again we find Regiomontanus as the intermediary in the transmission of the new mathematical knowledge from the Italians to the South German *cossists*.

*10.2.* It should be mentioned here that Regiomontanus states examples of the remainder problem in two places in his correspondence. The first runs [Curtze 1902: 219, no. 8]:

$$N \equiv 15 \bmod 17 \equiv 11 \bmod 13 \equiv 3 \bmod 10\,.$$

40 [Curtze 1902: 262, no. 18], and [Curtze 1902: 296, no. 8].

41 [Curtze 1902: 262, no. 18]. His correspondent, Jacob von Speyer, finds the same solution: p. 300; Regiomontanus confirms its correctness: p. 305.

42 For this method and its history see [Libbrecht 1973: 213–413].

43 [Boncompagni 1857: 304]. Libbrecht [1973: 213–413] did not know the Plimpton text.

44 München, Clm 14908, fol. 124v–125r; ed. in [Curtze 1895: 65–66].

Bianchini gives two special solutions,[45] but Regiomontanus is not content with them, pointing out that there is an infinite number of solutions and showing how to find them.[46] From this example alone it would not be clear that Regiomontanus had a general solution to such problems.

*10.3.* The second problem of this kind is stated in the correspondence without proof [Curtze 1902: 295, no. 6]:

$$N \equiv 12 \bmod 23 \equiv 7 \bmod 17 \equiv 3 \bmod 10\,.$$

## 11. Square Root Approximation

*11.1.* In the Vienna *Rechenbuch* there is a passage on square root approximation.[47] It gives the procedure common in Greek writings — which, indeed, goes back to the Babylonians:

$$\sqrt{n} = \sqrt{a^2 + r} \approx a + \frac{r}{2a} = n_1$$

and then, as a better approximation,

$$n_2 = \left(\frac{n}{n_1} + n_1\right) : 2\,.$$

This method can be found in Hero's writings and later in Arabic texts.[48]

*11.2.* Another square root approximation is in the Plimpton collection (no. 56), $\sqrt{10}$ being taken as example. The method may be expressed by the formula

$$\sqrt{n} = \sqrt{a^2 + r} \approx a + \frac{r}{2a} \approx a + \frac{4a^2 + 2r - 1}{(4a^2 + 2r) \cdot 2a}\,.$$

The author considers the formula to be not particularly useful, for he writes: *Hec pro exercicio ingenii scripsi. Non enim est in his utilitas digna tantis laboribus, cum neque in omnibus numeris locum habeat illa invencio, neque quantum licet iri possit.*

## 12. Theory of Numbers

*12.1.* Regiomontanus was also interested in the theory of numbers. In his Vienna *Rechenbuch* he calculates the fifth perfect number $(2^{13} - 1) \cdot 2^{12}$ and gives the formula $(2^{17} - 1) \cdot 2^{16}$ for the sixth perfect number — in the tradition of Euclid's IX.36 — without calculating it.[49] Up to now it has been assumed that Regiomontanus was

45 $N = 1103$ or $3313$: [Curtze 1902: 237].

46 $N_1 = 1103$; $N_{i+1} = N_i + 17 \cdot 10 \cdot 13$: [Curtze 1902: 254].

47 Wien 5203, fol. 167v–168r. See [Curtze 1897].

48 For the history of this method, see [Tropfke 1980: 263–268, 277].

49 Wien 5203, fol. 167r. Edited in [Curtze 1899: 288–289].

the first in the West to determine these numbers. But it is not so: in Bessarion's 13th-century manuscript (Venice, Biblioteca Nazionale Marciana, f. a. 332), there is a short text on the first five perfect numbers.[50] The Regiomontanus text is almost identical with that in the Venice manuscript. Therefore we may assume that he made his copy from it.

In his correspondence he gives the following problems, without solving them:

*12.2.* To find four squares whose sum is a square.[51]

*12.3.* To find twenty squares whose sum is a square and of which the smallest is greater than 30,000 [Curtze 1902: 334, no. $\epsilon$].

*12.4.* To find three squares that are in harmonic progression [Curtze 1902: 296, no. 10].

*12.5.* To find three numbers that are in harmonic progression and of which the smallest is greater than 500,000 [Curtze 1902: 334, no. $\delta$].

*12.6.* To find three squares that are in arithmetic progression and of which the smallest is greater than 20,000 [Curtze 1902: 334, no. $\gamma$].

It is not clear whether Regiomontanus could solve these problems. In three cases his correspondents gave a special solution, but Regiomontanus demanded general solutions.[52]

## 13. Geometrical Problems

*13.1.* In the Plimpton collection there are three geometrical problems about solving triangles. In the first case (no. 61) $c$, one side, is 20, the corresponding height, $h_c$, is 6 and one of the remaining sides ($b$) is double the third side ($a$). Two applications of Pythagoras' theorem lead to a quadratic equation, from which the part of $c$ that is cut off by the height $h_c$ may be determined. Regiomontanus then tries to find a non-algebraic solution by using Euclid II.13, the equivalent of the cosine-theorem. He thereby finds the same quadratic equation, but needs to perform only a few algebraic operations. The same problem can be found in *De triangulis* II.12, where the ratio of the two sides is 3 : 5 [Regiomontanus 1533: 51]. This is reduced to the quadratic equation $25x^2 + 3125 - 500x = 9x^2 + 180x + 1125$, which is then solved algebraically. In the autograph algebraic symbols are used, but in the printed edition of 1533 they are written out in words.[53]

*13.2.* Problem 62 of the Plimpton collection differs from problem 61 only in that $b$ is put equal to $a + 4$ and not to $2a$. It is solved by the theorem in Euclid on intersecting chords. Here again there is a second solution.

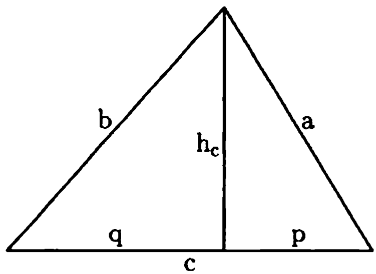
50 On folios 36v–37v.

51 [Curtze 1902: 296, no. 7]. Answer by Jacob von Speyer: 1, 4, 16, 100 or 4, 16, 49, 100: [Curtze 1902: 300]. Regiomontanus' comment: [Curtze 1902: 304].

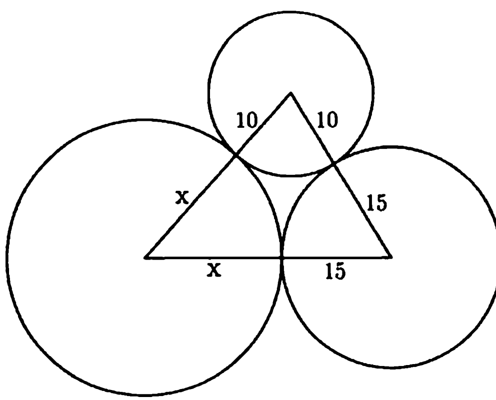
52 [Curtze 1902: 304]: *Difficulter tamen quispiam decem huiusmodi societates numerorum quadratorum inveniet, ...nisi artem id parandi calleat, quam ego petivi.*

53 See [Kaunzner 1980: 134–135 (with Plate 2)].

*13.3.* A slightly altered problem is in *De triangulis*, II.23: the height of a triangle ($h_c$), the difference between two of the sides ($b - a$) and the difference between the parts of the base divided by the altitude ($q - p$) are given [Regiomontanus 1533: 55–56]. In this case, the equation $\frac{1}{4}x^2 + 136 - 6x = 4x^2 + \frac{9}{4} - 6x$ must be solved.[54]




*13.4.* In the Plimpton collection another geometrical problem is proposed (no. 63). Of three mutually touching circles, the radii of two are given, and the area of the triangle whose vertices are the centres of the circles is also given: to find the radius of the third circle. The solution is based upon Hero's formula on the area of a triangle, which is explicitly taken from the *libellus trium fratrum*.[55] This procedure leads to a cubic equation.

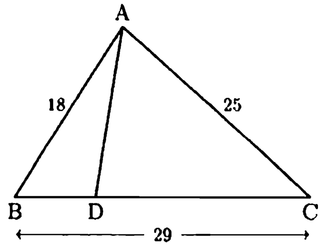



*13.5.* In two other cases Regiomontanus formulates problems on triangles which lead to cubic equations. One of them [Curtze 1902: 331, no. h] is not solved by him. The other one is treated in more detail. The three sides are given. The problem is to

54 See [Kaunzner 1980: 134–135 (with note 48 and Plate 3)].

55 The *Verba filiorum* of the Banū Mūsā; ed. in [Clagett 1964: 223–367].

find a division $BD$ of the base $BC$ so that $BD^2 + DA \cdot AB = AB^2$.[56] This leads to the cubic equation $x^3/324 + 540/29 = 3x$.[57]




Regiomontanus gives no solution, but recognizes the similarity to the problem of finding the chord of 1° when the chord of 3° is known.[58] Thus he understood the correspondence of this problem with the trisection of the angle. It should be noted that Regiomontanus treated the Greek problem of the trisection of an angle in the same letter to Bianchini in 1464 [Curtze 1902: 258–259], giving a proof that is similar to that in the *Verba filiorum* and in the *De triangulis* of Jordanus Nemorarius. It seems more likely that Regiomontanus consulted Jordanus' treatise than that of the Banū Mūsā.[59]

In the correspondence we also find problems about a cyclic quadrilateral, a tetrahedron and two maximum problems:

*13.6.* To determine the area of a cyclic quadrilateral when the diameter and the ratio of the four sides to each other are given.[60] Regiomontanus rejects Bianchini's opinion that the problem is insoluble [Curtze 1902: 235–236] and suggests a method of solution using Ptolemy's theorem [Curtze 1902: 245–252]. By solving this problem Regiomontanus finds the relation between the diagonals $e$, $f$ of the cyclic quadrilateral and its sides $a, b, c, d$:

$$\frac{e}{f} = \frac{ad + bc}{ab + cd}.$$

This formula was already known by Brahmagupta (598–after 665), but there is no reason to assume that Regiomontanus was acquainted with Brahmagupta's solution. Regiomontanus' results may have influenced the German *Rechenmeister* Simon Jacob (1510–1564) and the professor of the University of Altdorf, Johannes Praetorius (1537–1616), who dealt with the construction of the cyclic quadrilateral.[61]

56 See [Folkerts 1977: 228]. The text of the problem is edited in [Curtze 1902: 262, no. 17].

57 See [Cantor 1900: 284–285].

58 *Si dabitis lineam BD, dabo cordam unius gradus.*

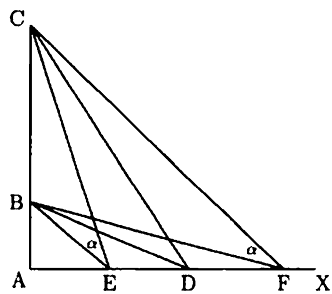
59 See [Clagett 1978: 352–353].

60 The problem is given twice: [Curtze 1902: 219, no. 3] (1464); and [Curtze 1902: 331, no. i] (1471). Another problem given by Bianchini is [Curtze 1902: 236] (Regiomontanus' remarks on p. 253).

61 The history of this problem is presented in [Tropfke 1940: 150–169].

*13.7.* An extension of the previous problem to three dimensions is the problem of determining the volume of a tetrahedron from the ratios of its six sides and the diameter of the circumscribed sphere [Curtze 1902: 219, no. 4]. Bianchini could not solve this problem, either [Curtze 1902: 236]. Regiomontanus knew the solution, but in his correspondence he only gives hints about the way to solve the problem, with the remark that he has written down all theorems necessary for it elsewhere.[62] The history of this problem has not yet been fully investigated. J. Tropfke erroneously wrote that N. Tartaglia (1499?–1557) was the first to determine the volume of a general tetrahedron from the lengths of the sides [Tropfke 1924: 26].

*13.8.* Maximum problem: A 10-foot-long stick hangs vertically, so that it is 4 feet above the floor. To find the point from which the stick subtends the greatest angle [Curtze 1902: 333, no. $\alpha$]. No solution is given.[63]




*13.9.* To inscribe the greatest possible square in a given triangle [Curtze 1902: 333, no. $\beta$]. Only the problem is given.

I now come to the end. We have seen that in his Vienna period, when he (partially) wrote the text in the Plimpton collection, Regiomontanus already had a good knowledge of algebra and geometry. Almost all the problems of the collection go back to Italian sources. I have shown that some of them were given by Leonardo Fibonacci, others by Paolo Gerardi, and there might also be other sources for them. Fridericus Amann must have learnt something of the contents of MS Plimpton 188 soon after it was finished — he was at least acquainted with the treatment of the six al-Khwārizmī types of equation. The same can be said of Johannes Widmann, who taught algebra at Leipzig University in the 1480s — as is witnessed by manuscript Dresden C 80. After writing this part of MS Plimpton 188, in the 1460s, when Regiomontanus was in Italy, he continued stating and solving mathematical problems, as can be seen from his correspondence. Thus Regiomontanus played a crucial role in transmitting mathematical knowledge from Italy to Central Europe in the 15th century.

62 *Hec autem omnia alibi demonstrata dedi*: [Curtze 1902: 252]. These texts do not appear to be extant.

63 Modern suggestions of how the problem could be solved are given in [Lorsch 1878] (also given in [Cantor 1900: 283–284]), and in [Günther 1908: 302–303].

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# Philippe de La Hire's Planetary Theories in Sanskrit

by DAVID PINGREE

On the basis of unpublished Sanskrit documents preserved at Jaipur, Kota and Baroda, an attempt is made to describe the process by which European astronomy, in the form of de La Hire's *Tabulae astronomicae*, was transmitted to Indian scientists in the 1730s. The first attempt was in a Sanskrit poem that presented the rules, without understanding, as variants of traditional siddhāntic computational procedures. The second, in prose, provided mathematically correct instructions, but bereft of understanding of the underlying geometry. The third attempted to offer geometrical demonstrations of how the mathematics also works. It is ironic that it was the first attempt, the poem, that was the most popular of these three.

The history of mathematical astronomy in India is a history of transmissions, concurrent transformations, and indigenous developments.[1] In this process, India, like many other centers of civilization, acted over several millennia as both recipient and transmitter. In this paper I will concentrate on its role as a receiver at the beginning of the last major transmission of external science to India, that which came from Europe. This phase had been preceded by the transmission of Mesopotamian mathematical astronomy in the fifth century B.C., of Greek mathematical astronomy in the second through fourth or fifth centuries A.D., and of Islamic mathematical astronomy in the eleventh till the eighteenth century.

In each of these transmissions the external science was transformed both by the existing Indian intellectual and scientific traditions, which shaped the ways in which the newly received science could be understood and could be expressed in Sanskrit, by the absence in India of the cultural assumptions that were familiar to those who had developed the newly introduced science elsewhere, and by the idiosyncracies of the individuals involved in the transmission. The exceptional value of the case I am about to discuss lies in the fact that considerable evidence concerning what transpired in the attempts to render part of de La Hire's *Tabulae astronomicae*[2] into Sanskrit still remains in the Palace Library at Jayapura; it provides a remarkably well documented instance of what must have happened many other times, both in India and in other countries.

1 See [Pingree 1978].

2 Philippe de La Hire, *Tabulae astronomicae*, Paris 1702; reprinted Paris 1727. I have used the first edition, the astronomers at Jayapura used the second. In the parts to be discussed in this paper there is no difference between them.

In November of 1730, Jayasiṃha,[3] Mahārājādhirāja of Jayapura, one of the principal cities of Rājasthān in Northwestern India, received a copy of the 1727 reprint of Philippe de La Hire's *Tabulae astronomicae* that came with a young Portuguese astronomer named Pedro da Silva. In the succeeding few years his astronomers trained in the Sanskrit tradition sought to understand the mathematics de La Hire employed in computing the longitudes (and latitudes where appropriate) of the Sun, the Moon, and the planets as well as lunar and solar eclipses. They had a very difficult time because the Frenchman's planetary models and his mathematics were very different from what they were accustomed to, and because the European "astronomers" at Jayasiṃha's court — da Silva and Joseph du Bois — were presumably incapable of explaining them to them; the explanations came only when Fathers Boudier and Pons arrived in Jayapura in August of 1734.

Fortunately, we are well informed concerning the process of solving the problem. We have, of course, de La Hire's book, which provides tables, instructions for their use, diagrams of the models he used in most cases, and worked examples; the tables give logarithms of several lengths appearing in the various computations of the coordinates of each planet, but not complete tables of the logarithms of numbers and of trigonometrical functions or any rules for using logarithms.

There exists in five manuscripts a Sanskrit poem in three chapters on computing with de La Hire's tables, the *Dṛkpakṣasāriṇī* composed by Kevalarāma,[4] who became Jayasiṃha's Jyotiṣarāja, or King of Astronomers, in 1725. The primitiveness of his work suggests that it was composed, perhaps on the basis of an earlier attempt to translate the Latin text of the instructions into Sanskrit, in about 1732. I suggest an earlier attempt to translate the instructions only because Kevalarāma seems not to have seen the diagram illustrating the portion I am going to discuss — that on computing the longitudes and the latitudes of the planets — nor, till the very end of his exposition of this in Chapter 1, does he appear to have examined de La Hire's worked example. Naturally, this clumsy effort, since it was in verse, the normal medium for expressing scientific ideas in Sanskrit, was far more popular than the more correct prose translation. I have used the edition of the *Dṛkpakṣasāriṇī* prepared from all five manuscripts by my student, Setsuro Ikeyama, and myself.

3 Jayasiṃha began his studies of astronomy in about 1706, when he acquired copies of at least a dozen standard works on traditional siddhāntic astronomy; by 1721 he had promised the Mughal emperor, Muḥammad Shāh, that he would prepare a definitive *zīj* in his honor, and had begun planning his first observatory, at Delhi; in 1726 he was studying Muslim astronomy through the Sanskrit expositions of it composed by Nityānanda a century earlier; in the next six years he acquired numerous astronomical manuscripts written in Arabic and Persian and sponsored the translation of a number of them into Sanskrit; and he finally published the Persian *Zīj-i Muḥammad Shāhī* in the late 1730s. See [Pingree 1999].

4 Kevalarāma wrote for Jayasiṃha, besides the *Dṛkpakṣasāriṇī*, a *Brāhmapakṣanirāsa*, in which he attempts to "prove" on very questionable grounds the superiority of the *Sūryasiddhānta* over the *Brāhmasphuṭasiddhānta* (my edition should appear shortly), and a *Bhāgavatajyotiḥśāstrayor bhūgolakhagolavirodhaparihāra*, in which he hypothesizes the coexistence of an enormous flat earth, as described in the *Bhāgavatapurāṇa*, and a tiny spherical earth, as assumed in the *siddhāntas* (an edition is being prepared by C. Minkowski and myself).

A year or two after Kevalarāma's effort — probably in 1734 — someone in Jayapura prepared a new version in Sanskrit prose of the section of de La Hire's instructions devoted to the Sun, the Moon, and the planets; unlike Kevalarāma he did not attempt to work with the instructions concerning lunar and solar eclipses, but he did include all of the tables. I found two copies of this in the Khasmohor collection at Jayapura, numbers 5292 and 5609; unfortunately the latter is missing the instructions and all of the tables prior to those concerning Jupiter. Recently I acquired a photocopy of a manuscript at the Oriental Institute at Baroda, accession number 3162, which was wrongly catalogued as the *Dṛkpakṣasāriṇī*; in fact, it is a copy of the new prose translation superior to the Jayapura manuscript. The author of this version, while influenced by Kevalarāma, has performed correct computations, though without understanding the models.

That understanding was finally provided by another Sanskrit prose treatise, the *Phiraṅgicandracchedyopayogika* that was composed in late 1734 or 1735, presumably following the instructions of Father Boudier. The two Khasmohor manuscripts of the new prose translation and a manuscript at the Rajasthan Oriental Research Institute in Kota, number 727, are the only known copies of this work.

I have provided, in the Appendix, English translations of Praeceptum VIII of de La Hire's *Usus tabularum*, a resume of his example for a superior planet omitting all numbers, and a facsimile of the pages containing his model for a superior planet and his identifications of its elements (Plate 1); a translation of de La Hire's identifications and explanations of the other symbols employed in this article are presented at the beginning of the Appendix. I have also provided a translation of verses 12 and 27–41 of Chapter 1 of Kevalarāma's *Dṛkpakṣasāriṇī* and an attempt to interpret his rules; and finally a translation of the Sanskrit prose version's rules and example for computing the longitude of Mars (ff. 6–7v of Khasmohor 5292 and ff. 6–8 of Baroda 3162) and an interpretation of its rules. What I intend to do now is to discuss Kevalarāma's inept and confusing version and then the far superior Sanskrit prose version, and to conclude with a short description of the models presented in the *Phiraṅgicandracchedyopayogika*. I begin with a step-by-step explanation of the verses in the *Dṛkpakṣasāriṇī*.

Kevalarāma I, 12. This verse corresponds to de La Hire's rule II in Praeceptum VIII, though it omits the two last steps. The Sanskrit technical vocabulary is that of the Classical Indian planetary model, where the mean longitude of the planet is measured on a deferent circle whose center is the center of the earth and the manda apogee is on the circumference of an epicycle whose center is the mean planet.

The next 14 verses deal with the computations of the longitude of the Sun and the Moon and of the latter's latitude.

I, 27. This begins with the last two steps of rule II; SP, the distance of the planet P from the Sun S, is found in the appropriate table with the true anomaly, κ, as the argument. However, the fact that SP is a logarithm is not mentioned; and the technical term used to refer to it, the manda hypotenuse, in Classical astronomy designates the distance of the manda apogee on the manda epicycle from the earth. It is only with de La Hire's text in front of him or her that the innocent reader would understand what Kevalarāma means. This is true of most of Kevalarāma's poem. At this point Kevalarāma skips rules III, IV and V, which deal with the computation of the planet's latitude, and proposes to go on to the computation of the angle at which the planet is seen from the earth by means of the angle at which it is seen from the Sun. Following Classical astronomy again, Kevalarāma calls this the śīghra equation, which is computed by means of a second epicycle whose center is the mean planet.

1,28. In conformity with this notion Kevalarāma finds a variant of the Classical śīghra argument, $\bar{\lambda}_S - \bar{\lambda}_P$, where $\bar{\lambda}_S$ is the mean longitude of the Sun and $\bar{\lambda}_P$ that of the planet (Kevalarāma in fact prescribes $\bar{\lambda}_P - \bar{\lambda}_S$, but still calls it the śīghra argument). This, of course, plays no role in de La Hire's astronomy, but is probably meant to represent the second step in rule VI where the smaller of the reduced place of the planet, $\angle \Upsilon SR$, and the place of the earth, $\angle \Upsilon ST$, is subtracted from the larger ($\Upsilon$ denotes the first point of Aries). This is followed by the first step in rule VI, $\angle ITS \pm 180° = \angle \Upsilon ST$, which is the angle of the earth seen from the Sun (the point I is defined in such a way that TI is always parallel to $S\Upsilon$).

At this point Kevalarāma returns to rules III and IV, telling his reader to look up the inclination in the appropriate table with the difference between the longitudes of the eccentric planet and the node as the argument. He calls the inclination by the Classical Sanskrit word for the latitude (i.e., he refers to the angle between the planet on its eccentric orbit and the plane of the ecliptic as seen from the earth rather than from the Sun); and he speaks of the planet on its orbit as if it were on the ecliptic. Indeed, he fails to mention at all rule V, the reduction of the planet's longitude on its orbit to its longitude on the ecliptic.

1,28–29. Kevalarāma now jumps to rule VIII, which gives a proportion involving logarithms ((log) Radius is (log) Tan 45°): $\log SR = \log \mathrm{Cos}\angle PSR + \log SP - \log \mathrm{Tan}\, 45°$. Note that the tabulated value of SP is a logarithm, and no other value for SP is given by de La Hire. Not understanding how to compute with logarithms, Kevalarāma treats the proportion as a normal Indian trairāśika:

$$SR = \frac{SP \times \mathrm{Cos}\angle PSR}{\mathrm{Radius}}.$$

From now on his computations become generally meaningless.

1,29. He now has one side of the triangle SRT; another is ST, the distance of the earth from the Sun in logarithms which de La Hire instructed one to find in rule I with $\angle ITS$. Kevalarāma names it "the hypotenuse from the Sun." He now seems to turn to rule VII, which deals with the angles r, s, and t in triangle

RST. The "sum and difference of these," on this interpretation, are t + r and t − r; of course, in de La Hire one is told to take $\frac{r}{2}$, $\frac{s}{2}$ and $\frac{t}{2}$, so that

$$\frac{t+r}{2} = 90° - \frac{s}{2}.$$

Kevalarāma then jumps to the first step of rule X, which defines the auxiliary angle X by:

$$\frac{\operatorname{Tan} 45°}{\operatorname{Tan} X} = \frac{\operatorname{Tan} \frac{s}{2}}{\operatorname{Tan} \frac{t-r}{2}}.$$

Curiously substituting $\operatorname{Tan}\frac{s}{2}$ for Tan 45° (perhaps thinking of $\operatorname{Tan}\frac{90°}{2}$ as $\operatorname{Tan}(90° - \frac{s}{2})$, and shortening it to $\operatorname{Tan}\frac{s}{2}$), Kevalarāma comes up with the formula:

$$\operatorname{Tan} X = \frac{(t-r) \times \operatorname{Tan}\frac{s}{2}}{t+r}.$$

In the second step of rule X de La Hire instructs one to find $\frac{t-r}{2}$ from

$$\log \operatorname{Tan} \frac{t-r}{2} = \log \operatorname{Tan} X + \log \operatorname{Tan} \frac{t+r}{2} - \log \operatorname{Tan} 45°.$$

I would guess that Kevalarāma just assumes that $\frac{t-r}{2}$ has been found, and that his next sentence means:

$$\operatorname{Tan}\frac{s}{2} + \frac{t-r}{2} = t, \quad \operatorname{Tan}\frac{s}{2} - \frac{t-r}{2} = r.$$

Of course, $\operatorname{Tan}\frac{s}{2}$ is an error for $(90° - \frac{s}{2}) = \frac{t+r}{2}$; substituting that in the two formulas, they are correct. Only one of the two angles, t, is an angle at the earth; it is what Kevalarāma calls the śīghra equation, again applying a Classical term to something very different from what is denoted by its Classical meaning.

1,30. The first and last sentences refer either incompletely or inaccurately to rule XII: $\angle ITS \pm t = \angle ITR$, where ∠ITR is the geocentric longitude of the planet. In between are given the variations from these rules when they are applied to an inferior rather than to a superior planet.

1,31–32. The first sentence in 1,31 refers to the Classical model, in which the longitude of the planet on the ecliptic is found as $\lambda_P = \bar{\lambda}_P \pm \delta_\mu \pm \delta_\sigma$ (without iteration), where $\delta_\mu$ is the manda equation and $\delta_\sigma$ the śīghra equation. It is obviously out of place here; Kevalarāma is just expressing his frustration at having to present de La Hire's more complicated rules, which he obviously did not understand. The remainder of this verse and the beginning of the next are based on rule ⟨XIII⟩, where de La Hire sets up the proportion:

$$\frac{\operatorname{Sin}\angle TSR}{\operatorname{Sin}\angle STR} = \frac{\operatorname{Tan}\angle PSR}{\operatorname{Tan}\angle PTR},$$

from which: $\log \operatorname{Tan}\angle PTR = \log \operatorname{Sin}\angle STR + \log \operatorname{Tan}\angle PSR - \log \operatorname{Sin}\angle TSR$, where ∠PTR is the planet's latitude.

Kevalarāma presents the formula:

$$\operatorname{Tan}\beta = \frac{\operatorname{Tan}\angle PSR \times \operatorname{Sin}\angle STR}{\operatorname{Sin}\angle TSR}.$$

Here, the same Sanskrit word meaning "latitude" is used to indicate both the heliocentric latitude $\angle PSR$ and the geocentric latitude $\angle PTR$.

1,32–33. The beginning of this section repeats in correct form what was incompletely or incorrectly stated in verse 30, namely that $\angle ITS \pm t = \angle ITR$, but mistakenly restricts its application to the superior planets. For the inferior planets, reflecting the Classical procedure as had been the case in 1,31, he states that $\lambda_P = \lambda_A \pm \delta_\sigma$. However, it is not the longitude of the śīghra apogee ($\lambda_A$, which is measured on the circumference of the śīghra epicycle) that is corrected by $\delta_\sigma$, but the mean longitude of the Sun corrected by the planet's manda equation.

1,33–34. The "rule of accuracy" is a misunderstanding of rule IX, where de La Hire sets up the proportion:

$$\frac{ST}{SR} = \frac{\operatorname{Tan} 45^\circ}{\operatorname{Tan} V}$$

for the superior planets (where V is again an auxiliary angle), from which he obtains: $\log \operatorname{Tan} V = \log SR + \log \operatorname{Tan} 45^\circ - \log ST$. For the inferior planets, he substitutes

$$\frac{SR}{ST}$$

on the left side of the equation. In his usual fashion, Kevalarāma writes:

$$\operatorname{Sin} V = \frac{ST \times \text{Radius}}{SR} \qquad \text{for the superior planets,}$$

$$\operatorname{Sin} V = \frac{SR \times \text{Radius}}{ST} \qquad \text{for the inferior planets.}$$

1,34–36. This repeats, with minor changes, the erroneous procedure given in verse 29.

1,36–38. Evidently in despair, Kevalarāma gives as "a second rule" the traditional Indian computation of the śīghra equation based on an epicyclic model. His statement at the end that he has seen this demonstrated with a globe having movable parts is curious; it implies a planetary model with the earth at the center, a concentric deferent that rotates in the plane of the planet's orbit, and a rotating epicycle whose center is on the circumference of the deferent. Where did he find such an instrument?

1,39–41. In this explanation of a rule mentioned by de La Hire (*laiyara* in Sanskrit), Kevalarāma first refers to the quantities computed in verses 28–29 (only some of which are used in the following computation), then summarizes rules VIII to X. In verse 40 he combines his trairāśika interpretations of rules VIII and IX:

$$\mathrm{SR} = \frac{\mathrm{Cos}\angle \mathrm{PSR} \times \mathrm{SP}}{\mathrm{Radius}} \quad \text{and} \quad \mathrm{Tan\,V} = \frac{\mathrm{SR} \times \mathrm{Radius}}{\mathrm{ST}}.$$

From these he forms a new trairāśika:

$$\mathrm{Tan\,V} = \frac{\mathrm{Cos}\angle \mathrm{PSR} \times \mathrm{SP}}{\mathrm{Radius}} \times \frac{\mathrm{Radius}}{\mathrm{ST}} = \frac{\mathrm{Cos}\angle \mathrm{PRS} \times \mathrm{SP}}{\mathrm{ST}}.$$

Of course, these formulae are wrong because some of the quantities referred to are logarithms. Then, continuing with IX, he has $\mathrm{V} - 45° = \mathrm{X}$.

In verse 41 he finally shows that he has learned in part how to compute with logarithms. This corresponds to rule X:

$$\frac{\mathrm{Tan}\,45°}{\mathrm{Tan\,X}} = \frac{\mathrm{Tan}\,\frac{t+r}{2}}{\mathrm{Tan}\,\frac{t-r}{2}},$$

from which

$$\log \mathrm{Tan}\,\frac{t-r}{2} = \log \mathrm{Tan\,X} + \log \mathrm{Tan}\,\frac{t+r}{2} - \log \mathrm{Tan}\,45°.$$

Kevalarāma has:

$$\mathrm{Tan}\,\frac{t-r}{2} = \frac{\mathrm{Tan\,X} + \mathrm{Tan}\,\frac{s}{2}}{\mathrm{Radius}}.$$

Radius, of course, is $\log \mathrm{Tan}\,45°$, and should be subtracted from the sum (not the product!) in the numerator taken as logarithms rather than divided into it. In the numerator $\frac{s}{2}$ is short for $(90° - \frac{s}{2}) = \frac{t+r}{2}$. But he almost got it right!

Presumably in triumph Kevalarāma ends his description of the computation of the longitudes and latitudes of the planets.

The rules in the Sanskrit prose version can be dealt with more summarily since they are essentially correct:

(a) The equation of time is called the *dyuphala*, "equation of the day"; it is called by its Classical name, *dvyaikya*, "sum of two" (referring to its two components), by Kevalarāma.

(f) The inclination is called the mean "latitude."

(g) The reduction is called the equation of the "latitude."

(j) $\angle \Upsilon\mathrm{ST}$ is called the *dhruvaka*, "fixed" (longitude).

(k) T is called the perigee of the śīghra (epicycle), that is, of the earth's orbit, so that T + 180° is the apogee.

(m) $\frac{s}{2}$ is called the śīghra argument, which term Kevalarāma had applied, in 1,28, to $\lambda_P - \lambda_S$.

(p) Here and hereafter, "shadow" means log Tangent, "Sine" means log Sine, and "Cosine" means log Cosine. They are "measured" by 1,000,000 — that is, they have 6 digits as in de La Hire's tables, and log Radius = log Tan 45° = 1,000,000.

(r) SR is called the Cosine of the equation of the "latitude"; that is Cosine Q, where Q is the planet's longitude on its orbit reduced to the ecliptic.

(v) In taking the log Tan X, the author of the example used a table of log Sines and log Tangents equivalent to that found in a manuscript in Jodhpur, Rajasthan Oriental Research Institute 23958, wherein the logarithms are given to eight figures and log Tan 45° is 100,000,000. The author correctly omits the last two digits in each logarithm.

(w) This rule is inspired by Kevalarāma's in 1,29.

Now I would just note briefly that the *Phiraṅgicandracchedyopayogika* explains that the orbit of the planet is an ellipse (the text calls it *matsyākāra*, "having the shape of a fish," with reference to the figures called "fish figures" in Classical Indian texts, used to erect a perpendicular in the middle of a straight line), and it gives in illustration a conic section producing an ellipse (see Plate 2) and an example of how one computes the equation of the center with such a figure (see Plate 3). This work also offers a copy, in Plate 4, of de La Hire's illustration of the heliocentric model for a superior planet, which is depicted on p. 14 of the *Usus tabularum* (Plate 1), and explains exactly how it works. This explanation begins with the Sanskrit words meaning: "in this it is imagined that the earth moves; the Sun is imagined to be fixed." Thus the Keplerian-Newtonian planetary theory was first introduced to scientists reading Sanskrit. Unfortunately, this text did not circulate.

If I were asked to guess, I would surmise that Joseph du Bois, the least trained of the European advisors at Jayasiṃha's court, advised Kevalarāma, and did so badly. Then Pedro da Silva was probably the one able to explain the computational rules of de La Hire, including the correct way to use logarithms, to the author of the Sanskrit prose translation, but was either ignorant of or, a more likely hypothesis, chose to remain silent about the heliocentric theory. Finally Père Boudier would have explained the use of ellipses and heliocentricity to the author of the *Phiraṅgicandracchedyopayogika*, even though de La Hire explicitly claimed in his prefatory epistle that Kepler's hypothesis was the cause of the failure of the Rudolphine Tables to agree completely with all the observed phenomena, and that he himself constructed his tables on observations alone without any particular hypothesis.

## Appendix

## Explanations of the Symbols

*1. Translation of de La Hire's explanations on Usus Tabularum, pp. 14–15 (cf. Plates 1a and 1b on pages 440–441)*

| | |
|---|---|
| S | Sun. |
| STRNBO | Plane of the ecliptic. |
| ANPO | Orbit of a planet, in which: |
| NO | Line of the nodes, or coincidence of the plane through the orbit with the plane of the ecliptic. |
| NROB | Reduction or projection of the orbit of the planet onto the plane of the ecliptic by mutually parallel lines (which are) perpendicular to the plane of the ecliptic. |
| P | The planet in its orbit. |
| T | The earth. |
| TM | The orbit of the earth above the plane of the ecliptic. |
| PR | The perpendicular straight line drawn from the planet P to the plane of the ecliptic at R. |
| R | The planet's place reduced to the ecliptic. |
| F | (This) point of the orbit indicates the first point of Aries in the orbit of the planet, so that angles OSF, OS♈ are mutually equal above the plane of the orbit and (that) of the ecliptic. The mean motion of the planet, however, is counted from the point F of the orbit through the angles FSP. |
| ♈SR | The planet's reduced angle or place from the first point of Aries. |
| PSR | The angle of the inclination of the planet seen from the Sun. |
| ♈ST | The true angle or place of the earth from the first point of Aries. |
| ♈SR | The angle of the planet's place reduced from the first point of Aries. |
| TSR | The so-called angle at the Sun. |
| STR | The so-called angle at the earth. |
| SR | The shortened distance of the planet. |
| BTI | The straight line parallel to S♈ drawn through the center of the earth. |
| ITR | The angle of the true longitude of the planet from the first point of Aries. |
| V, X, Y, Z | are angles which, having risen from the operations of the calculation, serve it alone. |
| PTR | The angle of the true latitude of the planet. |

The same is to be understood with regard to the inferior planets as with regard to the superior ones.

## 2. *Other Roman letter symbols and Greek letters*

| | |
|---|---|
| ∠ITS | The "true place" of the Sun as seen from the earth. |
| Q | The reduction of the planet's longitude on its orbit to the ecliptic. |
| r | The angle SRT. |
| s | The angle TSR. |
| t | The angle STR. |
| $\beta$ | The planet's latitude (angle PTR). |
| $\delta_\mu$ | The manda equation, corresponding to Ptolemy's equation of the center. |
| $\delta_\sigma$ | The śīghra equation, corresponding to Ptolemy's equation of the anomaly. |
| $\kappa$ | The true anomaly. |
| $\kappa_\mu$ | The manda argument. |
| $\bar{\kappa}_\mu$ | The uncorrected manda argument. |
| $\kappa_\sigma$ | The śīghra argument. |
| $\lambda_A$ | The longitude of the apogee. |
| $\lambda_N$ | The longitude of the node. |
| $\lambda_P$ | The longitude of the planet on its orbit or on the ecliptic as specified. |
| $\bar{\lambda}_P$ | The mean longitude of the planet. |
| $\lambda_S$ | The longitude of the Sun. |
| $\bar{\lambda}_S$ | The mean longitude of the Sun. |
| $\omega$ | The argument of latitude ($\lambda_P - \lambda_N$). |
| ♈ | The first point of Aries. |

De La Hire, *Tabulae astronomicae*, Usus tabularum, *Praeceptum* VIII

*On Longitude*

I. Let the apparent proposed time be corrected and made mean, as we taught in the precept on finding the place of the Sun, and let the true place of the Sun in the ecliptic be found together with the distance of the earth from the Sun from Table 14 in logarithmic numbers.

II. Let the eccentric place of the planet, which is the place of the planet seen from the Sun, be sought by the same method by which the true place of the Sun (was sought), and also the place of the planet's ascendent node, to which a correction will be applicable in the case of Saturn, and the true anomaly together with the distance of the planet from the Sun in logarithmic numbers as they are in the tables of the individual planets.

III. Let the place of the planet's node be subtracted from the eccentric place, and the remainder will be the argument of the planet's latitude.

IV. With the argument of the latitude, let there be found in their own tables the inclination of (each) planet and (its) reduction to the ecliptic. However, the reduction will be additive or subtractive as the apposite title shall indicate.

V. If it is additive let the reduction be added to the true eccentric place of the planet or let it be subtracted from it if it is subtractive, and the place of the planet reduced to the ecliptic will be obtained, which will be the longitude of the planet from the Sun.

VI. When six signs have been added to or subtracted from the true place of the Sun, the sum or difference will be the true place of the earth seen from the Sun. Let the reduced place of the planet be subtracted from the place of the earth or the latter from the former, that is, the smaller from the larger; but if the larger should exceed the smaller by an amount greater than six signs, let twelve signs be added to the smaller before the subtraction is made; but then the smaller becomes the larger. The remainder will be the angle at the Sun, in all cases less than six signs.

VII. Let half of the angle at the Sun be taken; its complement to a quadrant or to three signs will be half the sum of the angles, which (half-sum) we use in the resolution of the triangle whose angles are at the Sun, at the earth, and at the planet reduced on the plane of the ecliptic.

VIII. If the Radius should be to the Sine of the complement of the inclination of the planet as the planet's distance from the Sun taken from the table is to a fourth term, the last will be the distance of the reduced planet from the Sun, which they call the shortened distance.

IX. Besides, let the distance of the earth from the Sun be to the shortened distance in the cases of the superior planets (namely Saturn, Jupiter, and Mars), but the shortened distance to the distance of the earth from the Sun in the cases of the inferior planets (namely Venus and Mercury), as the Tangent of 45 degrees is to the Tangent of a certain angle which I call V, from which, when 45 degrees have been subtracted, I call the remaining angle X.

X. Finally let the Tangent of 45 degrees be to the Tangent of angle X as the Tangent of half the sum of the unknown angles which was found in the seventh article is to the Tangent of half the difference of the same unknown angles.

XI. Let this half-difference be added to the half-sum of the same angles in the cases of the superior planets, but let it be subtracted from it in the cases of the inferior planets; the angle at the earth will be obtained.

XII. But let the angle at the earth be added to the true place of the Sun if the distance of the earth from the reduced place of the planet in the order of the signs is less than a semicircle or six signs; on the contrary, if this distance is greater, the angle at the earth is subtracted; the true longitude of the planet will be obtained.

*On the Latitude*

⟨XIII.⟩ Let the Sine of the angle at the Sun be to the Sine of the angle at the earth as the Tangent of the planet's inclination is to the Tangent of its latitude. The latitude will be northern in the first six signs of the argument of the latitude, but southern in the last.

14 *Usus Tabularum.*

Cum Anomaliâ verâ ex Tabulâ 18. habemus,

| | | |
|---|---|---|
| Diametrum horizontalem ☾ | 30′ | 55″ |
| Parallaxim horizont. | ′56 | 41 |
| Distantiam ☾ à centro terræ in centes. semid. terræ 6060: | | |
| Sed in Tabula 23. reperientur, | | |
| Correctio subtrahenda diametri ☾ horizontalis, | | 8 |
| Correctio subtrahenda Parall. horizont. | | 15 |
| Correctio addenda distantiæ lunæ à centro terræ, 27 Cent. | | |
| Erunt igitur vera Diameter ☾ horizont. | 30. | 47 |
| Vera Parallaxis horizont. | 56. | 26 |
| Vera distantia lunæ à centro Terræ, | 6087. Cent. semid Terræ. | |

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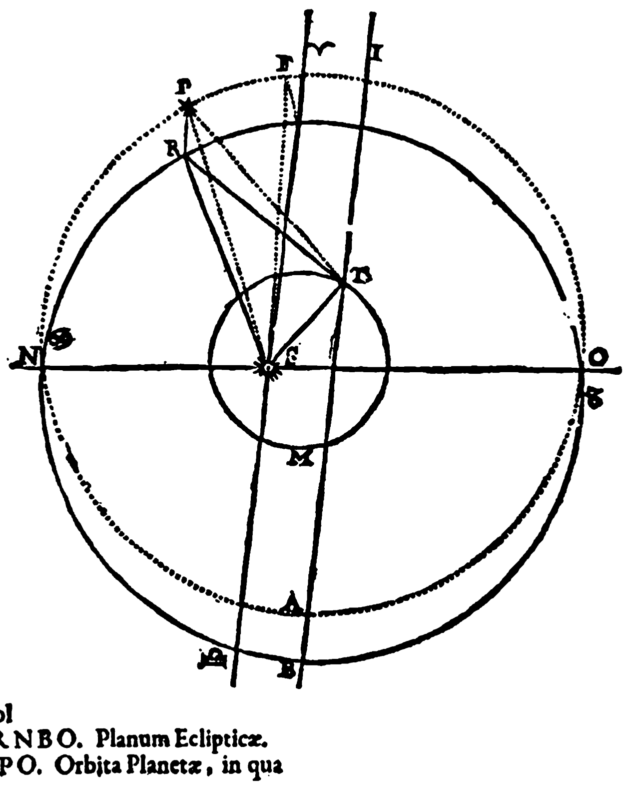
In hac figura Sole posito immobili, Planetarum superiorum Systema oculis subjicitur.




S. Sol
STRNBO. Planum Eclipticæ.
ANPO. Orbita Planetæ, in qua

Plate 1a. De La Hire's *Tabulae astronomicae*, page 14 (*Usus tabularum*).

NO. Linea Nodorum, vel occursus plani per orbitam cum Eclipticæ plano.

NROB. Orbitæ planetæ reductio vel projectio in planum Eclipticæ per lineas parallelas inter se & ad planum Eclipticæ perpendiculares.

P. Planeta in orbita sua.

T. Terra.

TM. Orbita Terræ super planum Eclipticæ

PR. Linea recta perpendicularis à planeta P ducta ad Eclipticæ planum in R.

R. Locus planetæ ad Eclipticam reductus.

F. Orbitæ punctum indicat primum Arietis punctum in orbitâ planetæ, ita ut anguli OSF, OS♈ tam super orbitæ quàm Eclipticæ planum sint inter se æquales. Numeratur autem medius motus Planetæ ab orbitæ puncto F per angulos FSP.

♈SR. Angulus vel locus Planetæ reductus à primo puncto Arietis.

PSR. Angulus inclinationis Planetæ ex Sole visi.

♈ST. Angulus vel locus verus Terræ à primo puncto Arietis.

♈SR Angulus loci planetæ reducti à primo puncto Arietis.

TSR. Angulus dictus ad Solem.

STR. Angulus dictus ad Terram.

SR. Planetæ distantia curtata.

BTI. Linea recta parallela S♈ per centrum Terræ ducta.

ITR. Angulus veræ longitudinis Planetæ à primo puncto Arietis.

V, X, Y, Z sunt anguli qui ex calculi operationibus orti, ei tantum inserviunt.

PTR. Angulus veræ Latitudinis Planetæ.

Idem intelligendum de Planetis inferioribus quod de superioribus.

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# PRÆCEPTUM VIII.

*Planetarum superiorum & inferiorum veram Longitudinem & Latitudinem ad datum tempus reperire.*

## DE LONGITUDINE.

I°. Propositum tempus apparens corrigatur & fiat medium, ut in præcepto de loco solis inveniendo docuimus, unàque verus locus Solis in Ecliptica cum distantia terræ à sole ex tabula 14. in Logarithmicis numeris reperiatur.

II. Quæratur locus excentricus Planetæ, qui planetæ locus est ex Sole visus, eàdem methodo quâ verus locus solis, nec non planetæ locus nodi ☊ ascendentis, cui adhibenda erit correctio in ♄, & anomalia vera, cum distantiâ planetæ à Sole in numeris Logarithmicis, ut se habent in Tabulis singulorum planetarum.

Plate 1b. De La Hire's *Tabulae astronomicae*, page 15 (*Usus tabularum*).

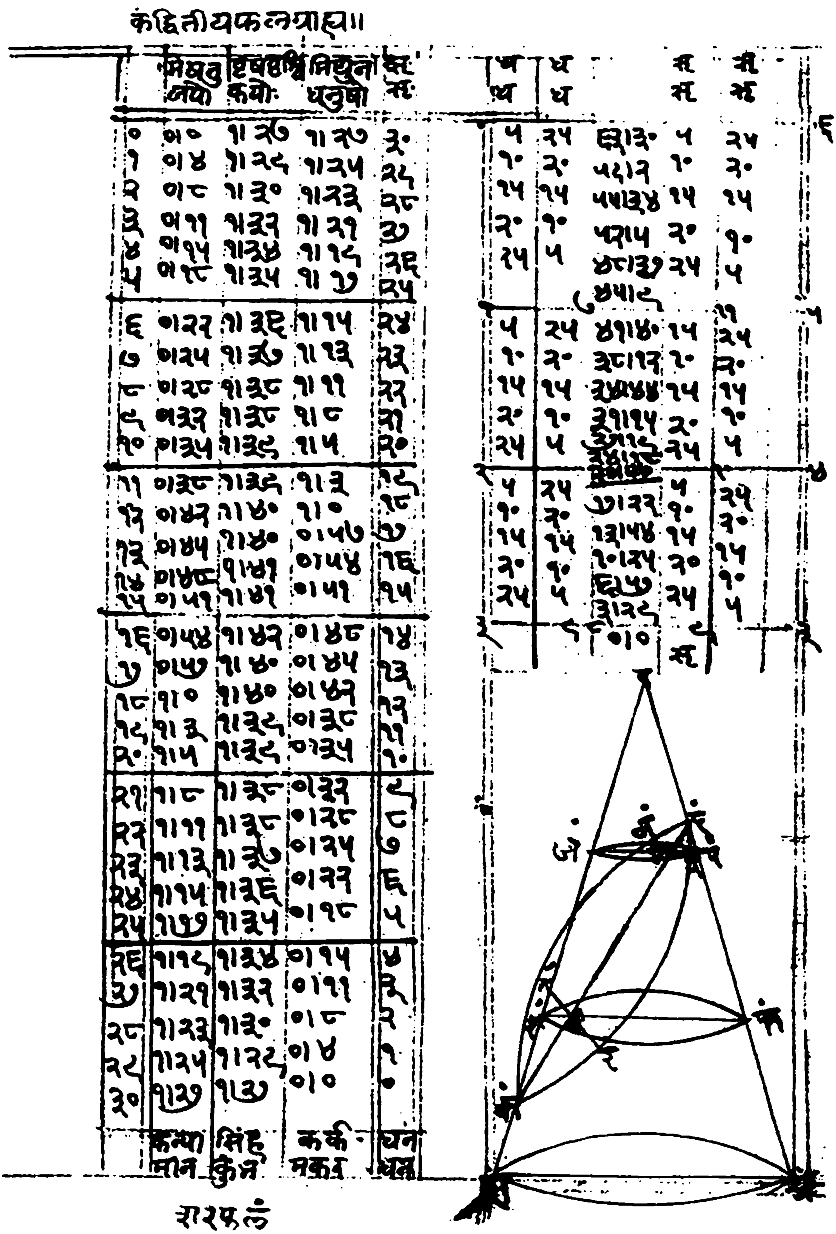

**Plate 2.** *Phiraṅgicandracchedyopayogika*, Jayapura, Khasmohor 5292, f. 39.

तेन तमं ज्ञातं भविष्यति अस्य छायासु चापं तउमको
भवति पुनः उवमे रेखा कार्या सा शपर्यंतं वर्द्धनीया यथा
सवी रेखा द्विगुणित त्रिज्यातुल्या भवेत्॥ तथा कार्या श
उसकोणः ज्ञातोस्ति उशभुजउसभुजौ वै ज्ञातौस्तः तेन र
उशसकोणः ज्ञातो भविष्यति अयं द्विगुणः उवसकोणः॥
मंदफलरूपः ज्ञातो भविष्यति कुतः उवभुजसवभुजयो
र्योगः द्विगुण त्रिज्यातुल्योस्ति उशभुजोपि उभयोस्तुल्ययोः
उवंशोधितं शेषं सवंशाव समानं भविष्यति॥ सवशत्रिभुजे
सकोणौ समानौ भविष्यतः अनयोर्योगतुल्य उवसको
णो भविष्यति इदमेवास्माकमिष्टं इदं मंदफलं मेषादि
केंद्रस्यांतुल्याद्दीघनं कार्यं अथ मंदकर्णसाधनं उव
सकोणज्यया इष्टमंदफलज्या रूपाया उसभुजो लभ्यते॥
तदा सउवकोणज्ययाकिमिति लब्धं मंदकर्णो भवति इ
ति भौमादीनां मंदफलसाधनोपपत्तिः॥ अथ मंदफलस्वरूपं भौमस्य

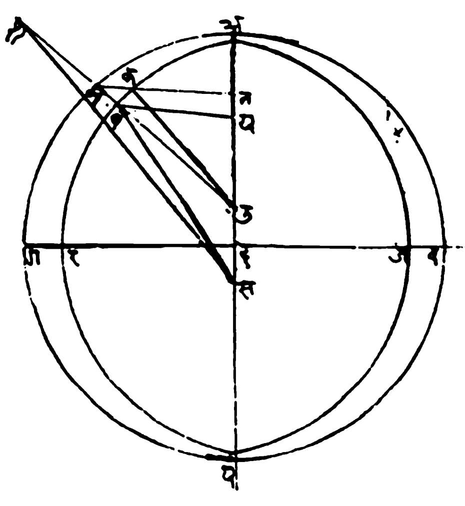


Plate 3. *Phiraṅgicandracchedyopayogika*, Jayapura, Khasmohor 5292, f. 52.

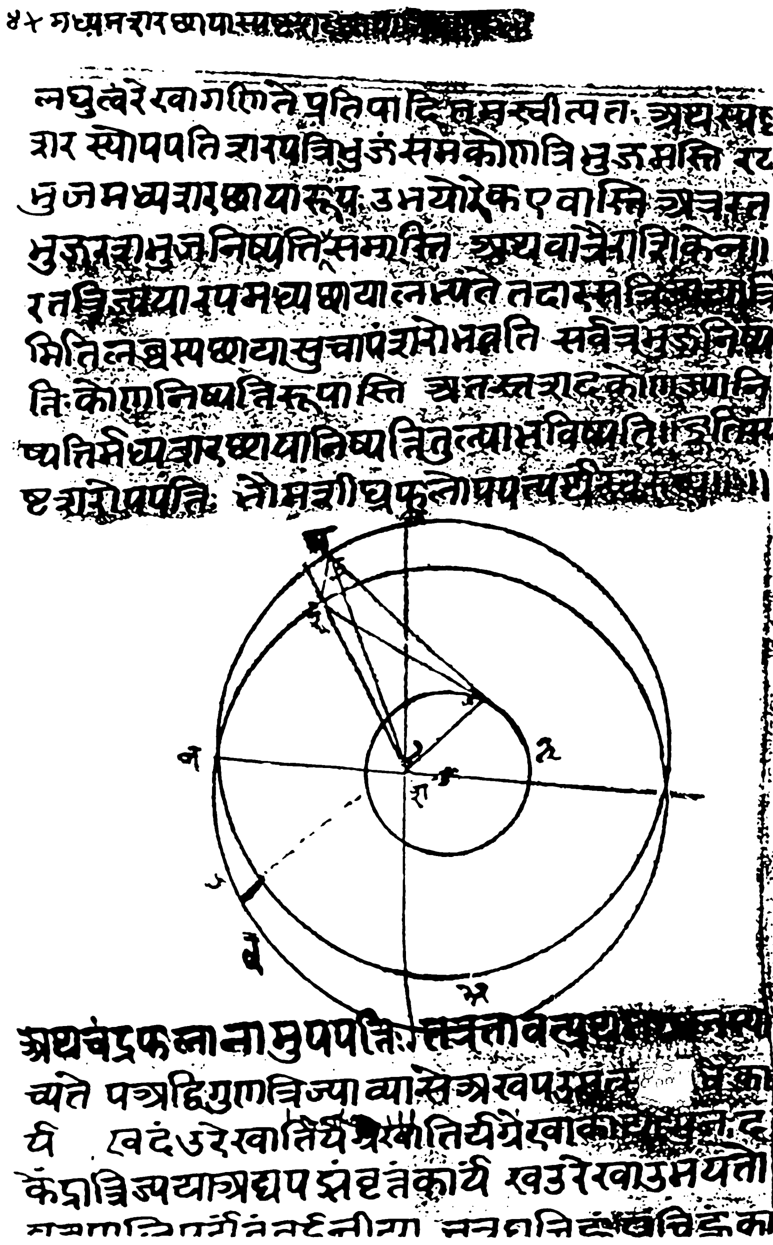

**Plate 4.** *Phiraṅgicandracchedyopayogika*, Jayapura, Khasmohor 5292, f. 43v.

## Summary of de La Hire's Rules from the Praeceptum and his Example for a Superior Planet

*Explanation of abbreviations used:*

(a) This is the statement in the Praeceptum and the example even though SP and SR are known only in logarithms.
(b) This is the computation in the example.
(c) This is the statement in the Praeceptum and the example.

I. find $\lambda_S$, and ST in logarithms

II. $\bar{\lambda}_P - \lambda_A = \bar{\kappa}_\mu$

with $\bar{\kappa}_\mu$ find $\delta_\mu$

$\bar{\lambda}_P \pm \delta_\mu = \lambda_P$ on the planet's orbit

$\bar{\kappa}_\mu \pm \delta_\mu = \kappa_\mu$

find $\lambda_N$, and SP in logarithms

III. $\lambda_P - \lambda_N = \omega$

IV. find $\angle$PSR and the reduction, Q

V. $\lambda_P \pm Q = \angle \Upsilon SR$ ($\lambda_P$ on the ecliptic)

VI. $\angle ITS \pm 180° = \angle \Upsilon ST$

$\angle \Upsilon ST - \angle \Upsilon SR$ (or $\angle \Upsilon SR - \angle \Upsilon ST$) $= \angle RST = s$

VII. $\frac{r}{2} + \frac{s}{2} + \frac{t}{2} = 90°$

$90° - \frac{s}{2} = \frac{t+r}{2}$

VIII. (a) $\frac{\text{Radius}}{\text{Cos}\angle PSR} = \frac{SP}{SR}$

(b) $\log SR = \log \text{Cos}\angle PSR + \log SP - \log \text{Radius}$

IX. (a) $\frac{ST}{SR}(\text{or}\frac{SR}{ST}) = \frac{\text{Tan}45°}{\text{Tan}V}$

(b) $\log \text{Tan } V = \log SR + \log \text{Radius} - \log ST$ (superior planets)

(b) $\log \text{Tan } V = \log ST + \log \text{Radius} - \log SR$ (inferior planets)

$V - 45° = X$

X. (c) $\frac{\text{Tan } 45°}{\text{Tan } X} = \frac{\text{Tan}\frac{t+r}{2}}{\text{Tan}\frac{t-r}{2}}$

(b) $\log \text{Tan}\frac{t-r}{2} = \log \text{Tan } X + \log \text{Tan}\frac{t+r}{2} - \log \text{Radius}$

XI. $\frac{t+r}{2} + \frac{t-r}{2} = t = \angle STR$

XII. $\angle ITS \pm \angle STR = \angle ITR$

⟨XIII.⟩ (b) $\dfrac{\operatorname{Sin}\angle TSR}{\operatorname{Sin}\angle STR} = \dfrac{\operatorname{Tan}\angle PSR}{\operatorname{Tan}\angle PTR}$

(c) $\log \operatorname{Tan}\angle PTR = \log \operatorname{Sin}\angle STR + \log \operatorname{Tan}\angle PSR - \log \operatorname{Sin}\angle TSR$

$\angle PTR = \beta$

Kevalarāma, *Dṛkpakṣasāriṇī* I, 12 and 27–41

12. The mean (longitude) diminished by the manda apogee is the manda argument. Whatever equation is to be obtained from that by proportion is (applied) positively or negatively to the mean (longitude) as the argument is in the six (signs) beginning with Libra or with Aries (respectively). The result is the (mean longitude) corrected by the manda (equation) at that time.
27. The manda hypotenuse of Mars and so on is to be obtained from the argument of the (mean longitude) corrected by the manda (equation). Now I will explain the derivation of the śīghra equation from a method (employing) good mathematics.
28. (The longitudes of) Jupiter, Mars, and Saturn are diminished by the corrected (longitude of the) Sun; that is the śīghra argument. If six (signs) are subtracted from (the longitude of) the Sun (or) added to it, (the result) is the angle (of the earth seen) from the Sun. The "latitude" is to be taken from the (appropriate) table by means of (the longitude of) the planet corrected by the manda equation and diminished by (the longitude of) its node. The manda hypotenuse is multiplied by its (the "latitude's") Cosine and divided by the Radius.
29. (The result is) the first side; the other is the hypotenuse from the Sun. (Take) the sum and the difference of these; (the difference is) multiplied by the "shadow" of half the angle at the Sun and divided by the sum; (take) its (the result's) arc. Compute the (half) portion of the previous angle increased and decreased by that in (terms of) the "shadow." (The two results) are respectively the two angles of the equation at the earth. The śīghra equation is the angle called "equation."
30. It is (applied) negatively or positively as the argument is in (the six signs) beginning with Aries or with Libra. In (the cases of ) Mercury and Venus the śīghra apogee diminished by the accurate (longitude of the) Sun is the śīghra argument for the sake of (determining) the śīghra equation. The two equations at the earth, a small and a large angle respectively, (are derived) from that by the previously mentioned rule. As the śīghra argument is in (the six signs) beginning with Aries or with Libra these two accurate (angles) are (applied) positively or negatively to the accurate (longitude of the) Sun.
31. The (śīghra) equation (is applied) as was previously mentioned to the (longitude of a superior) planet corrected by the manda (equation); the three (superior) planets beginning with Mars become accurate. The "shadow" of the

"latitude" is multiplied by the Sine of the angle at the earth and divided by the Sine of the angle at the Sun; (take) the arc of the result.

32. The accurate "latitude" of the planets in (terms of) a "shadow" results. Now I mention an accurate calculation from another rule. The śīghra argument is computed; if it is less or greater than six (signs), then there is increased and decreased by the angle at the earth
33. the true (longitude of the) Sun. (The superior planets) beginning with Mars are correct. Mercury and Venus are correct when the apogee of each is diminished or increased by what is called the (śīghra) "equation." Again I will describe a rule of accuracy. The hypotenuse of the Sun is multiplied by the Radius
34. and divided by the hypotenuse (of the planet's mean longitude) corrected by the manda (equation); in (the cases of) Mercury and Venus the hypotenuse (of the mean longitude of each) corrected by the manda (equation) is multiplied by the Radius and divided by the hypotenuse of the Sun; this is the Sine of the "other." The small(er) arc is to be imagined to be that of the equation of the śīghra,
35. the large(r arc) is (the arc of) the Radius. The difference and the sum of these (two are to be taken). The "shadow" of half the angle at the Sun is multiplied by the difference and divided by the sum. The arc of the result, (which is) in (terms of) a "shadow," is added to and subtracted from the computed previous (half) portion;
36. (the results are) in order the two "equations" at the earth (in the cases of the superior planets), in reverse order in the cases of Mercury and Venus. From this (comes) accuracy. There is a second rule. By the previous statement the Sine and the Cosine from the śīghra argument are multiplied by the Sine of the "other," which is the (maximum) Sine of the śīghra equation, and both are divided by the Radius.
37. They are called respectively the *bhujaphala* and the *koṭiphala*. The Radius is increased or decreased by the *koṭiphala* as the argument is in the six (signs beginning with) Cancer or Capricorn; whatever (the result) is, the square-root from its square increased by the square of the *bāhuphala* is the hypotenuse.
38. The product of the *bhujaphala* and the Radius, when divided by the hypotenuse, is (the Sine of) the śīghra equation; it is positive or negative as previously. Since I have seen this computation to have applicability with a sphere (whose parts) when rotated move like bees, what fault is there in this for me?
39. Now I will explain a rule mentioned by (de) La Hire. Taking as previously the śīghra argument, the angle at the Sun, its half, the manda (hypotenuse), and its (the planet's) "latitude," one should determine the Cosine of its "latitude."
40. Multiply this by the hypotenuse of the planet and divide (the product) by the hypotenuse of the Sun; if (it pertains to) Mercury and Venus, the hypotenuse of the Sun is divided by it (the Cosine of the "latitude") and multiplied by the manda hypotenuse. The arc of the result, (which is) in (terms of) a "shadow," is diminished by 45.

41. The "shadow" of that remainder is to be made; the "shadow" of the (half) portion of the angle of the Sun is to be added, and (the result) is divided by the Radius. The arc of the result, (which is) in (terms of) a "shadow," is the measure of half the difference.

## Summary of Kevalarāma's Rules

Rules in brackets have no connection with de La Hire; rules in parentheses are necessary, but were omitted by Kevalarāma. De La Hire's rules, referred to by their Roman numerals, when they were misunderstood by Kevalarāma are in parentheses.

| | | |
|---|---|---|
| 12. | $\bar{\lambda}_P - \lambda_A = \bar{\kappa}_\mu$ | II |
| | with $\bar{\kappa}_\mu$ find $\delta_\mu$ | II |
| | $\bar{\lambda}_P \pm \delta_\mu = \lambda_P$ | II |
| 27. | $\bar{\kappa}_\mu \pm \delta_\mu = \kappa_\mu$ | II |
| | with $\kappa_\mu$ find SP | II |
| 28. | $[\lambda_P - \lambda_S = \kappa_\sigma]$ | |
| | $\angle ITS \pm 180° = \angle \Upsilon ST$ | VI |
| | $\lambda_P - \lambda_N = \omega$ | III |
| | with $\omega$ find $\angle PSR$ | IV |
| | $SR = \frac{SP \times \text{Cos}\angle PSR}{\text{Radius}}$ | (VIII) |
| 29. | find ST | I |
| | (find $\frac{t+r}{2} = 90° - \frac{s}{2}$) | (VII) |
| | (find $\frac{t-r}{2}$) | (X) |
| | $\text{Tan}\,X = \frac{(t-r)\cdot \text{Tan}\frac{s}{2}}{t+r}$ | (X) |
| | $\text{Tan}\frac{s}{2} + \frac{t-r}{2} = t$ | (XI) |
| | $\left[\text{Tan}\frac{s}{2} - \frac{t-r}{2} = r\right]$ | |
| 30. | $\angle ITS \pm t = \angle ITR$ | (XII) |
| 31.–32. | $[\lambda_P \pm \delta_\sigma]$ | |
| | $\text{Tan}\,\beta = \frac{\text{Tan}\angle PSR \times \text{Sin}\angle STR}{\text{Sin}\angle TSR}$ | (⟨XIII⟩) |
| 32.–33. | $\angle ITS \pm t = \angle ITR$ | XII |

33.–34. $\operatorname{Sin} V = \dfrac{ST \times \text{Radius}}{SR}$ (superior planets) IX

$\operatorname{Sin} V = \dfrac{SR \times \text{Radius}}{ST}$ (inferior planets) IX

34.–36. $\approx$ 29.

36.–38. [Classical Indian computation of $\delta_\sigma$]

39. $\approx$ 28.–29.

40. $\operatorname{Tan} V = \dfrac{\operatorname{Cos}\angle PSR \times SP}{ST}$ (VIII and IX)

$V - 45° = X$ IX

41. $\log \operatorname{Tan} \frac{t-r}{2} = \dfrac{(\log)\operatorname{Tan} X + (\log)\operatorname{Tan} \frac{s}{2}}{\text{Radius}}$ (X)

## The Sanskrit Prose Version

Now is described the computation of the corrected (longitudes) of the five planets beginning with Mars. In that at the beginning is the computation of the corrected (longitude) of Mars.

(a) The mean (longitude of) Mars is to be corrected for the minutes of the equation of time. (b) The apogee subtracted from that is the manda argument. (c) From that the manda equation in degrees and so on is to be obtained by proportion. (d) When the argument is (in the six signs) beginning with Aries, this (equation) is (applied) negatively to the mean (longitude of) Mars corrected for the minutes of the equation of time; when the argument is in (the six signs) beginning with Libra, it is (applied) positively; (the result) is (the mean longitude of) Mars corrected by the manda (equation). (e) This manda equation is also to be applied to the manda argument as it was to the (mean longitude of the) planet. (f) Again, from the degrees of the arc (modulo 90°) of (the mean longitude) corrected by the manda (equation) and diminished by (the longitude of) the node is to be obtained by proportion the mean "latitude" in degrees and so on; this is to be put down separately. (g) Again, from (the mean longitude of) Mars corrected by the first equation and diminished by (the longitude of) the node is to be obtained the equation of the "latitude" in seconds; this is positive in an even quadrant, negative in an odd quadrant. (h) It is to be applied to (the mean longitude of) Mars corrected by the first equation; and it is also to be applied to the corrected manda argument. (i) Again, from the corrected manda argument of the Sun is to be obtained by proportion the hypotenuse of the Sun. (j) Again, (the longitude of) the Sun plus six signs is to be computed; (the result) is the fixed (longitude) of the earth. (k) This is the perigee of the śīghra (epicycle). (l) Again, the (mean longitude) corrected by the manda (equation) is diminished by (or subtracted from) the fixed (longitude) of the earth. (m) Half of this is the śīghra argument. If the (mean longitude) corrected by the manda (equation) and diminished by the fixed (longitude) of the earth is less than six (signs), half of the śīghra argument is to be taken; if the (mean longitude) corrected by the manda (equation) and diminished by the fixed (longitude) of the earth is greater than six (signs), half of the śīghra argument subtracted from a circle is to be computed. (n) Now the degrees of its Cosine are to be computed. (o) Again, from the corrected manda argument is to be obtained in the table the manda hypotenuse of the planet. (p) Again, the Sine of the degrees of the Cosine of the mean "latitude" is (to be obtained) by proportion from the table of the "shadow." (q) This is to be added to the manda hypotenuse of the planet. (r) The "shadow" of 45 degrees measured by 1,000,000 is to be subtracted; the remainder is the measure of the Cosine of the equation of the "latitude," called the "line." (s) Again, the "line" is to be added to the "shadow" of 45 degrees measured by 1,000,000 and is to be diminished by the hypotenuse of the Sun; the remainder is the measure of the "shadow" of the first angle. (t) The arc from this is to be obtained in the tables of the "shadow"; this, when diminished by 45 degrees, is called the "second angle." (u) Again, the "shadow" of the degrees of the Cosine of the śīghra argument is to be added to the "shadow" of the second angle. (v) The "shadow" of 45 degrees measured by 1,000,000 is to be subtracted from this; the arc of the remainder is to be obtained in the tables of the "shadow." This is the measure of the angle of half the difference. (w) This is to be subtracted from (the degrees of) the Cosine of the śīghra argument; (the result) is the angle of the equation. (x) If it is added to (the degrees of) the Cosine of the śīghra argument, then it is the difference between the corrected (longitudes) of the planet and of the Sun. (y) This difference, if (the mean longitude of) Mars corrected by the manda (equation) is less than six (signs), is to be subtracted from the corrected (longitude of the) Sun, (but) if it is greater than six (signs), it is to be added; (the result) is the corrected (longitude of Mars). (z) The corrected (longitudes) of Jupiter and Saturn are to be computed in the same way.

## Example Given in the Sanskrit Prose Version.

Now an example. In that the mean (longitude of) Mars corrected for the minutes of the equation of time is $8^s3;51,9°$. This diminished by the apogee — $5^s1;10,9°$ — is $3^s2;41,0°$. This is the manda argument of Mars. From that the manda equation of Mars in degrees and so on (obtained) by proportion is $10;39,18°$. Because the manda argument is in (the six signs) beginning with Aries, it is (applied) negatively to the mean (longitude of) Mars, $8^s3;51,9°$; (the mean longitude of) Mars corrected by the manda (equation) — $7^s23;11,51°$ — is produced. The manda argument corrected by the manda equation is $2^s22;1,51°$.[5] Now the (mean longitude)

5 The computer used $3^s2;41,9°$ instead of $3^s2;41,0°$ for the manda argument. The corrected manda argument should be $2^s2;1,42°$.

corrected by the manda (equation) diminished by the (longitude of) the node — $1^s17;44,36°$ — is $6^s5;27,15°$. Its arc (modulo 90°) is $0^s5;27,15°$. From this the mean "latitude" in degrees and so on (obtained) by proportion is 0;10,33°. This is put down separately. Now from (the mean longitude of) Mars corrected by the first equation and diminished by (the longitude of) the node (is obtained) the equation of the "latitude" in seconds, −10. (The mean longitude of) Mars corrected by the manda (equation) corrected by this is $7^s23;11,41°$. The manda argument corrected by the manda (equation) corrected by the equation of the "latitude" is $2^s22;1,41°$. Now the corrected manda argument of the Sun is $10^s7;57,28°$. From that the hypotenuse of the Sun (obtained) by proportion is 400,462. Now (the longitude of) the Sun plus six signs is $7^s18;5,41°$. This is the fixed (longitude) of the earth. This is the perigee of the śīghra (epicycle). Now (the mean longitude) corrected by the manda (equation) and the second equation — $7^s23;11,41°$ — diminished by the fixed (longitude) of the earth is $0^s5;6,10°$.[6] Half of this is the śīghra argument. Because (the mean longitude) corrected by the manda (equation) diminished by the fixed (longitude) of the earth is less than six signs, half of it — $0^s2;33,5°$ — (is taken). Where (the mean longitude) corrected by the manda (equation and) diminished by the fixed (longitude) of the earth is greater than six signs, half of it subtracted from a circle is to be set down as the śīghra argument. (The degrees of) the Cosine of its half — $0^s2;33,5°$ — arc $2^s27;26,55°$. Its degrees are 87;26,55°. Now the corrected manda argument is $2^s22;1,51°$.[7] From that (is obtained) the hypotenuse of Mars, 418,476. Now the degrees of the Cosine of the mean "latitude" - 0;10,33° - are 89;49,27°. Their Sine is 999,999. The sum of this and the manda hypotenuse of Mars — 418,476 — is 1,418,475. This is the measure of the Cosine of the "latitude" called the "line." Here, as many as are the digits of the hypotenuse, so many digits of the Sine are to be obtained. Again, this 1,418,475 diminished by the hypotenuse of the Sun — 400,462 — is 1,018,013. This is the measure of the "shadow" of the first angle. The arc from that (obtained) from the table of "shadows" is 56;33°. In deriving the arc, the remainder is multiplied by sixty and divided by the (tabular) difference; the quotient is in minutes and so on. The result which is below the number 56 in the table of "shadows" is subtracted from the number of the measure of the "shadow" of the first angle, 1,016,013; 56 degrees (are represented) by that. The remainder is multiplied by sixty and divided by the tabular difference; (the result) is in minutes and so on. The angle 56;33° diminished by 45 degrees is 11;33°. This is called the "second angle." The "shadow" of this is 93,103,985. The śīghra argument is $0^s2;33,5°$, (its) Cosine is 2;27,26,55; (its) degrees are 87;26,55°; (its) "shadow" is 113,512,956. The sum of the two "shadows" is 206,616,941. The terminal (first number) is diminished by 1; (the result) is 106,616,941. Its arc (found) in the (table of) "shadows" is the angle of half of the difference. Here, because of the magnitude of the number, the sum of the two "shadows" is to be computed by a different method; it is thus. Subtract the (last)

6 This is a mistake for $0^s5;6,0°$.

7 See Note 5.

pair of numbers from the number of the "shadow" of the "second angle"; six digits are to be obtained — 931,039. Subtract the (last) pair of numbers also from the measure of the "shadow" of the degrees of the Cosine of the śīghra argument; seven digits are to be obtained, 1,135,129. The sum of the two "shadows" is 2,066,168. The terminal (number) is diminished by 1; (the result is) 1,066,168. Its arc in the tables of "shadows" is 77;42°. This is the angle of half of the difference. This is subtracted from the degrees of the Cosine of the śīghra argument, 87;26,55°, the remainder is 9;45°. This is the angle of the equation; another name is the śīghra equation. Because the argument is in (the six signs) beginning with Aries, this is added to (the mean longitude of) Mars corrected by the two (previous) equations, $7^{s}23;11,41°$; the corrected (longitude of) Mars is $8^{s}2;56,41°$. Now the degrees of the Cosine of the śīghra argument, 87;26,55°, are added to the angle of half of the difference, 77;42°, (the result) is 165;9°. This is the difference in degrees and so on between the accurate (longitudes) of the Sun and of Mars; when divided by thirty, it is in signs and so on — $5^{s}15;9°$. This, since it is opposite to (the longitude of) the planet, is subtracted from the corrected (longitude of the) Sun, $1^{s}18;5,41°$; the remainder is $8^{s}2;56,41°$. This is the corrected (longitude of) Mars.

## Summary of the Sanskrit Prose Version's Example

Letters in parentheses refer to the sentences of the rules.

| | | |
|---|---|---|
| (a) | $\bar{\lambda}_P$ for mean time | (Praeceptum I, correction 3; cf. Kevalarāma 1,7) |
| (b) | $\bar{\lambda}_P - \lambda_A = \bar{\kappa}_\mu$ | II |
| (c) | with $\bar{\kappa}_\mu$ find $\delta_\mu$ | II |
| (d) | $\bar{\lambda}_P \pm \delta_\mu = \lambda_P$ in the planet's orbit | II |
| (e) | $\bar{\kappa}_\mu \pm \delta_\mu = \kappa_\mu$ | II |
| (f) | $\lambda_P - \lambda_N = \omega$ | III |
| | with $\omega$ find $\angle$PSR | IV |
| (g) | with $\omega$ find Q | IV |
| (h) | $\lambda_P \pm Q = \lambda_P$ in the ecliptic | V |
| | $[\kappa_\mu \pm Q]$ | |
| (i) | with $\kappa_\mu$ of the Sun find log ST | I |
| (j) | $\angle$ITS ± 180° = $\angle$♈ST | VI |
| (k) | [$\angle$♈ST is the perigee of the śīghra (epicycle)] | |
| (l) | $\angle$♈ST − $\angle$♈SR (or $\angle$♈SR − $\angle$♈ST) = $\angle$TSR = s [8] | VI |

8 One expects the sum of $\angle$♈ST and $\angle$♈SR rather than their difference.

| | | |
|---|---|---|
| (m) | [ $\frac{s}{2}$ is the śīghra argument] | VII |
| | take $\frac{s}{2}$ | |
| (n) | take $90° - \frac{s}{2}$ | VII |
| (o) | with $\kappa_\mu$ find $\log SP$ | II |
| (p) | take $\log \mathrm{Cos} \angle PSR$ | VIII |
| (q+r) | $\log \mathrm{Cos} \angle PSR + \log SP - \log \mathrm{Tan}\, 45° = \log SR$ | VIII |
| (s) | $\log SR + \log \mathrm{Tan}\, 45° - \log ST = \log \mathrm{Tan}\, V$ | IX |
| (t) | $V - 45° = X$ | IX |
| (u+v) | $\log \mathrm{Tan}\,(90° - \frac{s}{2}) + \log \mathrm{Tan}\, X - \log \mathrm{Tan}\, 45° = \log \mathrm{Tan}\, \frac{t-r}{2}$ | X |
| (w) | $\left[\left(90° - \frac{s}{2}\right) - \frac{t-r}{2} = r\right]$ | |
| (x) | $\left(90° - \frac{s}{2}\right) + \frac{t-r}{2} = t = \angle STR$ | XI |
| (y) | $\angle ITS \pm \angle STR = \angle ITR$ | XII |

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The author is extremely grateful to Jan Hogendijk and Benno van Dalen for their careful reading and constructive criticism at various stages of this paper, and for their numerous improvements to it. All remaining errors, of course, are completely my own.

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# Index of Proper Names

This index contains all proper names found in the text and footnotes of this volume, including the authors of all referenced literature. Excluded are names mentioned in the titles of bibliography items, as well as fictitious persons occurring in only one particular source. Generally, Greek names have been inserted in their Latin form (e.g., Menelaus for Menelaos). Medieval persons in the Latin West (up to ca. 1475) are usually listed under their first names and all other persons under their surnames or family names. Cross references are supplied in all cases where multiple forms of a name are in use. In ordering Arabic and Persian names the article *al-* is disregarded, but *Ibn* ("son of") and *Abū* ("father of") are treated as part of the name. Diacritics on letters in Indian, Arabic and Persian names were ignored when alphabetizing them.

**T**

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